

## Sufficient Conditions for Absolute Cesàro Summable Factor

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### Abstract

Quasi- $f$ -power increasing sequence has been used for infinite series to establish a theorem on a minimal set of sufficient conditions for absolute Cesàro  $\varphi - |C, \alpha; \delta; l|_k$  summable factor. Further, a set of new and well-known arbitrary results have been obtained by using the main theorem. The presented main result has been validated by the previous result under suitable conditions. In this way, the Bounded Input Bounded Output (BIBO) stability of impulse response has been improved by finding a minimal set of sufficient conditions for absolute summability because absolute summable is the necessary and sufficient condition for BIBO stability.

**Keywords-** Absolute summability, Cesàro summability, Infinite series, Quasi- $f$ -power increasing sequence.

### 1. Introduction

Summability is a field in which we study the non-converging series/integrals and assign a value (number) to it. In mathematics analysis, summability method is an alternative formulation of convergence of a series which is divergent in conventional sense. Throughout the 19<sup>th</sup> century, many mathematicians studied various divergent series and defined numerous summability methods namely Abel summability, Cesàro summability, Euler summability, Hausdorff summability, Nörlund summability, Riesz summability etc. The present paper is devoted to the study of Cesàro summability methods and a theorem has been established for the absolutely Cesàro summable factor of an infinite series. With the help of a minimal set of sufficient conditions for a system, the error can be minimized by using summability methods so that the output data is filtered and according to the user's interest. Let  $\sum_{n=0}^{\infty} a_n$  be an infinite series with sequence of partial sums,  $s_n = \sum_{k=0}^n a_k$  and  $n^{\text{th}}$  mean of the sequence  $\{s_n\}$  is given by  $u_n$ , s.t.,

$$u_n = \sum_{k=0}^{\infty} u_{nk} s_k. \quad (1)$$

An infinite series  $\sum_{n=0}^{\infty} a_n$  is absolute summable, if

$$\lim_{n \rightarrow \infty} u_n = s \quad (2)$$

and

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}| < \infty. \quad (3)$$

**Definition 1 (Flett, 1957).** Let  $t_n^\alpha$  is  $n^{\text{th}}$   $(C, \alpha)$  mean of order  $\alpha$  ( $0 < \alpha \leq 1$ ) of the sequence  $(na_n)$ , then the infinite series is  $|C, \alpha; \delta|_k$  summable for  $k \geq 1$  and  $\delta \geq 0$ , if

$$\sum_{n=1}^{\infty} n^{\delta k-1} |t_n^\alpha|^k < \infty, \quad (4)$$

where  $t_n^\alpha$  is given by

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{p=1}^n A_{n-p}^{\alpha-1} p a_p, \quad (5)$$

And

$$A_n^\alpha = \begin{cases} 0, & n < 0, \\ 1, & n = 0, \\ O(n^\alpha), & n > 0. \end{cases}$$

**Definition 2.** If  $\{\varphi_n\}$  is a sequence of positive real number and the sequence of the mean  $\{t_n^\alpha\}$  satisfies the following condition:

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{k-1}}{n^{k-\delta k}} |t_n^\alpha|^k < \infty, \quad (6)$$

then the infinite series is  $\varphi - |C, \alpha; \delta|_k$  summable for  $k \geq 1$  and  $\delta \geq 0$ .

**Definition 3.** If the mean  $\{t_n^\alpha\}$  satisfies:

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} |t_n^\alpha|^k < \infty, \quad (7)$$

then infinite series is  $\varphi - |C, \alpha; \delta; l|_k$  summable for  $k \geq 1$ ,  $\delta \geq 0$  and  $l$  is a real number.

Bor has derived several theorems (1993, 2011a, 2011b, 2014, 2015, 2016) on applications of the various wider class sequences by applying the absolute summability techniques. In the same direction, Özarşlan and Ari (2011), Özarşlan and Yuvuz (2013) used the absolute matrix summability and Parashar (1981) applied  $(K, 1, \alpha)$  summability for infinite series. Chandra and Jain (1988) used absolute product summability for Fourier series. Sonker and Munjal (2016a, 2017) determined the theorems on generalized absolute Cesàro summability and in (2016b); they used triangle matrices for the minimal set of sufficient conditions of an infinite series to be bounded. Summability is very useful technique for solving the problem of discrete time signal (Richa and Kumar, 2019), Heat transfer problem (Singh et al., 2018) formulation with the help of Fourier series and for formation of iteration methods for differential equations (Chauhana and Srivastava, 2019) problems.

