International Journal of Mathematical, Engineering and Management Sciences
Vol. 3, No. 2, 136–150, 2018
https://dx.doi.org/10.33889/IJMEMS.2018.3.2-011

A Summary of Replacement Policies with Number of Failures

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(Received March 22, 2017; Accepted September 27, 2017)

Abstract
Periodic replacement models with minimal repairs have been studied extensively. In order to prevent high repair costs after failures, this paper considers the number of failures as another decision variable for replacement policies. We begin with the standard periodic replacement model, and then summarize several extended replacement models with number of failures, using the respective assumptions of whichever occurs first and last, and replacing over planned measure, respectively. Optimum policies of replacement times T, number of periodic times N, and number of failures K are discussed analytically. From the optimization of the integrated models of T, N and K, comparisons among these policies are made in analytical ways to determine which policy should be more economical.

Keywords- Minimal repair, Periodic replacement, Replacement last, Random failure, Cumulative hazard rate.

1. Introduction

Maintenance policies are commonly performed at periodic times in applications. For example, maintenance in total productive maintenance (TPM) consists of periodical inspection, cleaning and servicing equipment and replacing deteriorated units to prevent serious breakdown problems (Venkatesh, 2007). The original model of periodic replacement was formulated by using the cumulative hazard rate, in which minimal repairs can fix failures without disturbing the failure rate (Barlow and Proschan, 1996; Nakagawa, 2005). Other theoretical research works on periodic patterns in maintenance plans have been studied extensively (Taghipour et al., 2010; Huynh et al., 2011; Taghipour and Banjevic, 2012; Golmakani and Moakedi, 2012; Wang and Banjevic, 2012; Taghipour and Kassaei, 2015; Kim et al., 2015).

Minimal repairs that cost less are always considered at minor failures for the large and complex systems with many kinds of units (Barlow and Proschan, 1996). Models for repairable system subjected to minimal repair (Pulcini, 2003), imperfect repair considering time-dependent repair effectiveness (Fuqing and Kumar, 2012), age-based replacement with repair for shocks and

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https://dx.doi.org/10.33889/IJMEMS.2018.3.2-011

degradation (Huynh et al., 2012), post-warranty maintenance with repair time threshold (Park et al., 2013), etc., have been studied extensively. Most recently, random working models with replacement and minimal repair (Zhao et al., 2016), and a case study of periodic maintenance policies for off-road engines has been surveyed (Toledo et al., 2016).

In order to minimize the repair and replacement cost for a long run and maximize the utilization of the operating unit, this paper begins with the standard periodic replacement model with minimal repairs (Nakagawa, 2005) and then summarizes several extended replacement policies with failures, using the assumptions of replacement first (Nakagawa, 2005), replacement last (Zhao and Nakagawa, 2012), and replacement overtime (Nakagawa and Zhao, 2015; Zhao et al., 2015). That is, assumptions of whichever occurs first and last, and replacing over planned measure will be taken into considerations for replacement policies, respectively. It will show that the extended models include the standard periodic replacement with minimal repairs. It has been indicated in (Zhao and Nakagawa, 2012; Nakagawa and Zhao, 2015; Zhao et al., 2015) that the last and overtime modes in replacement policies could let the unit operate for longer time with minimum replacement cost rates, which will also be discussed and compared with replacement first in this paper.

Throughout the paper, we assume that failures occur randomly at a non-homogeneous Poisson process with a hazard rate $H(t)$, then the probability that a number j of failures occur during time interval $[0, t]$ is

$$p_j(t) \equiv \frac{H(t)^j}{j!} e^{-H(t)} \qquad (j = 0, 1, 2, \dots).$$

Denote that $P_j(t) \equiv \sum_{i=j}^{\infty} p_i(t)$, $\bar{P}_j(t) \equiv 1 - P_j(t) = \sum_{i=0}^{j-1} p_i(t)$, then $P_0(t) \equiv 1$ and $\bar{P}_0(t) = 0$.

In addition, the failure distribution is obtained as $F(t) \equiv \sum_{j=1}^{\infty} p_j(t) = 1 - p_0(t) = 1 - e^{-H(t)}$ with mean time $\mu \equiv \int_0^{\infty} \bar{F}(t)dt$, and the failure rate is denoted as $h(t) \equiv dH(t)/dt$, where $h(t)$ increases with t from 0 to $h(\infty) \equiv \lim_{t\to\infty} h(t)$. We suppose the failures are fixed by minimal repairs without disturbing the failure rate. In order to prevent high repair costs after failures, we begin with the standard periodic model with minimal repairs in Section 2, and then summarize several extended replacement models with number of failures from Section 3.

