

Analysis of an $M^{[X]}/G(a, b)/1$ Unreliable G-queue with Loss, Instantaneous Bernoulli Feedback, Vacation and Two Delays of Verification

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(Received November 7, 2018; Accepted February 4, 2019)

Abstract

This paper deals with a batch arrival that customers arrive to the system according to a compound Poisson process. The customer's behavior is incorporated according to which loss with a certain probability and the server begins to provide a service only when a queue size minimum say 'a' and maximum service capacity is 'b'. Once the server completes the service, the unsatisfied customers may get the same service under Bernoulli schedule is termed as instantaneous Bernoulli feedback. The occurrence of negative customer cause the server to fail and removes a group of customers or an amount of work if present upon its arrival. As soon as the failure instant, the service channel send to the two delays of verification, the first verification delay starts before the repair process and the second verification delay begins after the repair process. We use the generating function method to derive the stationary queue size distribution. Some important performance measures such as different states of the system and the expected length of the queue explicitly. Some important special cases and numerical examples are determined.

Keywords- Bulk service, Feedback service, Vacation, G-queue, Two delays of verification.

1. Introduction

Bulk arrival and batch service queueing system are quite common in many real life situations such as the arrival of aircraft passengers, elevators, manufacturing systems, communication network, giant wheel, tourism etc. First work on bulk service queues was discussed by Bailey (1954). Neuts (1967) introduced the "General Bulk Service Rule" for which service begins only when an adequate number of customers in the queue is presented. Lee et al. (1992) have presented the bulk service queue with single vacation and derived the queue size distribution at a service completion epoch. Haridass and Arumuganathan (2008) analyzed a batch arrival, bulk service queue with the random breakdown and single vacation. Recently, Jeyakumar and Senthilnathan (2016) considered the bulk queue with multiple working vacations.

The concept of impatient customers in queueing models often occurs. We see the uses of the balking queue in emergency services in the hospital dealing with serious patients, web access, inventory system and production, etc. Haight (1957) was the first person who examined the concept of balking. Altman and Yechiali (2006) analyzed the queue with customers impatience and the server vacations. The Big-M penalty function technique has been used to transform the constrained optimization problem into an unconstrained optimization problem was discussed by Sahoo (2017). Ayyappan and Shyamala (2013) studied the batch arrival queueing model with vacation and balking.

An additional feature widely discussed in Bernoulli's customer feedback. Many queueing situations have the feature that the customers may be served again and again for a specific reason. If a customer's service is unsatisfied, it can be tried on a Bernoulli schedule. This type of re-service is termed as instantaneous Bernoulli feedback. Ke and Chang (2009) studied a non-Markovian retrial queue with modified vacation policy and feedback. Bhunia et al. (2017) considered the penalty function technique for dealing with the constraints and hybrid algorithm. Tamura and Yamada (2017); Bose and Pain (2018) used the different methods to obtain the optimum solution. Some of the authors like Rajadurai et al. (2014) and Krishnakumar et al. (2013) have contributed the work on Bernoulli feedback.

Queueing systems with server vacations have been found to be useful to design the models in which the server performs certain additional tasks, being checked for maintenance or simply taking a break. A single server queueing system with Bernoulli vacation schedule was introduced by Keilson and Servi (1986). Retrial queue with two stage service under Bernoulli vacation schedule and random failure was greatly discussed by Rajadurai et al. (2018). A Bernoulli vacation model under RA-policy was analyzed by Madan and Choudhury (2004). Ram and Manglik (2016) have discussed the reliability measures of the system such as availability, reliability, mean time to failure (MTTF), cost analysis and sensitivity analysis with the help of Laplace transformation, supplementary variable technique and Markov technique.

In real life situations, service interruptions are unpredictable cases and it makes the system to be more realistic. G-queues are the queues which are formed by negative customers and these types of customers have the effect of removing a positive customer from the queue and as a result, the customer loses his service and leaves the system. Two different types of the service model subject to the server breakdown and delay time to repair were studied by Choudhury and Tadj (2009). Wu and Lian (2013); Zhang and Liu (2015) considered the queue with negative customers and used the different methods to obtain the queue size distribution at an arbitrary epoch. An $M^{[X]}/G/1$ queueing system with loss, vacation and two delays of verification have been derived by Saggou et al. (2017). Terfas et al. (2018) have analyzed the queueing system with one unreliable server and two types of verification. Zirem et al. (2018) have discussed the batch arrival unreliable queue with balking and general retrial time.

The remainder part of this paper is organized in the following way: Section 2 provides a mathematical description of the model and definitions. In Section 3, we present the Kolmogorov forward equations of our model. The system's steady state behavior and the probability generating function of the queue size were explicitly derived in Section 4. Stability condition is found in Section 5. The system related performance measures like the various states of the system and expected length of the queue are derived in Section 6. Some of the existing results are deduced as particular cases are shown in Section 7. Numerical results of the system are acquired, the corresponding graph trend is given in Section 8. In Section 9, this research work is concluded with the proposed future work.

2. Model Description

A batch of positive customers arrive at the system according to a compound Poisson process with mean arrival rate Λ^+ . Let C_k indicate the number of customers belonging to the k^{th} arrival batch, where $C_k, k=1,2,3,\dots$ are independently and identically distributed (i.i.d) random variables (r.v) with a common distribution $\Pr[C_k = n] = c_n, n \geq 1$. If an arriving batch of customers find the server being busy, feedback service, vacation, two delays of verification, repair or idle period, the

arrivals either leave the service area with probability $1 - m$ or join the queue with probability m . The server starts service under “General Bulk Service Rule” only if a specified minimum say ‘a’ of customers have accumulated in the queue and he does not take more than ‘b’ customers for service in one batch. As soon as a batch of customers completes the service, he may repeat the same service under Bernoulli schedule with probability π or may leaves the system with complementary probability $(1 - \pi)$. Upon completion of service, the server may take a Bernoulli vacation with probability θ and wait for the next batch of customers with complementary probability $1 - \theta$. While the busy server is subjected to breakdown due to the arrival of negative customers. That time server will breakdown for a short interval of time. At the time of failure, the server takes some time before the repair process is term as First Delay of Verification (FDV). After the first delay of verification, it enters into the repair. At the end of the repair process the Second Delay of Verification (SDV) begins before the server will treat as good as new. Pictorial representation of the model under consideration is depicted in Figure 1.

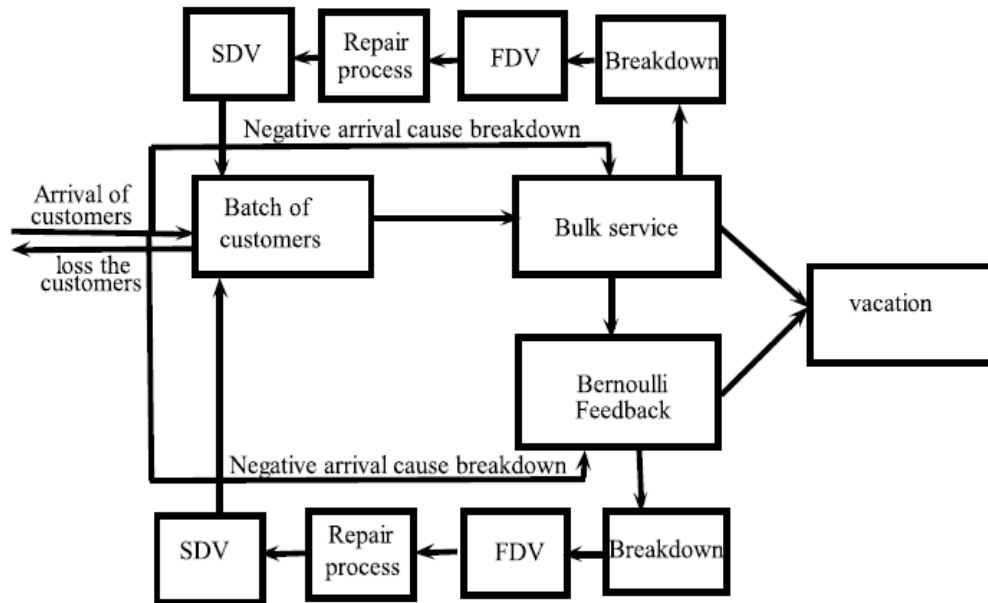


Figure 1. Pictorial representation of the model

2.1 Definitions

In this section, we solve the equations and derive the PGFs of the stationary queue size distribution. Let $\psi(t)$ be the queue length at time t , $M_1^0(t)$ be the elapsed service time at time t , $M_2^0(t)$ be the elapsed feedback service time at time t , $V^0(t)$ be the elapsed vacation time at time t , $D_1^0(t)$ and $D_2^0(t)$ be the elapsed first and second delay time at time t and $R^0(t)$ be the elapsed repair time at time t .