## 2. Known Result

A quasi- $f$ -power increasing sequence (Sulaiman, 2006) is a positive sequence  $B = \{B_n\}$  with a constant  $K = K(B, f) \geq 1$  for all  $1 \leq m \leq n$  such that

$$K f_n B_n \geq f_m B_m \tag{8}$$

and

$$f = [f_n(\zeta, \eta)] = \{n^\zeta (\log n)^\eta, 0 < \zeta < 1, \eta \geq 0\}. \tag{9}$$

A quasi- $f$ -power increasing sequence converted to quasi- $\zeta$ -power increasing sequence (Leindler, 2001), if  $\eta$  has a certain value as  $\eta=0$  in the condition (9). With the help of Cesàro summability of order  $\alpha$ , Bor (2015) has proved the following theorem for an infinite series.

**Theorem 2.1** Let  $(B_n)$  be a quasi- $f$ -power increasing sequence for some  $\zeta$  ( $0 < \zeta < 1$ ). Also suppose that there exists a sequence of numbers  $(D_n)$  such that it is  $\xi$ -quasi-monotone satisfying the following:

$$\sum n^{\xi_n} B_n = O(1), \tag{10}$$

$$\Delta D_n \leq \xi_n, \tag{11}$$

$$|\Delta \lambda_n| \leq |D_n|, \tag{12}$$

$$\sum D_n B_n \text{ is convergent for all } n. \tag{13}$$

If the conditions

$$|\lambda_n| B_n = O(1) \text{ as } n \rightarrow \infty, \tag{14}$$

$$\sum_{n=1}^m \frac{(w_n^\alpha)^k}{n} = O(B_m) \text{ as } m \rightarrow \infty, \tag{15}$$

are satisfied, then the series  $\sum a_n \lambda_n$  is  $|C, \alpha|_k$  summable for  $0 < \alpha \leq 1$  and  $k \geq 1$ .

### 3. Main Result

Considering absolute Cesàro summability  $\varphi - |C, \alpha; \delta, l|_k$  factors and a quasi- $f$ -power increasing sequence, we established the following theorem.

**Theorem 3.1** Let  $(B_n)$  be a quasi- $f$ -power increasing sequence for some  $\zeta$  ( $0 < \zeta < 1$ ) and  $(D_n)$  be a  $\xi$ -quasi-monotone sequence of numbers, s.t.,

$$\sum n^{\xi_n} B_n = O(1), \tag{16}$$

$$\Delta D_n \leq \xi_n, \tag{17}$$

$$|\Delta\lambda_n| \leq |D_n|, \tag{18}$$

$$\sum D_n B_n < \infty \text{ for all } n. \tag{19}$$

If the conditions

$$|\lambda_n| B_n = O(1) \text{ as } n \rightarrow \infty, \tag{20}$$

$$\sum_{n=v}^m \frac{\varphi_n^{l(k-1)}}{n^{(\alpha-l\delta+l)k}} = O\left(\frac{\varphi_v^{l(k-1)}}{v^{(\alpha-l\delta+l)k-1}}\right), \tag{21}$$

$$\sum_{n=1}^m \frac{\varphi_n^{l(k-1)} (w_n^\alpha)^k}{n^{l(k-\delta k)}} = O(B_m) \text{ as } m \rightarrow \infty, \tag{22}$$

are satisfied, then the series  $\sum a_n \lambda_n$  is  $\varphi - |C, \alpha; \delta, l|_k$  summable for  $0 < \alpha \leq 1$ ,  $\delta \geq 0$ ,  $k \geq 1$ ,  $l$  be a real number and  $w_n^\alpha$  is given by

$$w_n^\alpha = \begin{cases} \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \\ |t_n^\alpha|, & \alpha = 1. \end{cases} \tag{23}$$