2. Standard Periodic Model

Suppose the unit is replaced at periodic times nT $(n = 1, 2, \dots; 0 < T < \infty)$. Then (Nakagawa, 2005)

$$C_T(T) = \frac{c_T + c_M H(T)}{T}, \tag{1}$$

where c_T = replacement cost at each periodic time and c_M = cost of minimal repair at each failure. Clearly,

$$C_T(0) \equiv \lim_{T\to 0} C_T(T) = \infty, C_T(\infty) \equiv \lim_{T\to\infty} C_T(T) = c_M h(\infty). \tag{2}$$

Optimum T^* to minimize $C_T(T)$ satisfies

$$Th(T)-H(T)=\frac{c_T}{c_M},\ i.e.,\int_0^T t\,dh(t)=\frac{c_T}{c_M}. \tag{3}$$

The resulting cost rate is

$$C_T(T^*)=c_M h(T^*). \tag{4}$$

This optimum policy indicates that minimal repairs are made if some failures occur during $(0,T^*)$, and the unit is replaced at time T^*.

When $H(t)=t^m$ $(m>1)$, optimum T^* is given by a solution of $(m-1)T^m=c_T/c_M$. Table 1 presents optimum T^* and its cost rate for m and c_T/c_M. This shows that T^* increases with c_T/c_M and decreases with m, and its cost rate $C_T(T^*)$ increases with replacement cost c_T and the failure numbers for repairs.

Table 1. Optimum T^* and its cost rate when $H(t)=t^m$

c_T/c_M	$m=2.0$		$m=3.0$	
	T^*	$C_T(T^*)/c_M$	T^*	$C_T(T^*)/c_M$
2.0	1.41	2.83	1.00	3.00
3.0	1.73	3.46	1.14	3.93
4.0	2.00	4.00	1.26	4.76
5.0	2.24	4.47	1.36	5.53
6.0	2.45	4.90	1.44	6.24
7.0	2.65	5.29	1.52	6.91
8.0	2.83	5.66	1.59	7.56
9.0	3.00	6.00	1.65	8.18
10.0	3.16	6.32	1.71	8.77

3. Failure Over Periodic Time

As the first extension of the standard periodic replacement model in Section 2, we suppose the unit cannot be replaced strictly at periodic times, but is replaced at the first failure over the above each periodic time nT $(n=1,2,\ldots;0<T<\infty)$. Now the problem become to reconsider the periodic interval T with over time mode. Then (Nakagawa and Zhao, 2015)

$$C_O(T)=\frac{c_O+c_M[H(T)+1]}{T+1/Q(T)} \tag{5}$$

where c_O = replacement cost over each time T, and

$$Q(T)\equiv\frac{1}{\int_T^\infty e^{-[H(t)-H(T)]}dt}=\frac{\bar{F}(T)}{\int_T^\infty \bar{F}(t)dt}\geq h(T).$$

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Clearly, (5) and (1) have similar forms.

When $T \to \infty$,

$$C_O(\infty) \equiv \lim_{T\to\infty} C_O(T) = c_M h(\infty),$$

and when $T \to 0$,

$$C_O(0) \equiv \lim_{T \to 0} C_O(T) = \frac{c_O + c_M}{\mu}. \tag{6}$$

Differentiating $C_O(T)$ with respect to T and setting it equal to zero,

$$TQ(T) - H(T) = \frac{c_o}{c_M},\ i.e., \int_0^T [Q(T) - h(t)]dt = \frac{c_o}{c_M}. \tag{7}$$

Therefore, the following optimum policies are given:

a) If $\int_0^\infty t dh(t) > c_O/c_M$, then there exists a unique T_O^* $(0 < T_O^* < \infty)$ in (7) to minimize $C_O(T)$, and

$$C_O(T_O^*) = c_M Q(T_o^*) = \frac{c_O + c_M H(T_O^*)}{T_O^*}. \tag{8}$$

b) If $\int_0^\infty t dh(t) \le c_O/c_M$, $T_O^* = \infty$, and the cost rate has been given in (2).

Further, it can be found that $T_O^* \le T$ as $Q(T) \ \ge \ h(T)$ in (3) and (7). Thus, comparing of (1) and (8), when $c_O = c_T$, the standard periodic replacement is more economical.

When $H(t) = t^m$, Table 2 presents optimum optimum T_O^* and its cost rate for m and c_O/c_M. This indicates that $T_O^* < T$ and $C_O(T_O^*) > C_T(T^*)$ when $c_O = c_T$. However, the differences between two expected cost rates become smaller as c_O/c_M are larger.

Table 2. Optimum T_O^* and its cost rate when $H(t) = t^m$

c_O/c_M	$m = 2.0$		$m = 3.0$	
	T_O^*	$C_O(T_O^*)/c_M$	T_O^*	$C_O(T_O^*)/c_M$
2.0	1.15	2.89	0.78	3.18
3.0	1.51	3.50	0.96	4.05
4.0	1.81	4.02	1.10	4.84
5.0	2.07	4.49	1.22	5.58
6.0	2.31	4.91	1.33	6.28
7.0	2.53	5.30	1.42	6.94
8.0	2.74	5.66	1.51	7.58
9.0	2.93	6.00	1.59	8.19
10.0	3.12	6.33	1.66	8.78

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https://dx.doi.org/10.33889/IJMEMS.2018.3.2-011