The state of the server $\mathcal{Y}(t)$ at time t is given by, $\mathcal{Y}(t) = 0, 1, 2, 3, 4, 5, 6$ denotes if the server is idle, in service, in feedback service, on Bernoulli vacation, under first delay time, under second delay time, under repair respectively.

Joint distributions of the server state and queue size are defined as,

$$Q_r(t)dt = P\{\psi(t) = r, \mathcal{Y}(t) = 0\}, \text{ for } t \geq 0, 0 \leq r \leq a - 1$$

$$\mathcal{H}_{1,n}(w, t)dw = P\{\psi(t) = n, \mathcal{Y}(t) = 1; w \leq M_1^0(t) \leq w + dw\}, \text{ for } w, t \geq 0, n \geq 0$$

$$\mathcal{H}_{2,n}(w, t)dw = P\{\psi(t) = n, \mathcal{Y}(t) = 2; w \leq M_2^0(t) \leq w + dw\}, \text{ for } w, t \geq 0, n \geq 0$$

$$\mathcal{V}_n(w, t)dw = P\{\psi(t) = n, \mathcal{Y}(t) = 3; w \leq V^0(t) \leq w + dw\}, \text{ for } w, t \geq 0, n \geq 0$$

$$\mathcal{D}_{1,n}(w, t)dw = P\{\psi(t) = n, \mathcal{Y}(t) = 4; w \leq D_1^0(t) \leq w + dw\}, \text{ for } w, t \geq 0, n \geq 0$$

$$\mathcal{D}_{2,n}(w, t)dw = P\{\psi(t) = n, \mathcal{Y}(t) = 5; w \leq D_2^0(t) \leq w + dw\}, \text{ for } w, t \geq 0, n \geq 0$$

$$\mathcal{R}_n(w, t)dw = P\{\psi(t) = n, \mathcal{Y}(t) = 6; w \leq R^0(t) \leq w + dw\}, \text{ for } w, t \geq 0, n \geq 0$$

Service time, vacation time, feedback service time, repair time, first and second delay time follow general (arbitrary) distribution and the notations used for their probability density functions (pdf), Cumulative Distribution Function (CDF), Laplace Stieltjes Transform are given in Table 1.

Table 1. Notations

Time	CDF	Hazard rate	pdf	LST
Service	$M_1(w)$	$\mu_1(w)$	$m_1(h) = \mu_1(h)e^{-\int_0^h \mu_1(w)dw}$	$m_1^*(s)$
Bernoulli Vacation	$V(w)$	$\gamma(w)$	$v(r) = \gamma(r)e^{-\int_0^r \gamma(w)dw}$	$v^*(s)$
Feedback Service	$M_2(w)$	$\mu_2(w)$	$m_2(h) = \mu_2(h)e^{-\int_0^h \mu_2(w)dw}$	$m_2^*(s)$
Repair	$R(w)$	$\beta(w)$	$r(g) = \beta(g)e^{-\int_0^g \beta(w)dw}$	$r^*(s)$
Two delays of verification	$D_i(w)$	$\xi_i(w)$	$d_i(f) = \xi_i(f)e^{-\int_0^f \xi_i(w)dw}$	$d_i^*(s)$

3. Mathematical Formulation of Model

The given model is ruled by the following set of Kolmogorov forward difference differential equations.

The server is in busy state

$$\frac{\partial}{\partial w} \mathcal{H}_{1,n}(w, t) + \frac{\partial}{\partial t} \mathcal{H}_{1,n}(w, t) + (m\Lambda^+ + \Lambda^- + \mu_1(w))\mathcal{H}_{1,n}(w, t) = m\Lambda^+(1 - \delta_{n,0}) \sum_{k=1}^n c_k \mathcal{H}_{1,n-k}(w, t), n \geq 0 \quad (1)$$

The server is in busy state

$$\frac{\partial}{\partial w} \mathcal{H}_{2,n}(w, t) + \frac{\partial}{\partial t} \mathcal{H}_{2,n}(w, t) + (m\Lambda^+ + \Lambda^- + \mu_2(w))\mathcal{H}_{2,n}(w, t) = m\Lambda^+(1 - \delta_{n,0}) \sum_{k=1}^n c_k \mathcal{H}_{2,n-k}(w, t), n \geq 0 \quad (2)$$

The server is in vacation state

$$\frac{\partial}{\partial w} \mathcal{V}_n(w, t) + \frac{\partial}{\partial t} \mathcal{V}_n(w, t) + (m\Lambda^+ + \gamma(w))\mathcal{V}_n(w, t) = m\Lambda^+(1 - \delta_{n,0}) \sum_{k=1}^n c_k \mathcal{V}_{n-k}(w, t), n \geq 0 \quad (3)$$

The server is in the delay state

$$\frac{\partial}{\partial w} \mathcal{D}_{i,n}(w, t) + \frac{\partial}{\partial t} \mathcal{D}_{i,n}(w, t) + (m\Lambda^+ + \xi_i(w))\mathcal{D}_{i,n}(w, t) = m\Lambda^+(1 - \delta_{n,0}) \sum_{k=1}^n c_k \mathcal{D}_{i,n-k}(w, t), n \geq 0 \quad (4)$$

The server is in repair state

$$\frac{\partial}{\partial w} \mathcal{R}_n(w, t) + \frac{\partial}{\partial t} \mathcal{R}_n(w, t) + (m\Lambda^+ + \beta(w))\mathcal{R}_n(w, t) = m\Lambda^+(1 - \delta_{n,0}) \sum_{k=1}^n c_k \mathcal{R}_{n-k}(w, t), n \geq 0 \quad (5)$$

The server is in an idle state

$$\begin{aligned} \frac{d}{dt} \mathcal{Q}_r(t) = & -m\Lambda^+ \mathcal{Q}_r(t) + m\Lambda^+(1 - \delta_{r,0}) \sum_{k=1}^r c_k \mathcal{Q}_{r-k}(t) \\ & + (1 - \theta)[(1 - \pi) \int_0^\infty \mathcal{H}_{1,r}(w, t)\mu_1(w)dw + \int_0^\infty \mathcal{H}_{2,r}(w, t)\mu_2(w)dw] \\ & + \int_0^\infty \mathcal{V}_r(w, t)\gamma(w)dw + \int_0^\infty \mathcal{D}_{2,r}(w, t)\xi_2(w)dw, 0 \leq r \leq a - 1. \end{aligned} \quad (6)$$

where $\delta_{i,j}$ denotes Kronecker's delta function.

The following boundary conditions are considered to be $w = 0$ to solve the equations (1) to (6)

$$\begin{aligned} \mathcal{H}_{1,0}(0, t) = & m\Lambda^+ \sum_{r=a}^b \sum_{k=0}^{a-1} c_{r-k} \mathcal{Q}_k(t) + \sum_{r=a}^b \int_0^\infty \mathcal{V}_r(w, t)\gamma(w)dw \\ & + (1 - \theta)[(1 - \pi) \sum_{r=a}^b \int_0^\infty \mathcal{H}_{1,r}(w, t)\mu_1(w)dw \\ & + \sum_{r=a}^b \int_0^\infty \mathcal{H}_{2,r}(w, t)\mu_2(w)dw] + \sum_{r=a}^b \int_0^\infty \mathcal{D}_{2,r}(w, t)\xi_2(w)dw, \end{aligned} \quad (7)$$

$$\begin{aligned} \mathcal{H}_{1,n}(0, t) = & m\Lambda^+ \sum_{k=0}^{a-1} c_{b+n-k} \mathcal{Q}_k(t) + (1 - \theta)[(1 - \pi) \\ & \int_0^\infty \mathcal{H}_{1,n+b}(w, t)\mu_1(w)dw + \int_0^\infty \mathcal{H}_{2,n+b}(w, t)\mu_2(w)dw] \\ & + \int_0^\infty \mathcal{V}_{n+b}(w, t)\gamma(w)dw + \int_0^\infty \mathcal{D}_{2,n+b}(w, t)\xi_2(w)dw, \end{aligned} \quad (8)$$

$$\mathcal{H}_{2,n}(0, t) = \pi \int_0^\infty \mathcal{H}_{1,n}(w, t) \mu_1(w) dw, n \geq 0 \quad (9)$$

$$\mathcal{V}_n(0, t) = (1 - \pi) \theta \int_0^\infty \mathcal{H}_{1,n}(w, t) \mu_1(w) dw + \theta \int_0^\infty \mathcal{H}_{2,n}(w, t) \mu_2(w) dw, n \geq 0 \quad (10)$$

$$\mathcal{D}_{1,n}(0, t) = \Lambda^- \int_0^\infty \mathcal{H}_{1,n}(w, t) dw + \Lambda^- \int_0^\infty \mathcal{H}_{2,n}(w, t) dw, n \geq 0 \quad (11)$$

$$\mathcal{R}_n(0, t) = \int_0^\infty \mathcal{D}_{1,n}(w, t) \xi_1(w) dw, n \geq 0 \quad (12)$$

$$\mathcal{D}_{2,n}(0, t) = \int_0^\infty \mathcal{R}_n(w, t) \beta(w) dw, n \geq 0 \quad (13)$$