#### 4. Proof of the Theorem

The series  $\sum a_n \lambda_n$  will be  $\varphi - |C, \alpha; \delta, l|_k$  summable, if the  $n^{\text{th}}$  mean  $T_n^\alpha$  of order  $\alpha$  of the sequence  $\{na_n \lambda_n\}$  satisfies,

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} |T_n^\alpha|^k < \infty. \tag{24}$$

Using Abel's transformation, the  $n^{\text{th}}$  mean  $T_n^\alpha$  of the sequence  $\{na_n \lambda_n\}$  is given by

$$\begin{aligned} T_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \lambda_v \\ &= \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \\ |T_n^\alpha| &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{p=1}^v A_{n-p}^{\alpha-1} p a_p \right| + \frac{|\lambda_n|}{A_n^\alpha} \left| \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v \right| \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} |\Delta\lambda_v| A_v^\alpha w_v^\alpha + |\lambda_n| w_n^\alpha \\ &= |T_{n,1}^\alpha| + |T_{n,2}^\alpha|. \end{aligned} \tag{25}$$

Applying Minkowski's inequality, we have

$$|T_n^\alpha|^k = |T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k (|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k). \quad (26)$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} |T_{n,r}^\alpha|^k < \infty, \quad \text{for } r = 1, 2. \quad (27)$$

By using Holder's inequality and Abel's transformation, we have

$$\begin{aligned} \sum_{n=2}^{m+1} \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} |T_{n,1}^\alpha|^k &\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} \frac{1}{(A_n^\alpha)^k} \left( \sum_{v=1}^{n-1} |\Delta \lambda_v| A_v^\alpha w_v^\alpha \right)^k \\ &\leq \sum_{n=2}^{m+1} \frac{\varphi_n^{l(k-1)}}{n^{(\alpha-l\delta+l)k}} \sum_{v=1}^{n-1} v^{\alpha k} |D_v| (w_v^\alpha)^k \left( \sum_{v=1}^{n-1} |D_v| \right)^{k-1} \\ &= O(1) \sum_{v=1}^m v^{\alpha k} |D_v| (w_v^\alpha)^k \sum_{n=v+1}^{m+1} \frac{\varphi_n^{l(k-1)}}{n^{(\alpha-l\delta+l)k}} \\ &= O(1) \sum_{v=1}^m v^{\alpha k} |D_v| (w_v^\alpha)^k \frac{\varphi_v^{l(k-1)}}{v^{(\alpha-l\delta+l)k-1}} \\ &= O(1) \sum_{v=1}^m v |D_v| (w_v^\alpha)^k \frac{\varphi_v^{l(k-1)}}{v^{l(k-\delta k)}} \\ &= O(1) \sum_{v=1}^{m-1} \Delta(v|D_v|) \sum_{r=1}^v \frac{\varphi_r^{k-1}}{r^{k-\delta k}} (w_r^\alpha)^k + O(1)m|D_m| \sum_{v=1}^m \frac{\varphi_v^{k-1}}{v^{k-\delta k}} (w_v^\alpha) \\ &= O(1) \sum_{v=1}^{m-1} |(v+1)\Delta|D_v| - |D_v||B_v + O(1)m|D_m|B_m \\ &= O(1) \sum_{v=1}^{m-1} v|\Delta D_v|B_v + O(1) \sum_{v=1}^{m-1} |D_v|B_v + O(1)m|D_m|B_m \\ &= O(1) \sum_{v=1}^{m-1} v\xi_v B_v + O(1) \sum_{v=1}^{m-1} |D_v|B_v + O(1)m|D_m|B_m \\ &= O(1) \quad \text{as } m \rightarrow \infty, \end{aligned} \quad (28)$$

$$\begin{aligned}
 \sum_{n=2}^m \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} |T_{n,2}^\alpha|^k &= O(1) \sum_{n=1}^m |\lambda_n| (w_n^\alpha)^k \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} \\
 &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^n (w_v^\alpha)^k \frac{\varphi_v^{l(k-1)}}{v^{l(k-\delta k)}} + O(1) |\lambda_m| \sum_{n=1}^m (w_n^\alpha)^k \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} \\
 &= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| B_n + O(1) |\lambda_m| B_m \\
 &= O(1) \sum_{n=1}^{m-1} |D_n| B_n + O(1) |\lambda_m| B_m \\
 &= O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{29}$$