4. Failure Number

In order to extend the following replacement policies with failures, we firstly in this section consider the case when only failure number is used for replacement decision. That is, the unit is replaced at a number K $(K = 1, 2, ...)$ of failures. Then (Nakagawa, 2005; Nakagawa and Zhao, 2015)

$$C_N(K) = \frac{c_K + c_M K}{\int_0^\infty \bar{P}_K(t)dt} (K = 1, 2, ...), \tag{9}$$

where $c_K =$ replacement cost at failure K. Clearly,

$$C_N(\infty) \equiv \lim_{K\to\infty} C_N(K) = c_M h(\infty),$$

in (2), and

$$C_N(1) = C_O(0) = \frac{c_K + c_M}{\mu}$$

in (6) when $c_O = c_K$. Forming the inequality $C_N(K+1) - C_N(K) \geq 0$,

$$\frac{\int_0^\infty \bar{P}_K(t)dt}{\int_0^\infty p_K(t)dt} - K \geq \frac{c_K}{c_M}. \tag{10}$$

The left-hand side of (10) increases with K (Nakagawa, 2005; Nakagawa, and Zhao, 2015). Letting $L_1(K)$ be the left-hand side of (10), if $L_1(\infty) > c_K/c_M$, then there exists a unique K^* $(1 \leq K^* < \infty)$ in (10) to minimize $C_N(K)$, and the resulting cost rate is

$$\frac{c_M}{\int_0^\infty p_{K^*-1}(t)dt} < C_N(K^*) \leq \frac{c_M}{\int_0^\infty p_{K^*}(t)dt}. \tag{11}$$

When $H(t) = t^m$, Table 3 presents optimum optimum K^* and its cost rate for m and c_K/c_M. In this case, optimum K^* is given by $K^* = [(c_K - c_M)/(m-1)c_M] + 1$, where $[x]$ means the greatest integer contained in x. It is nature that K^* can be increasing when the repair cost c_M decrease and the cost rate $C_N(K^*)$ increases with the number of failures.

Table 3. Optimum K^* and its cost rate when $H(t) = t^m$

c_K/c_M	$m = 2.0$		$m = 3.0$	
	K^*	$C_N(K^*)/c_M$	K^*	$C_N(K^*)/c_M$
2.0	2	2.89	2	3.36
3.0	3	3.50	2	4.20
4.0	5	4.02	3	5.04
5.0	5	4.49	3	5.76
6.0	6	4.91	3	6.48
7.0	7	5.30	4	7.13
8.0	8	5.66	4	7.77
9.0	10	6.00	5	8.37
10.0	10	6.33	6	8.97

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https://dx.doi.org/10.33889/IJMEMS.2018.3.2-011

5. Next Periodic Time

Suppose the unit is replaced at the next periodic time $(n+1)\ T$ over periodic time nT $(n = 0, 1, 2, ...; 0 < T < \infty)$ when more than K $(K = 1, 2, ...)$ failures have arrived, and we obtain an optimum K for given. Then (Nakagawa, 2005)

$$C_P(K,T) = \frac{c_T + c_M \sum_{n=0}^{\infty}[H((n+1)T) - H(nT)]\bar{P}_K(nT)}{T\sum_{n=0}^{\infty}\bar{P}_K(nT)} \qquad (K = 1, 2, ...). \tag{12}$$

Forming the inequality $C_P(K+1,T) - C_P(K,T) \geq 0$,

$$Q_1(K,T)\sum_{n=0}^{\infty}\bar{P}_K(nT) - \sum_{n=0}^{\infty}\big[H\big((n+1)T\big) - H(nT)\big]\bar{P}_K(nT) \geq \frac{c_T}{c_M}, \tag{13}$$

where

$$Q_1(K,T) \equiv \frac{\sum_{n=0}^{\infty}[H((n+1)T) - H(nT)]p_K(nT)}{\sum_{n=0}^{\infty}\bar{P}_K(nT)}.$$

Thus, the left-hand side $L_2(K,T)$ of (13) increases strictly with K. If $L_2(K,T) > c_T/c_M$, then there exists a unique and minimum K_P^* $(1 \leq K_P^* < \infty)$ in (13) to minimize $C_P(K,T)$, and

$$c_M Q_1(K_P^* - 1, T) < C_P(K_P^*, T) \leq c_M Q_1(K_P^*, T). \tag{14}$$

When $H(t) = t^m$, Table 4 presents optimum K_P^* and its cost rate for m and c_T/c_M, and it shows the same tendencies with Table 4.

Table 4. Optimum K_P^* and its cost rate when $H(t) = t^m$ and $T = 0.1$

c_T/c_M	$m = 2.0$		$m = 3.0$	
	K_P^*	$C_P(K_P^*,T)/c_M$	K_P^*	$C_P(K_P^*,T)/c_M$
2.0	2	3.00	1	3.42
3.0	3	3.60	1	4.40
4.0	4	4.12	2	5.03
5.0	5	4.58	3	5.78
6.0	6	5.00	3	6.47
7.0	7	5.38	4	7.14
8.0	8	5.74	4	7.77
9.0	19	6.08	5	8.39
10.0	10	6.40	6	9.00

6. Time and Failure Number

6.1 Replacement First

Suppose the unit is replaced at planned time T $(0 < T \leq \infty)$ or at failure number K $(1 \leq K \leq \infty)$, whichever occurs first. Then

$$C_F(T,K) = \frac{c_T\bar{P}_K(T) + c_K P_K(T) + c_M\int_0^T \bar{P}_K(t)h(t)dt}{\int_0^T \bar{P}_K(t)dt}, \tag{15}$$

clearly, $C_F(T,\infty) = C_T(T)$ in (1) and $C_F(\infty,K) = C_N(K)$ in (9).