Further, the initial conditions are

$$\begin{aligned} Q_0(0) = 1, Q_r(0) = 0 \text{ for } 1 \leq r \leq a - 1, \\ \mathcal{H}_{1,n}(0) = \mathcal{H}_{2,n}(0) = \mathcal{D}_{i,n}(0) = \mathcal{R}_n(0) = \mathcal{V}_n(0) = 0 \text{ for } n \geq 0, i = 1, 2. \end{aligned} \quad (14)$$

The probability generating functions for (PGF) of the model are:

$$\begin{aligned} B_i(w, \chi, t) = \sum_{n=0}^\infty \chi^n B_{i,n}(w, t); B_i(\chi, t) = \sum_{n=0}^\infty \chi^n B_{i,n}(t); C(\chi) = \sum_{n=1}^\infty c_n \chi^n \\ G(w, \chi, t) = \sum_{n=0}^\infty \chi^n G(w, t); G(\chi, t) = \sum_{n=0}^\infty \chi^n G(t); Q(\chi) = \sum_{r=0}^{a-1} Q_r \chi^r \end{aligned} \quad (15)$$

Here $B = \mathcal{H}, \mathcal{D}$; $G = \mathcal{V}, \mathcal{R}$; $i = 1, 2$.

Taking Laplace transforms from equations (1) to (6) and multiplying by suitable powers of χ and use the equation (15), we get

$$\frac{\partial}{\partial w} \bar{\mathcal{H}}_1(w, \chi, s) + (s + m\Lambda^+(1 - C(\chi)) + \Lambda^- + \mu_1(w)) \bar{\mathcal{H}}_1(w, \chi, s) = 0, \quad (16)$$

$$\frac{\partial}{\partial w} \bar{\mathcal{H}}_2(w, \chi, s) + (s + m\Lambda^+(1 - C(\chi)) + \Lambda^- + \mu_2(w)) \bar{\mathcal{H}}_2(w, \chi, s) = 0, \quad (17)$$

$$\frac{\partial}{\partial w} \bar{\mathcal{V}}(w, \chi, s) + (s + m\Lambda^+(1 - C(\chi)) + \gamma(w)) \bar{\mathcal{V}}(w, \chi, s) = 0, \quad (18)$$

$$\frac{\partial}{\partial w} \bar{\mathcal{D}}_i(w, \chi, s) + (s + m\Lambda^+(1 - C(\chi)) + \xi_i(w)) \bar{\mathcal{D}}_i(w, \chi, s) = 0, i = 1, 2 \quad (19)$$

$$\frac{\partial}{\partial w} \bar{\mathcal{R}}(w, \chi, s) + (s + m\Lambda^+(1 - C(\chi)) + \beta(w)) \bar{\mathcal{R}}(w, \chi, s) = 0. \quad (20)$$

Similarly from equations (7) to (13), we get

$$\begin{aligned}
 \chi^b \bar{\mathcal{H}}_1(0, \chi, s) &= m\Lambda^+ \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n \bar{\mathcal{Q}}_r(s) (\chi^b - \chi^{n+r}) \\
 &\quad + m\Lambda^+ \sum_{r=0}^{a-1} C(\chi) \bar{\mathcal{Q}}_r(s) \chi^r - \chi^b \sum_{r=0}^{a-1} (s + m\Lambda^+) \bar{\mathcal{Q}}_r(s) + \chi^b \\
 &\quad + (1 - \theta) [(1 - \pi) \int_0^\infty \bar{\mathcal{H}}_1(w, \chi, s) \mu_1(w) dw \\
 &\quad + \int_0^\infty \bar{\mathcal{H}}_2(w, \chi, s) \mu_2(w) dw] + \int_0^\infty \bar{\mathcal{V}}(w, \chi, s) \gamma(w) dw \\
 &\quad + \int_0^\infty \bar{\mathcal{D}}_{2,r}(w, \chi, s) \xi_2(w) dw + \sum_{r=0}^{b-1} (\chi^b - \chi^r) [(1 - \theta)(1 - \pi) \\
 &\quad \int_0^\infty \bar{\mathcal{H}}_{1,r}(w, s) \mu_1(w) dw + (1 - \theta) \int_0^\infty \bar{\mathcal{H}}_{2,r}(w, s) \mu_2(w) dw \\
 &\quad + \int_0^\infty \bar{\mathcal{V}}_r(w, s) \gamma(w) dw + \int_0^\infty \bar{\mathcal{D}}_{2,r}(w, s) \xi_2(w) dw], \tag{21}
 \end{aligned}$$

$$\bar{\mathcal{H}}_2(0, \chi, s) = \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)], \tag{22}$$

$$\begin{aligned}
 \bar{\mathcal{V}}(0, \chi, s) &= \theta \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \bar{M}_2[\Pi_b(\chi, s)] \\
 &\quad + (1 - \pi) \theta \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 \bar{\mathcal{D}}_1(0, \chi, s) &= \Lambda^- \bar{\mathcal{H}}_1(0, \chi, s) \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \\
 &\quad + \Lambda^- \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \tag{24}
 \end{aligned}$$

$$\begin{aligned}
 \bar{\mathcal{R}}(0, \chi, s) &= \Lambda^- \bar{\mathcal{H}}_1(0, \chi, s) \bar{\mathcal{D}}_1[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \\
 &\quad + \Lambda^- \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \bar{\mathcal{D}}_1[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \tag{25}
 \end{aligned}$$

$$\begin{aligned}
 \bar{\mathcal{D}}_2(0, \chi, s) &= \Lambda^- \bar{\mathcal{H}}_1(0, \chi, s) \bar{\mathcal{D}}_1[\Omega_a(\chi, s)] \bar{\mathcal{R}}[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \\
 &\quad + \Lambda^- \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \bar{\mathcal{D}}_1[\Omega_a(\chi, s)] \bar{\mathcal{R}}[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \tag{26}
 \end{aligned}$$

Solving the equations (16) to (20) and multiplying two sides of equations by $\mu_1(w)$, $\mu_2(w)$, $\gamma(w)$, $\xi_i(w)$, and $\beta(w)$ respectively, and integrate with respect to w , we get

$$\int_0^\infty \bar{\mathcal{H}}_1(w, \chi, s) \mu_1(w) dw = \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \tag{27}$$

$$\int_0^\infty \bar{\mathcal{H}}_2(w, \chi, s) \mu_2(w) dw = \bar{\mathcal{H}}_2(0, \chi, s) \bar{M}_2[\Pi_b(\chi, s)] \tag{28}$$

$$\int_0^\infty \bar{\mathcal{V}}(w, \chi, s) \gamma(w) dw = \bar{\mathcal{V}}(0, \chi, s) \bar{\mathcal{V}}[\Omega_a(\chi, s)] \tag{29}$$

$$\int_0^\infty \bar{\mathcal{D}}_i(w, \chi, s) \xi_i(w) dw = \bar{\mathcal{D}}_i(0, \chi, s) \bar{\mathcal{D}}_i[\Omega_a(\chi, s)], i = 1, 2 \tag{30}$$

$$\int_0^\infty \bar{\mathcal{R}}(w, \chi, s) \beta(w) dw = \bar{\mathcal{R}}(0, \chi, s) \bar{\mathcal{R}}[\Omega_a(\chi, s)] \tag{31}$$