Collecting (24) - (29), we have

$$\sum_{n=1}^{\infty} \frac{\varphi_n^{l(k-1)}}{n^{l(k-\delta k)}} |T_n^\alpha|^k < \infty. \tag{30}$$

Hence, the proof of the theorem is complete.

## 5. Corollaries

**Corollary 5.1 (Bor, 2015)** Let  $(B_n)$  be a quasi- $f$ -power increasing sequence for some  $\zeta$  ( $0 < \zeta < 1$ ) and  $(D_n)$  be a  $\xi$ -quasi-monotone sequence of numbers satisfying (16-20) and the following conditions:

$$\sum_{n=v}^m \frac{\varphi_n^{(k-1)}}{n^{(\alpha+1)k}} = O\left(\frac{\varphi_v^{(k-1)}}{v^{(\alpha+1)k-1}}\right), \tag{31}$$

$$\sum_{n=1}^m \frac{\varphi_n^{(k-1)} (w_n^\alpha)^k}{n^k} = O(B_m) \quad \text{as } m \rightarrow \infty, \tag{32}$$

then the series  $\sum a_n \lambda_n$  is  $\varphi - |C, \alpha|_k$  summable for  $0 < \alpha \leq 1, k \geq 1$  and  $w_n^\alpha$  is given by

$$w_n^\alpha = \begin{cases} \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \\ |t_n^\alpha|, & \alpha = 1. \end{cases} \tag{33}$$

**Proof:** By using  $l = 1$  and  $\delta = 0$  in main theorem, we will get (31) and (32). We omit the details of the proof as it is similar to that of the main theorem 3.1.

**Corollary 5.2** Let  $(B_n)$  be a quasi- $f$ -power increasing sequence for some  $\zeta$  ( $0 < \zeta < 1$ ) and  $(D_n)$  be a  $\xi$ -quasi-monotone sequence of numbers satisfying (16-20) and the following condition:

$$\sum_{n=1}^m \frac{(w_n^\alpha)^k}{n} = O(B_m) \quad \text{as } m \rightarrow \infty, \tag{34}$$

then the series  $\sum a_n \lambda_n$  is  $|C, \alpha|_k$  summable for  $0 < \alpha \leq 1, k \geq 1$  and  $w_n^\alpha$  is given by

$$w_n^\alpha = \begin{cases} \max_{1 \leq v \leq n} |t_v^\alpha|, & 0 < \alpha < 1, \\ |t_n^\alpha|, & \alpha = 1. \end{cases} \quad (35)$$

**Proof:** By using  $\varphi_n = n, l = 1$  and  $\delta = 0$  in main theorem, we will get (34). We omit the details of the proof as it is similar to that of the main theorem 3.1.

## 6. Conclusion

The main result of this research article is a problem on generalized absolute summability factor of infinite series which make the system stable. A necessary and sufficient condition for a system to be BIBO (Bounded Input Bounded Output) stable is that the impulse response be absolutely summable, i.e.,

$$\text{BIBO stable} \iff \sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Summability techniques are trained to minimize the error. With the use of summability Technique, the output of the signals can be made stable, bounded and used to predict the behavior of the input data, the initial situation and the changes in the complete process.

Further, this study has a number of direct applications in rectification of signals in FIR filter (Finite impulse response filter) and IIR filter (Infinite impulse response filter). In a nut shell, the absolute summability methods have vast potential in dealing with the problems based on infinite series. Based on the investigation, it can be concluded that our result is a generalized form which can be reduced for several well-known summabilities. Our theorem is validated by Corollary 5.1, which is a mean result of Bor (2015) on the minimal set of sufficient condition for infinite series to be absolute summable.

## Conflict of Interest

The authors declare that there is no conflict of interest for this publication.

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