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https://dx.doi.org/10.33889/IJMEMS.2018.3.2-011

When $c_T = c_K$, differentiating $C_F(T,K)$ with respect to T and setting it equal to zero,

$$\int_0^T \bar{P}_K(t)[h(T)-h(t)]dt = \frac{c_T}{c_M}. \tag{16}$$

Letting $L_3(T,K)$ be the left-hand side of (16), $L_3(T,K)$ increases strictly with T from 0. Thus, if $L_3(\infty,K) > c_T/c_M$, then there exists a unique T_F^* $(0 < T_F^* < \infty)$ which satisfies (16), and

$$C_F(T_F^*,K) = c_M h(T_F^*). \tag{17}$$

The optimum policy to minimize $C_F(T,K)$ is $(T_F^* = T^*, K_F^* = \infty)$. The optimum policy indicates that replacement done at time T is more economical than that at failure K.

Next, we find optimum K_F^* to minimize $C_F(T,K)$ for given T. Forming the inequality $C_F(T,K+1) - C_F(T,K) \geq 0$,

$$H_1(T,K)\int_0^T \bar{P}_K(t)dt - \int_0^T \bar{P}_K(t)h(t)dt \geq \frac{c_T}{c_M}\ , \tag{18}$$

where

$$H_1(T,K) = \frac{\int_0^T H(t)^K dF(t)}{\int_0^T H(t)^K \bar{F}(t)dt}.$$

Therefore, if $T \leq T^*$, then $K_F^* = \infty$, and conversely, if $T > T^*$, then optimum K_F^* $(1 \leq K_F^* < \infty)$ which satisfies (18) exists.

6.2 Replacement Last

Suppose the unit is replaced at planned time T $(0 \leq T \leq \infty)$ or at failure number K $(0 \leq K \leq \infty)$, whichever occurs last. The mean time to replacement is

$$TP_K(T) + \int_T^\infty t dP_K(t) = T + \int_T^\infty \bar{P}_K(t)dt. \tag{19}$$

The expected number of failures until replacement is

$$H(T) + \int_T^\infty \bar{P}_K(t)h(t)dt, \tag{20}$$

which agrees with (19) when $h(t) = 1$. Thus

$$C_L(T,K) = \frac{c_T P_K(T) + c_K \bar{P}_K(T) + c_M[H(T) + \int_T^\infty \bar{P}_K(t)h(t)dt]}{T + \int_T^\infty \bar{P}_K(t)dt}. \tag{21}$$

Clearly, $C_L(0,K) = C_F(\infty,K) = C_N(K)$ and $C_L(T,0) = C_F(T,\infty) = C_T(T)$.

When $c_T = c_K$, differentiating $C_L(T,K)$ with respect to T and setting it equal to zero,

$$\int_0^T [h(T)-h(t)]dt - \int_T^\infty \bar{P}_K(t)[h(t)-h(T)]dt = \frac{c_T}{c_M}. \tag{22}$$

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https://dx.doi.org/10.33889/IJMEMS.2018.3.2-011

There exists a unique T_L^* $(0 < T_L^* < \infty)$ which satisfies (22), and the resulting cost rate is

$$C_L(T_L^*, K) = c_M h(T_L^*). \tag{23}$$

The optimum policy to minimize $C_L(T,K)$ is $(T_L^* = T^*, K_L^* = \infty)$. The optimum policy indicates that replacement done at time T is more economical than that at failure K.

Next, we find optimum K_L^* to minimize $C_L(T,K)$ for given T. Forming the inequality $C_L(T,K+1) - C_L(T,K) \geq 0$,

$$\int_0^T [\widetilde{H}_1(T,K) - h(t)]dt - \int_T^\infty \bar{P}_K(t)[h(t) - h(T)]dt \geq \frac{c_T}{c_M}, \tag{24}$$

where

$$\widetilde{H}_1(T,K) \equiv \frac{\int_T^\infty H(t)^K dF(t)}{\int_T^\infty H(t)^K \bar{F}(t)dt}.$$

Thus, the left-hand side of (24) increases strictly from that of (7). Therefore, if $T \geq T_O^*$, then $K_L^* = 0$, and conversely, if $T < T_O^*$, then optimum K_L^* $(1 \leq K_L^* < \infty)$ exists.

The following results are obtained by comparing the policies of periodic replacement, replacement first and last in Sections 2 and 6:
a) If $T < T_O^*$, then replacement last should be adopted.
b) If $T_O^* \leq T \leq T^*$, then standard periodic replacement should be adopted.
c) If $T > T^*$, then replacement first should be adopted.