Again integrating equations (16) to (20) with respect to w and using the equation (22), (23), (24), (25) and (26), we get

$$\bar{\mathcal{H}}_1(\chi, s) = \bar{\mathcal{H}}_1(0, \chi, s) \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right], \quad (32)$$

$$\bar{\mathcal{H}}_2(\chi, s) = \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right], \quad (33)$$

$$\begin{aligned} \bar{V}(\chi, s) &= \theta \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \bar{M}_2[\Pi_b(\chi, s)] \left[\frac{1 - \bar{V}[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right] \\ &+ (1 - \pi) \theta \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \left[\frac{1 - \bar{V}[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right] \end{aligned} \quad (34)$$

$$\begin{aligned} \bar{D}_1(\chi, s) &= \Lambda^- \bar{\mathcal{H}}_1(0, \chi, s) \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \left[\frac{1 - \bar{D}_1[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right] \\ &+ \Lambda^- \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \left[\frac{1 - \bar{D}_1[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right], \end{aligned} \quad (35)$$

$$\begin{aligned} \bar{R}(\chi, s) &= \Lambda^- \bar{\mathcal{H}}_1(0, \chi, s) \bar{D}_1[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \left[\frac{1 - \bar{R}[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right] \\ &+ \Lambda^- \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \bar{D}_1[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \left[\frac{1 - \bar{R}[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right], \end{aligned} \quad (36)$$

$$\begin{aligned} \bar{D}_2(\chi, s) &= \Lambda^- \bar{\mathcal{H}}_1(0, \chi, s) \bar{D}_1[\Omega_a(\chi, s)] \bar{R}[\Omega_a(\chi, s)] \left[\frac{1 - \bar{M}_1[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \\ &\left[\frac{1 - \bar{D}_2[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right] + \Lambda^- \pi \bar{\mathcal{H}}_1(0, \chi, s) \bar{M}_1[\Pi_b(\chi, s)] \bar{D}_1[\Omega_a(\chi, s)] \bar{R}[\Omega_a(\chi, s)] \\ &\left[\frac{1 - \bar{M}_2[\Pi_b(\chi, s)]}{\Pi_b(\chi, s)} \right] \left[\frac{1 - \bar{D}_2[\Omega_a(\chi, s)]}{\Omega_a(\chi, s)} \right], \end{aligned} \quad (37)$$

Inserting the equations (27), (28), (29) and (30) into the equation (21), we get

$$\bar{\mathcal{H}}_1(0, \chi, s) = \frac{\left(\begin{aligned} &m\Lambda^+ \sum_{r=0}^{a-1} C(\chi) \bar{Q}_r(s) \chi^r - \chi^b (s + m\Lambda^+) \sum_{r=0}^{a-1} \bar{Q}_r(s) \\ &+ m\Lambda^+ \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n \bar{Q}_r(s) (\chi^b - \chi^{n+r}) + \chi^b \\ &+ \sum_{r=0}^{b-1} (\chi^b - \chi^r) [(1 - \theta)(1 - \pi) \int_0^\infty \bar{\mathcal{H}}_{1,r}(w, s) \mu_1(w) dw \\ &+ (1 - \theta) \int_0^\infty \bar{\mathcal{H}}_{2,r}(w, s) \mu_2(w) dw + \int_0^\infty \bar{V}_r(w, s) \gamma(w) dw \\ &+ \int_0^\infty \bar{D}_{2,r}(w, s) \xi_2(w) dw], \end{aligned} \right)}{Dr(\chi, s)} \quad (38)$$

where

$$\begin{aligned} Dr(\chi, s) &= \Pi_b(\chi, s) \chi^b - [(1 - \pi)(1 - \theta) + (1 - \pi)\theta \bar{V}[\Omega_a(\chi, s)] \\ &+ [(1 - \theta)\pi + \theta \pi \bar{V}[\Omega_a(\chi, s)]] \bar{M}_2[\Pi_b(\chi, s)] \Pi_b(\chi, s) \bar{M}_1[\Pi_b(\chi, s)] \\ &+ \Lambda^- \bar{D}_1[\Omega_a(\chi, s)] \bar{R}[\Omega_a(\chi, s)] \bar{D}_2[\Omega_a(\chi, s)] [[1 - \bar{M}_1[\Pi_b(\chi, s)]] \\ &+ \pi \bar{M}_1[\Pi_b(\chi, s)] [1 - \bar{M}_2[\Pi_b(\chi, s)]]]] \end{aligned}$$

$$\Pi_b(\chi, s) = s + \Lambda^- + m\Lambda^+(1 - C(\chi)); \quad \Omega_a(\chi, s) = s + m\Lambda^+(1 - C(\chi))$$

substituting the equation (38) into the equations (32) to (37), we get $\bar{\mathcal{H}}_i(\chi, s)$, $\bar{\mathcal{V}}(\chi, s)$, $\bar{\mathcal{D}}_i(\chi, s)$ and $\bar{\mathcal{R}}(\chi, s)$. Taking the inversion of the above states, we get the transient state of the system.

4. The Steady State Analysis

To determine the steady-state probability distribution for the proposed model, we can use the Tauberian property,

$$\lim_{s \rightarrow 0} s \bar{f}(s) = \lim_{t \rightarrow \infty} f(t). \quad (39)$$

Using this property, we get the different states of the system and adding these, we get the PGF of number of customers in the queue at an arbitrary epoch.

$$P(\chi) = \mathcal{H}_1(\chi) + \mathcal{H}_2(\chi) + \mathcal{V}(\chi) + \mathcal{D}_1(\chi) + \mathcal{R}(\chi) + \mathcal{D}_2(\chi) + \mathcal{Q}(\chi)$$

$$P(\chi) = \frac{\left(\begin{aligned} & [m\Lambda^+ \sum_{r=0}^{a-1} \mathcal{Q}_r(C(\chi)\chi^r - \chi^b) + m\Lambda^+ \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} \\ & c_n \mathcal{Q}_r(\chi^b - \chi^{n+r}) + \sum_{r=0}^{b-1} (\chi^b - \chi^r) W_r] \times [\Omega_a(\chi) \\ & (1 - \bar{M}_1[\Pi_b(\chi)]) + \Omega_a(\chi) \pi \bar{M}_1[\Pi_b(\chi)] (1 - \bar{M}_2[\Pi_b(\chi)]) \\ & + \theta \pi \Pi_b(\chi) \bar{M}_1[\Pi_b(\chi)] \bar{M}_2[\Pi_b(\chi)] + (1 - \pi) \theta \Pi_b(\chi) \bar{M}_1[\Pi_b(\chi)] \\ & (1 - \bar{V}[\Omega_a(\chi)]) + \Lambda^- (1 - \bar{M}_1[\Pi_b(\chi)]) (1 - \bar{D}_1[\Omega_a(\chi)]) + \Lambda^- \pi \\ & \bar{M}_1[\Pi_b(\chi)] (1 - \bar{M}_2[\Pi_b(\chi)]) (1 - \bar{D}_1[\Omega_a(\chi)]) + \Lambda^- \bar{D}_1[\Omega_a(\chi)] \\ & \bar{R}[\Omega_a(\chi)] (1 - \bar{M}_1[\Pi_b(\chi)]) (1 - \bar{D}_2[\Omega_a(\chi)]) + \Lambda^- \pi \bar{M}_1[\Pi_b(\chi)] \\ & \bar{D}_1[\Omega_a(\chi)] \bar{R}[\Omega_a(\chi)] (1 - \bar{M}_2[\Pi_b(\chi)]) (1 - \bar{D}_2[\Omega_a(\chi)]) + \Lambda^- \\ & \bar{D}_1[\Omega_a(\chi)] (1 - \bar{M}_1[\Pi_b(\chi)]) (1 - \bar{R}[\Omega_a(\chi)]) + \Lambda^- \pi \bar{M}_1[\Pi_b(\chi)] \\ & \bar{D}_1[\Omega_a(\chi)] (1 - \bar{M}_2[\Pi_b(\chi)]) (1 - \bar{R}[\Omega_a(\chi)]) + Dr(\chi) \mathcal{Q}(\chi) \end{aligned} \right)}{[Dr(\chi) \Omega_a(\chi)]}. \quad (40)$$

where

$$\begin{aligned} \mathcal{H}_r &= (1 - \pi) \int_0^\infty \mathcal{H}_{1,r}(w) \mu_1(w) dw + \int_0^\infty \mathcal{H}_{2,r}(w) \mu_2(w) dw \\ \mathcal{W}_r &= (1 - \theta) \mathcal{H}_r + \int_0^\infty \mathcal{V}_r(w) \gamma(w) dw + \int_0^\infty \mathcal{D}_{2,r}(w) \xi_2(w) dw. \end{aligned}$$

5. Stability Conditions

The probability generating function $P(\chi)$ is easily seen that at $\chi = 1$ is zero in the numerator as well as the denominator. For that, we have to apply L'Hopital's rule and equating the expression to 1, we have

$$\begin{aligned} & X_1 \times [(1 - \bar{M}_1(\Lambda^-)) + \pi \bar{M}_1(\Lambda^-) (1 - \bar{M}_2(\Lambda^-)) + \theta \pi \Lambda^- E(V) \bar{M}_1(\Lambda^-) \bar{M}_2(\Lambda^-) \\ & + (1 - \pi) \theta \Lambda^- \bar{M}_1(\Lambda^-) E(V) + \Lambda^- (1 - \bar{M}_1(\Lambda^-)) (E(D_1) + E(R) + E(D_2)) \\ & + \Lambda^- \pi \bar{M}_1(\Lambda^-) (1 - \bar{M}_2(\Lambda^-)) (E(D_1) + E(R) + E(D_2))] + C_1 \times \sum_{r=0}^{a-1} \mathcal{Q}_r = C_1 \end{aligned} \quad (41)$$