6.3 Numerical Examples

Table 5 presents optimum T_F^* and T_L^* and their cost rates for given K when $H(t) = t^{2.0}$. Table 6 presents optimum K_F^* and K_L^* and their cost rates for given T when $H(t) = t^{2.0}$. In Table 5, when $K = 5$, it shows that $T_F^* < T_L^*$ and $C_F(T_F^*, K) < C_L(T_L^*, K)$ for $c_T/c_M \leq 4.0$. In Table 6, it shows that $C_F(T, K_F^*) > C_L(T, K_L^*)$ for $K_F^* = \infty$ and $C_F(T, K_F^*) < C_L(T, K_L^*)$ for 0.

Table 5. Optimum T_F^* and T_L^* and their cost rates for given K when $H(t) = t^{2.0}$

c_T/c_M	$K = 1$				$K = 5$			
	T_F^*	$C_F(T_F^*,K)/c_M$	T_L^*	$C_L(T_L^*,K)/c_M$	T_F^*	$C_F(T_F^*,K)/c_M$	T_L^*	$C_L(T_L^*,K)/c_M$
2.0	1.69	3.38	1.42	2.84	1.42	2.83	1.60	3.20
3.0	2.26	4.51	1.73	3.47	1.74	3.47	1.83	3.65
4.0	2.82	5.64	2.00	4.00	2.02	4.03	2.05	4.09
5.0	3.39	6.77	2.24	4.47	2.27	4.54	2.26	4.52
6.0	3.95	7.90	2.45	4.90	2.51	5.02	2.46	4.92
7.0	4.51	9.03	2.65	5.29	2.75	5.49	2.65	5.30
8.0	5.08	10.16	2.83	5.66	2.98	5.96	2.83	5.66
9.0	5.64	11.28	3.00	6.00	3.21	6.42	3.00	6.00
10.0	6.21	12.41	3.16	6.32	3.44	6.88	3.16	6.33

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Table 6. Optimum K_F^* and K_L^* and their cost rates for given T when $H(t) = t^{2.0}$

	$T = 1.0$				$T = 5.0$			
c_T/c_M	K_F^*	$C_F(T,K_F^*)/c_M$	K_L^*	$C_L(T,K_L^*)/c_M$	K_F^*	$C_F(T,K_F^*)/c_M$	K_L^*	$C_L(T,K_L^*)/c_M$
2.0	∞	3.00	2	2.94	3	3.01	0	5.40
3.0	∞	4.00	3	3.59	4	3.61	0	5.60
4.0	∞	5.00	5	4.12	5	4.13	0	5.80
5.0	∞	6.00	6	4.58	6	4.59	0	6.00
6.0	∞	7.00	7	5.00	7	5.00	0	6.20
7.0	∞	8.00	8	5.38	8	5.39	0	6.40
8.0	∞	9.00	9	5.74	9	5.75	0	6.60
9.0	∞	10.00	10	6.08	10	6.08	0	6.80
10.0	∞	11.00	11	6.40	11	6.40	0	7.00

7. Periodic Times and Failure Number

7.1 Replacement First

Suppose the unit is replaced at NT $(N = 1, 2, \dots; 0 < T < \infty)$ or at failure K $(1 \leq K < \infty)$, whichever occurs first. Then, replacing formally T with NT in (15), the expected cost rate is

$$C_F(N,K) = \frac{c_T \bar{P}_K(NT) + c_K P_K(NT) + c_M \int_0^{NT} \bar{P}_K(t) h(t) dt}{\int_0^{NT} \bar{P}_K(t) dt}. \tag{25}$$

When $c_T = c_K$, forming the inequality $C_F(N+1,K) - C_F(N,K) \geq 0$,

$$\int_0^{NT} \bar{P}_K(t)[H_1(N,K) - h(t)]dt \geq \frac{c_T}{c_M}\ , \tag{26}$$

where

$$H_1(N,K) \equiv \frac{\int_{NT}^{(N+1)T} \bar{P}_K(t) h(t) dt}{\int_{NT}^{(N+1)T} \bar{P}_K(t) dt}.$$

Therefore, if $\int_0^{\infty} \bar{P}_K(t)[h(\infty) - h(t)]dt > c_T/c_M$, then there exists a finite and unique N_F^* $(1 \leq N_F^* < \infty)$ which satisfies (26), and the resulting cost rate is

$$c_M H_1(N_F^* - 1, K) < C_F(N_F^*, K) \leq c_M H_1(N_F^*, K). \tag{27}$$

Furthermore, (26) becomes

$$N\{H[(N+1)T] - H(NT)\} - H(NT) \geq \frac{c_T}{c_M}\ , \tag{28}$$

as $K \to \infty$. Thus, if $\int_0^{\infty} t dh(t) > c_T/c_M$, then a unique and minimum N_F^* which satisfies (28) exists. Therefore, because N_F^* decreases strictly with K to N^*, from (27), optimum policy to minimize $C_F(N,K)$ is $(N_F^* = N^*, K_F^* = \infty)$, where N^* is given in (28).

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7.2 Replacement Last

Suppose the unit is replaced at time NT $(N = 0, 1, 2, \dots; 0 < T < \infty)$ or at failure K $(0 \le K < \infty)$, whichever occurs last. Then, replacing formally T with NT in (21), the expected cost rate is

$$C_L(N, K) = \frac{c_T P_K(NT) + c_K \bar{P}_K(NT) + c_M[H(NT) + \int_{NT}^{\infty} \bar{P}_K(t)h(t)dt]}{NT + \int_{NT}^{\infty} \bar{P}_K(t)dt}. \qquad (29)$$

When $c_T = c_K$, forming the inequality $C_L(N+1, K) - C_L(N, K) \ge 0$,

$$\int_0^{NT} [\widetilde{H}_1(N, K) - h(t)]dt - \int_{NT}^{\infty} \bar{P}_K(t)[h(t) - \widetilde{H}_1(N, K)]dt \ge \frac{c_T}{c_M},$$

where

$$\widetilde{H}_1(N, K) \equiv \frac{\int_{NT}^{(N+1)T} p_K(t)h(t)dt}{\int_{NT}^{(N+1)T} p_K(t)dt}. \qquad (30)$$

Therefore, if $\int_0^{\infty} tdh(t) > c_T/c_M$, then there exists a finite and unique minimum N_L^* $(0 \le N_L^* < \infty)$ which satisfies (30), and the resulting cost rate is

$$c_M \widetilde{H}_1(N_L^* - 1, K) < C_L(N_L^*, K) \le c_M \widetilde{H}_1(N_L^*, K). \qquad (31)$$

Furthermore, noting that the left-hand side of (30) increases strictly with K, (30) becomes (28).Therefore, because N_L^* increases strictly with K from N^*, from (31), optimum policy to minimize $C_L(N, K)$ is $(N_L^* = N^*, K_L^* = 0)$, where N^* is given in (28).

7.3 Numerical Examples

Table 7 presents optimum N_F^* and N_L^* and their cost rates for given K when $H(t) = t^{2.0}$ and $T = 0.1$. This shows that when $K = 1$, $C_L(N_L^*, K) < C_F(N_F^*, K)$, and for $K = 5$, $C_F(N_F^*, K) < C_L(N_L^*, K)$ when $c_T/c_M < 5.0$.

Table 7. Optimum N_F^* and N_L^* and their cost rates for given T when $H(t) = t^{2.0}$ and $T = 0.1$

c_T/c_M	$K = 1$				$K = 5$			
	N_F^*	$C_F(N_F^*, K)$	N_L^*	$C_L(N_L^*, K)$	N_F^*	$C_F(N_F^*, K)$	N_L^*	$C_L(N_L^*, K)$
2.0	15	3.38	14	2.84	14	2.83	16	3.20
3.0	20	4.51	17	3.47	17	3.47	18	3.65
4.0	21	5.64	20	4.00	20	4.03	20	4.09
5.0	22	6.77	22	4.47	22	4.54	21	5.52
6.0	22	7.90	24	4.90	25	5.02	24	4.92
7.0	22	9.03	26	5.29	27	5.49	26	5.30
8.0	23	10.16	27	5.66	27	5.96	28	5.66
9.0	27	11.28	29	6.00	28	6.42	30	6.00
10.0	27	12.41	30	6.33	29	6.88	30	6.33

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8. Over Time and Failure Number

8.1 Replacement First

Suppose the unit is replaced at failure K or at the first failure over T, whichever occurs first (Nakagawa and Zhao, 2015). Then, the mean time to replacement is

$$\int_0^T t dP_K(t) + \sum_{j=0}^{K-1} \int_0^T \left[\frac{1}{\bar{F}(t)} \int_T^\infty u dF(u)\right] d\,P_j(t) = \int_0^T \bar{P}_K(t)dt + \frac{\bar{P}_K(T)}{Q(T)}. \tag{32}$$

The expected number of failures until replacement is

$$KP_K(T) + \sum_{j=0}^{K-1}(j+1)p_j(T) = \int_0^T \bar{P}_K(t)h(t)dt + \bar{P}_K(T) = \sum_{j=0}^{K-1} P_j(T). \tag{33}$$

which agrees with (32) when $h(t) + Q(t) = 1$. Then

$$C_{OF}(T,K) = \frac{c_O \bar{P}_K(T) + c_K P_K(T) + c_M \sum_{j=0}^{K-1} P_j(T)}{\int_0^T \bar{P}_K(t)dt + \bar{P}_K(T)/Q(T)}, \tag{34}$$

which agrees with $C_O(T)$ in (5) as $K \to \infty$ and agrees with $C_N(K)$ in (9) as $T \to \infty$.

When $c_O = c_K$, differentiating $C_{OF}(T,K)$ with respect to T and setting it equal to zero,

$$\int_0^T \bar{P}_K(t)[Q(T) - h(t)]dt = \frac{c_O}{c_M}. \tag{35}$$

Thus, if $\int_0^\infty \bar{P}_K(t)[h(\infty) - h(t)]dt > c_O/c_M$, there exists a finite and unique T_{OF}^* $(0 < T_{OF}^* < \infty)$ which satisfies (35), and the resulting cost rate is

$$C_{OF}(T_{OF}^*, K) = c_M Q(T_{OF}^*). \tag{36}$$

Therefore, optimum policy to minimize $C_{OF}(T,K)$ is $(T_{OF}^* = T_O^*, K_{OF}^* = \infty)$, where T_O^* is given in (7). This optimum policy indicates that replacement done over time T is more economical than that at failure K.