Next, we need to determine the unknown probabilities, W_r , $r = 0, 1, 2, \dots, b - 1$. For this, we have to relate this expression to the idle-server probabilities, Q_r , r varies from 0 to $a-1$, then the L.H.S. of the expression above must be positive. Thus $P(1) = 1$ is fulfilled if

$$[\Pi_b(\chi)\chi^b - [(1 - \pi)(1 - \theta) + (1 - \pi)\theta\bar{V}(\Omega_a(\chi)) + [(1 - \theta) + \theta\pi\bar{V}(\Omega_a(\chi))]\bar{M}_2(\Pi_b(\chi))]\Pi_b(\chi)\bar{M}_1(\Pi_b(\chi)) + \Lambda^-\bar{D}_1(\Omega_a(\chi))\bar{R}(\Omega_a(\chi))\bar{D}_2(\Omega_a(\chi))][[1 - \bar{M}_1(\Pi_b(\chi))] + \pi\bar{M}_1(\Pi_b(\chi))[1 - \bar{M}_2(\Pi_b(\chi))]] > 0.$$

$$\text{If } \rho = \frac{[m\Lambda^+E(Y)[\theta\pi E(V)\bar{M}_1(\Lambda^-)\bar{M}_2(\Lambda^-) - \bar{M}'_2(\Lambda^-)\pi\bar{M}_1(\Lambda^-) + \bar{M}'_1(\Lambda^-) + \pi S_2 + (1 - \pi)\theta E(V)\bar{M}_1(\Lambda^-) + S_1(E(D_1) + E(R) + E(D_2)) - \bar{M}'_1(\Lambda^-)((1 - \pi) + \pi\bar{M}_2(\Lambda^-))]}{b} \quad (42)$$

is the condition to be satisfied by the model under consideration for the existence of steady state. Equation (40) has $b+a$ unknowns in the numerator. We have to get ‘ b ’ constants in the numerator for that we using this expression $\sum_{r=0}^{a-1} W_r = m\Lambda^+ \sum_{r=0}^{a-1} Q_r - m\Lambda^+ \sum_{r=0}^{a-1} Q_r \sum_{k=1}^{a-r-1} c_k$. Now equation (40) gets the expression involving ‘ b ’ unknowns.

6. Performance Measures

If the system is in a steady-state condition, then we have found $\mathcal{V}_q(1)$, $\mathcal{H}_q(1)$, $\mathcal{D}_q(1)$ and $\mathcal{R}_q(1)$ be the probabilities that the server is in vacation, busy, two delays of verification and repair state respectively. We have

$$\mathcal{V}_q(1) = \frac{X_1(\Lambda^-\theta\pi E(V)\bar{M}_1(\Lambda^-)\bar{M}_2(\Lambda^-) + (1-\pi)\theta\bar{M}_1(\Lambda^-)E(V)\Lambda^-}{C_1}$$

$$\mathcal{H}_q(1) = \mathcal{H}_1(1) + \mathcal{H}_2(1) = \frac{X_1 S_1}{C_1}$$

$$\mathcal{D}_q(1) = \mathcal{D}_1(1) + \mathcal{D}_2(1) = \frac{X_1 \Lambda^- E(D_1) S_1}{C_1}$$

$$\mathcal{R}_q(1) = \frac{X_1 \Lambda^- E(R) S_1}{C_1}$$

6.1 Mean Queue Size

Expected length of the queue L_q is obtained from the equation (40). That is $\lim_{\chi \rightarrow 1} \frac{d}{d\chi} P(\chi)$

$$L_q = \left[\frac{NR'''(1)DR''(1) - DR'''(1)NR''(1)}{3[(DR'')^2]} \right]$$

where

$$DR'' = -2m\Lambda^+E(Y)C_1$$

$$\begin{aligned}
 DR''' = & -3[(m\Lambda^+E(Y))(-m\Lambda^+E(Y(Y-1))) - 2m\Lambda^+E(Y)b + \Lambda^-b(b-1) \\
 & - [[\theta\pi\bar{M}_2(\Lambda^-)A_2 - 2\theta\pi(m\Lambda^+E(Y))^2E(V)\bar{M}'_2(\Lambda^-) - \pi G_2 + (1-\pi) \\
 & \theta A_2]\Lambda^-\bar{M}_1(\Lambda^-) + ((1-\pi) + \bar{M}_2(\Lambda^-)\pi)[-m\Lambda^+E(Y(Y-1))\bar{M}_1(\Lambda^-) \\
 & - \Lambda^-G_1 + 2(m\Lambda^+E(Y))^2\bar{M}'_1(\Lambda^-)] - 2(m\Lambda^+E(Y))^2(\theta\pi E(V)\bar{M}_2(\Lambda^-) \\
 & - \pi\bar{M}'_2(\Lambda^-) + (1-\pi)\theta E(V))(\bar{M}_1(\Lambda^-) + \Lambda^-\bar{M}'_1(\Lambda^-)) \\
 & + \Lambda^-G_1(1-\pi(1-\bar{M}_2(\Lambda^-))) - 2\Lambda^-\pi(m\Lambda^+E(Y))^2\bar{M}'_1(\Lambda^-)\bar{M}'_2(\Lambda^-) \\
 & + \Lambda^-\pi\bar{M}_1(\Lambda^-)G_2 + S_1m\Lambda^+E(Y(Y-1))(E(D_1) + E(R) + E(D_2)) \\
 & + S_1(m\Lambda^+E(Y))^2(2E(D_1)E(R) + 2E(R)E(D_2) + 2E(D_2)E(D_1) \\
 & + E(D_1^2) + E(R^2) + E(D_2^2)) + 2(m\Lambda^+E(Y))^2 \\
 & (E(D_1) + E(R) + E(D_2))(\bar{M}'_1(\Lambda^-) + \pi S_2)] + m\Lambda^+E(Y(Y-1))C_1]
 \end{aligned}$$

$$\begin{aligned}
 NR'' = & -2m\Lambda^+E(Y)(X_1[(1-\bar{M}_1(\Lambda^-)) + \pi\bar{M}_1(\Lambda^-)(1-\bar{M}_2(\Lambda^-)) \\
 & + \theta\pi\Lambda^-E(V)\bar{M}_1(\Lambda^-)\bar{M}_2(\Lambda^-) + \Lambda^-(1-\bar{U}_1(\Lambda^-))(E(D_1) + E(R) + E(D_2)) \\
 & + \Lambda^-\pi\bar{M}_1(\Lambda^-)(1-\bar{M}_2(\Lambda^-))(E(D_1) + E(R) + E(D_2)) \\
 & + (1-\pi)\theta\Lambda^-\bar{M}_1(\Lambda^-)E(V)] + C_1\sum_{r=0}^{\alpha-1} Q_r)
 \end{aligned}$$

$$\begin{aligned}
 NR''' = & 3[-X_2m\Lambda^+E(Y)[\pi\bar{M}_1(\Lambda^-)(1-\bar{M}_2(\Lambda^-)) + \theta\pi\Lambda^-E(V)\bar{M}_1(\Lambda^-)\bar{M}_2(\Lambda^-) \\
 & + (1-\bar{M}_1(\Lambda^-)) + \Lambda^-(1-\bar{M}_1(\Lambda^-))(E(D_1) + E(R) + E(D_2)) \\
 & + \Lambda^-\pi\bar{M}_1(\Lambda^-)(1-\bar{M}_2(\Lambda^-))[E(D_1) + E(R) + E(D_2)] \\
 & + E(V)(1-\pi)\theta\Lambda^-\bar{M}_1(\Lambda^-)] + X_1(-2(m\Lambda^+E(Y))^2\bar{M}'_1(\Lambda^-) \\
 & - m\Lambda^+E(Y(Y-1))(1-\bar{M}_1(\Lambda^-)) - \pi[2(m\Lambda^+E(Y))^2\bar{M}'_2(\Lambda^-)\bar{M}_1(\Lambda^-) \\
 & + (1-\bar{M}_2(\Lambda^-))A_1] + (1-\pi)\theta[2\Lambda^-(m\Lambda^+E(Y))^2E(V)\bar{M}'_1(\Lambda^-) \\
 & + \bar{M}_1(\Lambda^-)[2(m\Lambda^+E(Y))^2E(V) - \Lambda^-A_2]] + \theta\pi 2(m\Lambda^+E(Y))^2\bar{M}_2(\Lambda^-) \\
 & E(V)[\bar{M}_1(\Lambda^-) + \Lambda^-\bar{M}'_1(\Lambda^-)] + \theta\pi\Lambda^-\bar{M}_1(\Lambda^-)(2(m\Lambda^+E(Y))^2\bar{M}'_2(\Lambda^-) \\
 & E(V) - \bar{M}_2(\Lambda^-)A_2) - \Lambda^-[2(m\Lambda^+E(Y))^2E(D_1)\bar{M}'_1(\Lambda^-) - (1-\bar{M}_1(\Lambda^-)) \\
 & A_3] + \Lambda^-\pi[-2(m\Lambda^+E(Y))^2E(D_1)\bar{M}'_2(\Lambda^-)\bar{M}_1(\Lambda^-) + (1-\bar{M}_2(\Lambda^-)) \\
 & [2(m\Lambda^+E(Y))^2\bar{M}'_1(\Lambda^-)E(D_1) - \bar{M}_1(\Lambda^-)A_3]] - 2\Lambda^-(m\Lambda^+E(Y))^2E(R) \\
 & [\bar{M}'_1(\Lambda^-) + (1-\bar{M}_1(\Lambda^-))E(D_1)] - \Lambda^-(1-\bar{M}_1(\Lambda^-))A_4 \\
 & + \Lambda^-\pi[-2(m\Lambda^+E(Y))^2E(R)\bar{M}_1(\Lambda^-)[\bar{M}'_2(\Lambda^-) + E(D_1)(1-\bar{M}_2(\Lambda^-))]] \\
 & + \Lambda^-\pi(1-\bar{M}_2(\Lambda^-))[2(m\Lambda^+E(Y))^2E(R)\bar{M}'_1(\Lambda^-) - \bar{M}_1(\Lambda^-)A_4]
 \end{aligned}$$