Next, we find optimum K_{OF}^* to minimize $C_{OF}(T,K)$ for given T. Forming the inequality $C_{OF}(T,K+1) - C_{OF}(T,K) \geq 0$,

$$H_2(T,K)\left[\int_0^T \bar{P}_K(t)dt + \frac{\bar{P}_K(T)}{Q(T)}\right] - \sum_{j=0}^{K-1} P_j(T) \geq \frac{c_O}{c_M}, \tag{37}$$

where

$$H_2(T,K) \equiv \frac{\int_0^T H(t)^{K-1} dF(t)}{\int_0^T H(t)^{K-1}\left\{\int_t^\infty e^{-[H(u)-H(t)]}du\right\}F(t)}.$$

Thus, the left-hand side of (37) increases with K to that of (7). Therefore, if $T \leq T_O^*$, then $K_{OF}^* = \infty$, and conversely, if $T > T_O^*$, then optimum K_{OF}^* $(1 \leq K_{OF}^* < \infty)$ exists.

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8.2 Replacement Last

Suppose that the unit is replaced at failure K or at the first failure over T, whichever occurs last (Nakagawa and Zhao, 2015). Thus, the mean time to replacement is

$$T + \int_T^\infty \bar{P}_K(t)dt + \frac{P_K(T)}{Q(T)}. \tag{38}$$

The expected number of failures is

$$K\bar{P}_K(T) + \sum_{j=K+1}^{\infty}(j+1)\,p_j(T) = H(T) + 1 + \sum_{j=1}^{K-1}\bar{P}_j(T). \tag{39}$$

Eq. (39) agrees with (38) when $h(t) = Q(t) = 1$. Then

$$C_{OL}(T,K) = \frac{c_O P_K(T) + c_K \bar{P}_K(T) + c_M\left[H(T) + 1 + \sum_{j=1}^{K-1}\bar{P}_j(T)\right]}{T + \int_T^\infty \bar{P}_K(t)dt + \frac{P_K(T)}{Q(T)}}. \tag{40}$$

which agrees with $C_O(T)$ in (5) as $K = 0$ and agrees with $C_N(K)$ in (9) as $T \to 0$.

When $c_O = c_K$, differentiating $C_{OL}(T,K)$ with respect to T and setting it equal to zero,

$$\int_0^T [Q(T) - h(t)]dt - \int_T^\infty \bar{P}_K(t)[h(t) - Q(T)]dt = \frac{c_O}{c_M}. \tag{41}$$

Thus, if $\int_0^T t dh(t) > c_O/c_M$, there exists a unique T_{OL}^* $(0 < T_{OL}^* < \infty)$ which satisfies (41), and the resulting cost rate is

$$C_{OL}(T_{OL}^*, K) = c_M Q(T_{OL}^*). \tag{42}$$

Furthermore, letting $L_1(K;T)$ be the left-hand side of (41), $L_1(K;T) - L_1(K+1;T) > 0$, $L_1(K;T)$ decreases strictly with K from the left-hand side of (7). Thus, T_{OL}^* increases with K to T_O^* given in (7). Therefore, optimum policy to minimize $C_{OL}(T,K)$ is $(T_{OL}^* = T_O^*, K_{OL}^* = 0)$, where T_O^* is given in (7). This optimum policy indicates that replacement done over time T is more economical than that at failure K.

Next, we find optimum K_{OL}^* to minimize $C_{OL}(T,K)$ for given T. Forming the inequality $C_{OL}(T,K+1) - C_{OL}(T,K) \geq 0$,

$$\widetilde{H}_2(T,K)\left[T + \int_T^\infty \bar{P}_K(t)dt + \frac{P_K(T)}{Q(T)}\right] - \left[H(T) + 1 + \sum_{j=1}^{K-1} P_j(T)\right] \geq \frac{c_O}{c_M}\ , \tag{43}$$

where

$$\widetilde{H}_2(T,K) \equiv \frac{\int_T^\infty H(t)^{K-1} dF(t)}{\int_T^\infty H(t)^{K-1}\left\{\int_t^\infty e^{-[H(u)-H(t)]}\right\}dF(t)}. \tag{44}$$

Thus, the left-hand side of (43) increases with K from

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$$\widetilde{H}_2(T,1)\left[T+\frac{1}{Q(T)}\right]-H(T)-1>TQ(T)-H(T),$$

which agrees with (7). Letting $\tilde{T}_O$ be a solution of

$$\widetilde{H}_2(T,1)\left[T+\frac{1}{Q(T)}\right]-H(T)-1=\frac{c_O}{c_M}, \tag{45}$$

then $\tilde{T}_O < T_O^*$. Therefore, if $T \geq \tilde{T}_O$, then $K_{OL}^* = 0$, and conversely, if $T < \tilde{T}_O$, then optimum K_{OL}^* $(1 \leq K_{OL}^* < \infty)$ exists.