$$\begin{aligned}
 & -2\Lambda^-(m\Lambda^+E(Y))^2E(D_2)[\bar{M}'_1(\Lambda^-) + (1 - \bar{M}_1(\Lambda^-))E(D_1)] \\
 & -\Lambda^-(1 - \bar{M}_1(\Lambda^-))[2(m\Lambda^+E(Y))^2E(D_2)E(R) - A_5] \\
 & -2\Lambda^-\pi(m\Lambda^+E(Y))^2\bar{M}_1(\Lambda^-)E(D_2)[\bar{M}'_2(\Lambda^-) + (1 - \bar{M}_2(\Lambda^-))(E(R) \\
 & + E(D_1))] + \Lambda^-\pi(1 - \bar{M}_2(\Lambda^-))[2(m\Lambda^+E(Y))^2E(D_2)\bar{M}'_1(\Lambda^-) \\
 & - \bar{M}_1(\Lambda^-)A_5] - m\Lambda^+E(Y) \sum_{r=0}^{a-1} Q_r(-m\Lambda^+E(Y(Y-1))) \\
 & -2m\Lambda^+E(Y)b + \Lambda^-b(b-1) - [[\theta\pi\bar{M}_2(\Lambda^-)A_2 \\
 & -2\theta\pi(m\Lambda^+E(Y))^2E(V)\bar{M}'_2(\Lambda^-) - \pi G_2 + (1 - \pi)\theta A_2]\Lambda^-\bar{M}_1(\Lambda^-) \\
 & + ((1 - \pi) + \bar{M}_2(\Lambda^-)\pi)[-m\Lambda^+E(Y(Y-1))\bar{M}_1(\Lambda^-) \\
 & -\Lambda^-G_1 + 2(m\Lambda^+E(Y))^2\bar{M}'_1(\Lambda^-)] - 2(m\Lambda^+E(Y))^2(\theta\pi E(V)\bar{M}_2(\Lambda^-) \\
 & -\pi\bar{M}'_2(\Lambda^-) + (1 - \pi)\theta E(V))(\bar{M}_1(\Lambda^-) + \Lambda^-\bar{M}'_1(\Lambda^-)) \\
 & + \Lambda^-G_1(1 - \pi(1 - \bar{M}_2(\Lambda^-))) - 2\Lambda^-\pi(m\Lambda^+E(Y))^2\bar{M}'_1(\Lambda^-)\bar{M}'_2(\Lambda^-) \\
 & + \Lambda^-\pi\bar{M}_1(\Lambda^-)G_2 + S_1m\Lambda^+E(Y(Y-1))(E(D_1) + E(R) + E(D_2)) \\
 & + S_1(m\Lambda^+E(Y))^2(2E(D_1)E(R) + 2E(R)E(D_2) + 2E(D_2)E(D_1) \\
 & + E(D_1^2) + E(R^2) + E(D_2^2)) + 2(m\Lambda^+E(Y))^2 \\
 & (E(D_1) + E(R) + E(D_2))(\bar{M}'_1(\Lambda^-) + \pi S_2)] \\
 & -[m\Lambda^+E(Y(Y-1)) \sum_{r=0}^{a-1} Q_r + 2m\Lambda^+E(Y) \sum_{r=0}^{a-1} rQ_r]C_1]
 \end{aligned}$$

and

$$\begin{aligned}
 X_1 &= m\Lambda^+ \sum_{r=0}^{a-1} Q_r(E(Y) + r - b) + m\Lambda^+ \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r(b - n - r) + \\
 & \sum_{r=0}^{b-1} (b - r)W_r \\
 X_2 &= m\Lambda^+ \sum_{r=0}^{a-1} \sum_{n=1}^{b-r-1} c_n Q_r(b(b-1) - (n+r)(n+r-1)) \\
 & + m\Lambda^+ \sum_{r=0}^{a-1} Q_r(E(Y(Y-1)) + 2E(Y)r + r(r-1) - b(b-1)) \\
 & + \sum_{r=0}^{b-1} (b(b-1) - r(r-1))W_r \\
 C_1 &= -m\Lambda^+E(X) + \Lambda^-b - [m\Lambda^-\Lambda^+E(Y)[\theta\pi E(V)\bar{M}_2(\Lambda^-)\bar{M}_1(\Lambda^-) - \bar{M}'_2(\Lambda^-) \\
 & \pi\bar{M}_1(\Lambda^-) + (1 - \pi)\theta E(V)\bar{M}_1(\Lambda^-) - \bar{M}'_1(\Lambda^-)[(1 - \pi) + \pi\bar{M}_2(\Lambda^-)] \\
 & \bar{M}'_1(\Lambda^-) + \pi S_2 + S_1(E(D_1) + E(R) + E(D_2))] \\
 & -m\Lambda^+E(Y)[(1 - \pi) + \pi\bar{M}_2(\Lambda^-)]] \\
 S_1 &= 1 - \bar{M}_1(\Lambda^-) + \pi\bar{M}_1(\Lambda^-)[1 - \bar{M}_2(\Lambda^-)] \\
 S_2 &= -\bar{M}'_1(\Lambda^-)(1 - \bar{M}_2(\Lambda^-)) + \bar{M}_1(\Lambda^-)\bar{M}'_2(\Lambda^-) \\
 G_1 &= m\Lambda^+E(Y(Y-1))\bar{M}'_1(\Lambda^-) - (m\Lambda^+E(Y))^2\bar{M}''_1(\Lambda^-) \\
 G_2 &= m\Lambda^+E(Y(Y-1))\bar{M}'_2(\Lambda^-) - (m\Lambda^+E(Y))^2\bar{M}''_2(\Lambda^-)
 \end{aligned}$$

$$\begin{aligned}
 A_1 &= 2(m\Lambda^+ E(Y))^2 \bar{M}'_1(\Lambda^-) + m\Lambda^+ E(Y(Y-1)) \bar{M}_1(\Lambda^-) \\
 A_2 &= m\Lambda^+ E(Y(Y-1))E(V) + (m\Lambda^+ E(Y))^2 E(V^2) \\
 A_3 &= m\Lambda^+ E(Y(Y-1))E(D_1) + (m\Lambda^+ E(Y))^2 E(D_1^2) \\
 A_4 &= m\Lambda^+ E(Y(Y-1))E(R) + (m\Lambda^+ E(Y))^2 E(R^2); E(Y) = C'(1) \\
 A_5 &= m\Lambda^+ E(Y(Y-1))E(D_2) + (m\Lambda^+ E(Y))^2 E(D_2^2); E(Y(Y-1)) = C''(1)
 \end{aligned}$$

7. Particular Cases

Case 1: No bulk service, No loss, No Bernoulli feedback and No breakdown. Let $a = b = 1$, $m = 1$, $\pi = 1$ and $\Lambda^- = 0$, then equation (40) becomes

$$P(\chi) = \frac{[1 - [(1 - \theta) + \theta \bar{V}[\Omega_a(\chi)]]] \bar{M}_1[\Omega_a(\chi)] \mathcal{Q}}{[(1 - \theta) + \theta \bar{V}[\Omega_a(\chi)]] \bar{M}_1[\Omega_a(\chi)] - \chi}$$

where $\Omega_a(\chi) = (1 - C(\chi))\Lambda^+$; $\mathcal{Q} = 1 - \rho$, $\rho = \Lambda^+ E(Y)(E(M_1) + \theta E(V))$

which coincides with Madan and Choudhury (2004).

Case 2: No bulk arrival, No bulk service, No loss, No breakdown and No vacation. Let $a = b = 1$, $m = 1$, $\pi = 1$ and $\Lambda^- = 0$, then equation (40) reduces to

$$P(\chi) = \frac{[1 - \bar{M}_1[\Omega_a(\chi)] + \pi \bar{M}_1[\Omega_a(\chi)][1 - \bar{M}_2[\Omega_a(\chi)]]] \mathcal{Q}}{[(1 - \pi) \bar{M}_1[\Omega_a(\chi)] + \pi \bar{M}_1[\Omega_a(\chi)] \bar{M}_2[\Omega_a(\chi)]] - \chi}$$

where $\Omega_a(\chi) = (1 - \chi)\Lambda^+$; $\mathcal{Q} = 1 - \rho$, $\rho = \Lambda^+ E(Y)(E(M_1) + \pi E(M_2))$.

These expressions are equivalent to the results by Choudhary and Tadj (2009) without considering breakdown.

8. Numerical Results

This section deals with the numerical illustration of the proposed queueing model through variations in the parameters using MATLAB to examine the performance measures of the system. We consider that the input batch size of the customers follows a geometric distribution with parameter $p=1/3$, where $q=1-p$. Feedback service time, vacation time, service time, two delays of verification times and repair times are Erlangianly and exponentially distributed. Each value was chosen arbitrarily in order to fulfill the stability condition.

Taking the value of the parameters as $a = 2$, $b = 5$, $\theta = 0.3$, $m = 0.3$, $\Lambda^+ = 2$, $\Lambda^- = 1.1$, $\mu_2 = 14$, $\gamma = 9$, $\beta = 1.2$, $\xi_1 = 2$, $\pi = 0.2$, $\xi_2 = 2.4$ and increase the service rate (μ_1) value from 1 to 10 in steps of 1. We determined the values of the utilization factor ρ , expected queue length L_q and expected waiting time W_q are tabulated in Table 2 and the corresponding graph is drawn in Figure 2.

Table 2. The impact of the rate of service (μ_1) on ρ , L_q , W_q

μ_1	Exponential			Erlang-2 stage		
	ρ	L_q	W_q	ρ	L_q	W_q
1	0.2267	8.9637	7.4697	0.2511	10.3300	8.6083
2	0.1581	5.0483	4.2069	0.1703	5.2843	4.4036
3	0.1229	3.8587	3.2156	0.1301	3.8717	3.2265
4	0.1016	3.3227	2.7690	0.1063	3.3044	2.7537
5	0.0872	3.0259	2.5216	0.0906	2.9996	2.4997
6	0.0769	2.8458	2.3715	0.0794	2.8189	2.3491
7	0.0691	2.7292	2.2743	0.0711	2.7041	2.2534
8	0.0631	2.6499	2.2083	0.0647	2.6273	2.1894
9	0.0582	2.5940	2.1617	0.0595	2.5738	2.1449
10	0.0542	2.5534	2.1278	0.0553	2.5355	2.1129

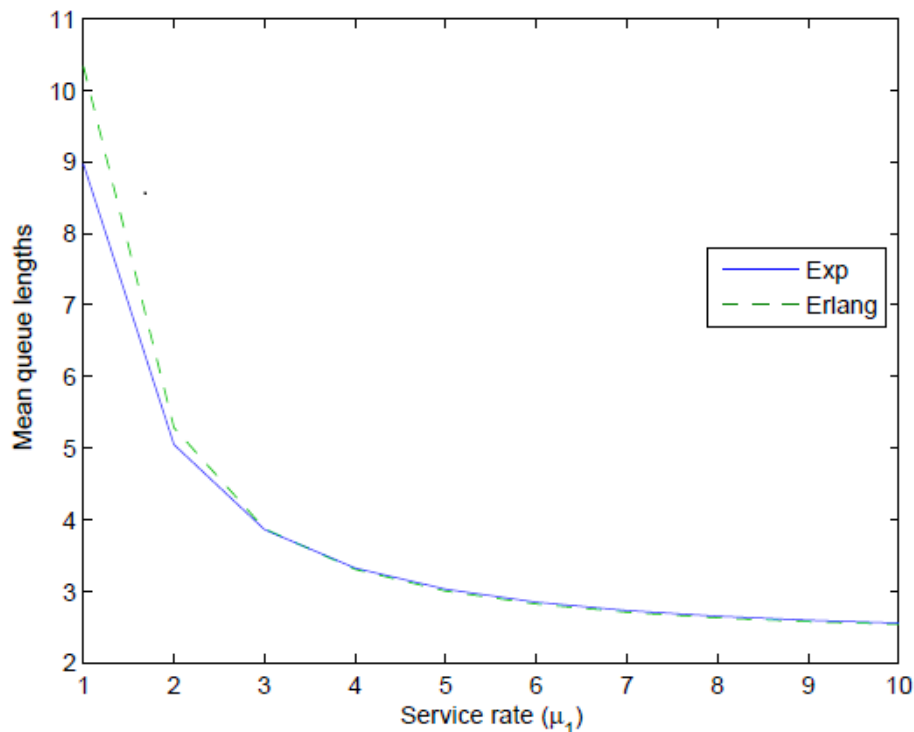


Figure 2. ρ , L_q , W_q versus service rate (μ_1)

Fixing the particular values as $a = 2$, $b = 5$, $\theta = 0.3$, $m = 0.3$, $\Lambda^+ = 2$, $\Lambda^- = 1.1$, $\mu_2 = 17$, $\gamma = 9$, $\mu_1 = 10$, $\xi_1 = 2$, $\pi = 0.2$, $\xi_2 = 2.4$ and increase the repair rate (β) value from 1 to 10 in steps of 1. We determined the values of the utilization factor ρ , expected queue length L_q and expected waiting time W_q are tabulated in Table 3 and the resultant graphical study is observed in Figure 3.

Table 3. The impact of the rate of repair (β) on ρ , L_q , W_q

β	Exponential			Erlang-2 stage		
	ρ	L_q	W_q	ρ	L_q	W_q
1	0.0577	2.8060	2.3384	0.0589	2.7922	2.3268
2	0.0445	2.0142	1.6785	0.0454	1.9886	1.6572
3	0.0401	1.8068	1.5057	0.0409	1.7785	1.4820
4	0.0379	1.7136	1.4280	0.0386	1.6840	1.4033
5	0.0366	1.6610	1.3842	0.0373	1.6308	1.3590
6	0.0357	1.6273	1.3561	0.0364	1.5967	1.3305
7	0.0351	1.6039	1.3366	0.0357	1.5730	1.3108
8	0.0346	1.5868	1.3223	0.0353	1.5556	1.2964
9	0.0343	1.5737	1.3114	0.0349	1.5423	1.2853
10	0.0340	1.5633	1.3027	0.0346	1.5319	1.2765

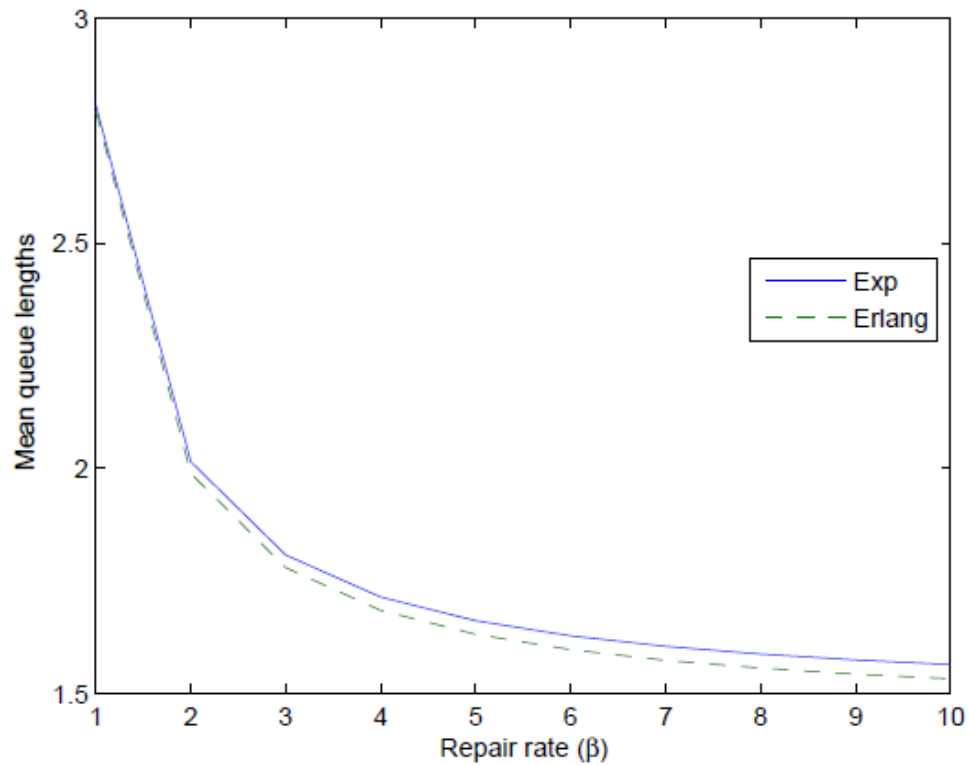


Figure 3. ρ , L_q , W_q versus repair rate (β)