8.3 Numerical Examples

Table 8 presents optimum T_{OF}^* and T_{OL}^* and their cost rates for given K when $H(t) = t^{2.0}$, This shows that when $K = 5$, $C_{OF}(T_{OF}^*, K) < C_{OL}(T_{OL}^*, K)$ for $c_O/c_M \leq 4.0$, and $C_{OF}(T_{OF}^*, K) > C_{OL}(T_{OL}^*, K)$ for $c_O/c_M \geq 5.0$.

Table 8. Optimum T_{OF}^* and T_{OL}^* and their cost rates for given K when $H(t) = t^{2.0}$

c_O $/c_M$	$K=1$				$K=5$			
	T_{OF}^*	$C_{OF}(T_{OF}^*,K)$ $/c_M$	T_{OL}^*	$C_{OL}(T_{OL}^*,K)$ $/c_M$	T_{OF}^*	$C_{OF}(T_{OF}^*,K)$ $/c_M$	T_{OL}^*	$C_{OL}(T_{OL}^*,K)$ $/c_M$
2.0	1.45	3.38	1.16	2.89	1.16	2.89	1.35	3.21
3.0	2.11	4.51	1.52	3.50	1.52	3.50	1.61	3.66
4.0	2.76	5.64	1.82	4.02	1.83	4.04	1.87	4.10
5.0	3.44	6.77	2.09	4.48	2.12	4.55	2.11	4.52
6.0	4.14	7.90	2.33	4.90	2.40	5.03	2.34	4.92
7.0	4.89	9.03	2.56	5.29	2.68	5.49	2.56	5.30
8.0	5.70	10.16	2.77	5.66	2.95	5.96	2.77	5.66
9.0	6.59	11.28	2.97	6.00	3.22	6.42	2.97	6.00
10.0	7.59	12.41	3.17	6.32	3.50	6.88	3.17	6.33

Table 9 presents optimum K_{OF}^* and K_{OL}^* and their cost rates for given T when $H(t) = t^{2.0}$. Noting that $1 \leq T^* \leq 3.16$ in Table 9 for $m = 2$, when $c_O/c_M = 10.0$, $K_{OF}^* = \infty$ and $K_{OL}^* = 0$, and for $2 \leq c_O/c_M \leq 10.0$, when $T = 1.0$, $K_{OF}^* = \infty$, and when $T = 5$, $K_{OL}^* = 0$.

Table 9. Optimum K_{OF}^* and K_{OL}^* and their cost rates for given T when $H(t) = t^{2.0}$

c_O $/c_M$	$T=1.0$				$T=5.0$			
	K_{OF}^*	$C_{OF}(T,K_{OF}^*)$ $/c_M$	K_{OL}^*	$C_{OL}(T,K_{OL}^*)$ $/c_M$	K_{OF}^*	$C_{OF}(T,K_{OF}^*)$ $/c_M$	K_{OL}^*	$C_{OL}(T,K_{OL}^*)$ $/c_M$
2.0	∞	2.89	0	2.89	2	3.01	0	5.48
3.0	∞	3.62	3	3.58	3	3.61	0	5.68
4.0	∞	4.34	4	4.12	4	4.13	0	5.87
5.0	∞	5.07	5	4.58	5	4.59	0	6.07
6.0	∞	5.79	6	5.00	6	5.00	0	6.26
7.0	∞	6.52	7	5.39	7	5.39	0	6.46
8.0	∞	7.24	8	5.75	8	5.75	0	6.66
9.0	∞	7.96	9	6.08	9	6.08	0	6.85
10.0	∞	8.69	10	6.40	10	6.40	0	7.05

9. Conclusions

We have summarized several extended replacement models with number of failures in this paper: (i) The standard periodic replacement model with times nT in (Barlow and Proschan, 1965; Nakagawa, 2005); (ii) Replacement is done at the first failure over T. (iii) Replacement is done at failure number K. (iv) Replacement is done at the next periodic time overt periodic times nT when more than K failures have arrived. (v) Replacement is done at time T or at failure number K, whichever occurs first, or whichever occurs last. (vi) Replacement is done at time NT or at failure number K, whichever occurs first, or whichever occurs last. (vii) Replacement is done at failure number K or at the first failure over T, whichever occurs first, and whichever occurs last. Decision variables T, N and K of replacement policies have been optimized analytically and computed numerically.

It has been shown that optimum time in (ii) is less than or equal to that in (i) and the policy in (i) is more economical than that in (ii). When the number of failures K has been considered in (iii), its comparison with (i) is discussed from the optimization in (v), which indicates that replacement done at T is more economical than that at K. The overtime replacement in (vii) also shows that replacement done at failure K is not so economical. Comparisons of the approaches of whichever occurs first and last in (v), (vi) and (vii) indicates that both have advantage and disadvantage in cost, and the analytical discussions show that we can compare the replacement policies from the optimizations of their integrated models.

Acknowledgement
This paper was made possible by National Natural Science Foundation of China under Grant No. 71371097.

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