Assigning particular values as $a = 2$, $b = 5$, $\theta = 0.3$, $m = 0.3$, $\Lambda^+ = 2$, $\Lambda^- = 1.1$, $\mu_2 = 17$, $\gamma = 9$, $\beta = 1.2$, $\xi_1 = 2$, $\pi = 0.2$, $\xi_2 = 2.4$ and increase the vacation rate (γ) value from 5 to 14 in steps of 1. We found the values of the utilization factor ρ , expected queue length L_q and expected waiting time W_q are tabulated in Table 4 and the corresponding graph is shown in Figure 4.

Table 4. The impact of the rate of vacation (γ) on ρ , L_q , W_q

γ	Exponential			Erlang-2 stage		
	ρ	L_q	W_q	ρ	L_q	W_q
5	0.0558	2.5872	2.1560	0.0601	2.5713	2.1428
6	0.0536	2.5456	2.1214	0.0580	2.5296	2.1080
7	0.0521	2.5177	2.0981	0.0564	2.5016	2.0847
8	0.0509	2.4978	2.0815	0.0553	2.4816	2.0680
9	0.0500	2.4829	2.0691	0.0544	2.4666	2.0555
10	0.0493	2.4714	2.0595	0.0537	2.4550	2.0458
11	0.0487	2.4622	2.0518	0.0531	2.4457	2.0381
12	0.0482	2.4547	2.0455	0.0526	2.4382	2.0318
13	0.0478	2.4484	2.0404	0.0522	2.4319	2.0266
14	0.0474	2.4432	2.0360	0.0519	2.4266	2.0222

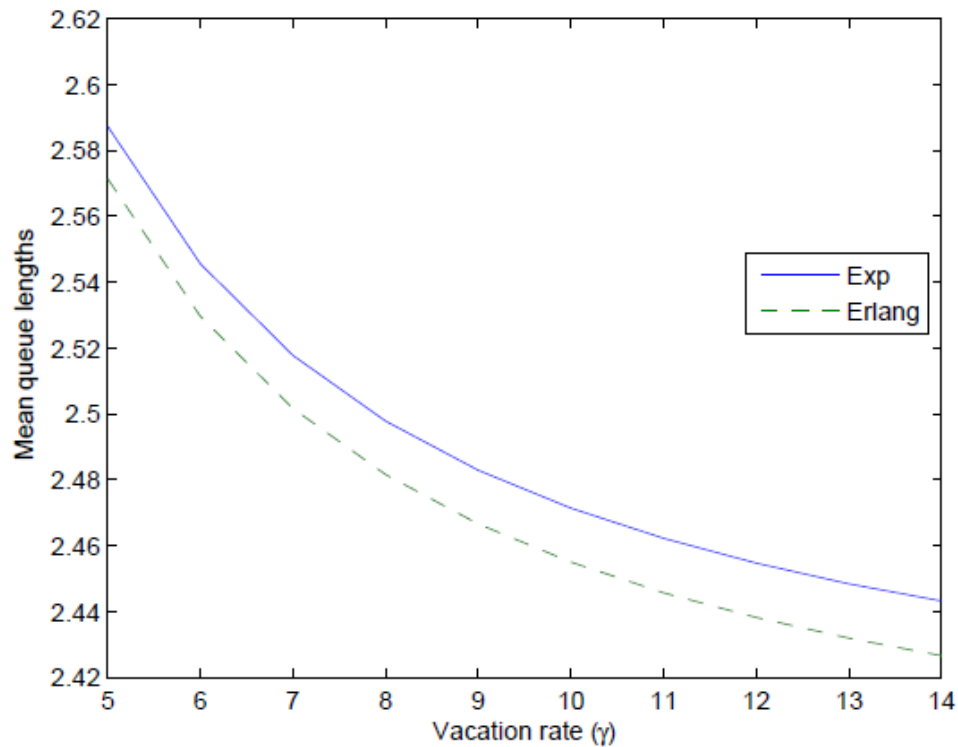


Figure 4. ρ , L_q , W_q versus vacation rate (γ)

Again assigning particular values as $a = 2$, $b = 5$, $\theta = 0.3$, $m = 0.3$, $\mu_1 = 10$, $\Lambda^- = 1.1$, $\mu_2 = 14$, $\gamma = 9$, $\beta = 1.2$, $\xi_1 = 2$, $\pi = 0.2$, $\xi_2 = 2.4$ and increase the arrival rate (Λ^+) value from 5 to 14 in steps of 1. We found the values of the utilization factor ρ , expected queue length L_q and expected waiting time W_q are tabulated in Table 5 and the graphical study is shown in Figure 5.

Table 5. The impact of the rate of arrival (Λ^+) on ρ , L_q , W_q

Λ^+	Exponential			Erlang-2 stage		
	ρ	L_q	W_q	ρ	L_q	W_q
3	0.0814	3.6947	2.0526	0.0830	3.6580	2.0322
4	0.1085	5.4675	2.2781	0.1107	5.4000	2.2500
5	0.1356	7.9945	2.6648	0.1383	7.8772	2.6257
6	0.1627	11.4014	3.1671	0.1660	11.2066	3.1130
7	0.1898	15.8237	3.7675	0.1937	15.5133	3.6936

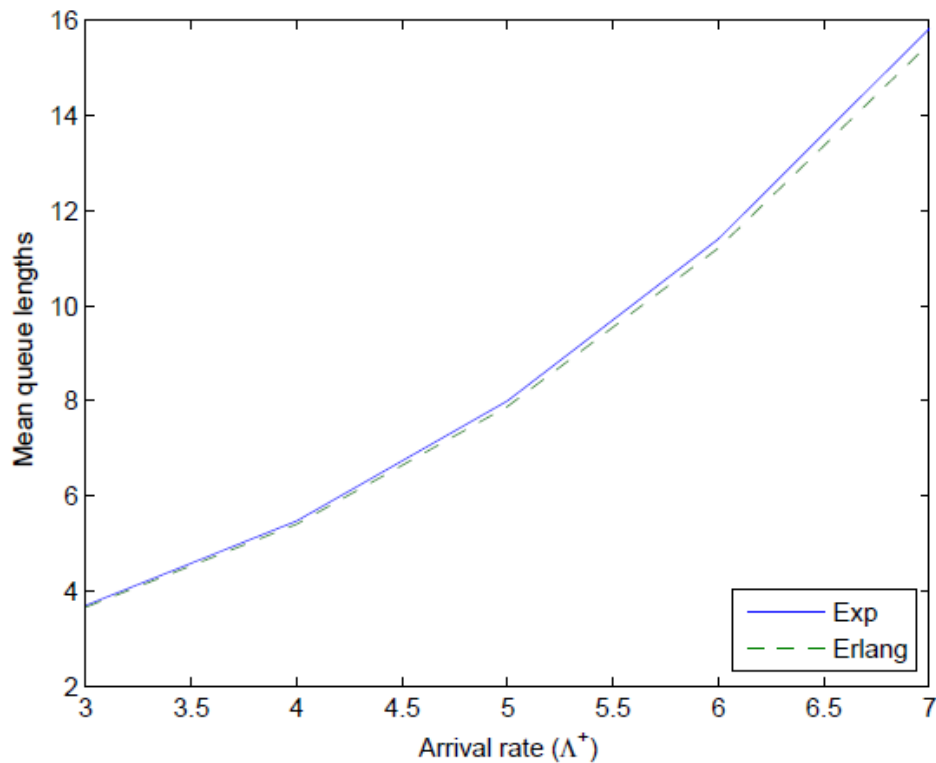


Figure 5. ρ , L_q , W_q versus arrival rate (Λ^+)

9. Conclusion and Further Work

In this paper, we derived the probability generating function of the number of customers in the queue at an arbitrary epoch. The performance measures of the system state probabilities and expected length of the queue are determined under steady state conditions. We have given numerical justification for this queueing model. In future this work may be extended into a queueing system with working vacations and vacation interruption.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgement

The authors like to thank the anonymous referees for their careful review for the improvement of this paper.

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