International Journal of Mathematical, Engineering and Management Sciences
Vol. 4, No. 4, 1051–1067, 2019
https://dx.doi.org/10.33889/IJMEMS.2019.4.4-084

DQM Based on the Modified Form of CTB Shape Functions for Coupled Burgers' Equation in 2D and 3D

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(Received July 23, 2018; Accepted May 7, 2019)

Abstract
This work concerns for solving of coupled Burgers' equations (CBEs) in 2D and 3D via DQM based on cubic trigonometric B-spline (CTB) shape functions. In the method, the shape functions are modified and used for the integration of space derivative. Consequently, the CBEs are transformed into the integral equations. These integral equations are solved by an "optimal strong stability-preserving Runge-Kutta method (SSP-RK54)". Three examples are taken for analysis. The assessment of the present results are done with a number of already presented results in the literature. We initiated that the present method generates more precise results. Straightforward algorithm, little amount of computational cost and less error norms are the major achievements of the method. Therefore, the present method possibly will be very valuable optional method for the computation of nonlinear PDEs. Moreover, the analysis of method's stability is also done.

Keywords- CBEs, DQM, CTB functions, SSP-RK54, Stability analysis.

1. Introduction
There is a significant role of the nonlinear Burgers' equations to model the fluid dynamics problems to study turbulence, shock wave structures, mass transport, etc. In view of appropriate applications of these equations, it is great need to find a suitable method, which provide better solutions economically as well as computationally. The Burgers' equation is a 2nd order nonlinear PDE which was established by Bateman (1915). Afterward a solution of this equation was suggested by Burger (1948). The Burgers' equation may be treated as a straightforward mathematical model, which is generally used for a range of applications. This equation is a special type of N-S equation. The inclusion of straight forwardness nonlinear convection and diffusion terms, makes the study of the solutions curiosity of this equation.

We consider the following CBEs:

(i) The CBEs in two-dimension

$$\begin{cases} u_t = -\left(u_x u + u_y v\right) + \left(u_{xx} + u_{yy}\right) \big/ \mathrm{Re}, \\ v_t = -\left(v_x u + v_y v\right) + \left(v_{xx} + v_{yy}\right) \big/ \mathrm{Re}, \end{cases} \tag{1}$$

with initial conditions (I.C's.)

$$u=\varphi_1 \text{ and } v=\varphi_2 \text{ at } t=0, \tag{2}$$

and boundary conditions (B.C's).

$$u(x,y,t)=\delta,\ v(x,y,t)=\gamma, \tag{3}$$

where $(x,y)\in\partial R$. $R=\{(x,y): x\in[a_1,a_2],\ y\in[b_1,b_2]\}$ is the rectangular domain and ∂R is the boundary of R.

(ii) The CBEs in three-dimension

$$\begin{cases} u_t=-u_x u-u_y v-u_z w+\left(u_{xx}+u_{yy}+u_{zz}\right)/\mathrm{Re}, \\ v_t=-v_x u-v_y v-v_z w+\left(v_{xx}+v_{yy}+v_{zz}\right)/\mathrm{Re}, \\ w_t=-w_x u-w_y v-w_z w+\left(w_{xx}+w_{yy}+w_{zz}\right)/\mathrm{Re}, \end{cases} \tag{4}$$

with I. C.'s.

$$u=\phi_1,\ v=\phi_2,\ w=\phi_3 \text{ at } t=0, \tag{5}$$

and B. C.'s.

$$u(x,y,z,t)=\xi_1,\ v(x,y,z,t)=\xi_2,\ w(x,y,z,t)=\xi_3, \tag{6}$$

where $(x,y,z)\in\partial\Omega$. $\Omega=\{(x,y,z): x\in[a_1,a_2],\ y\in[b_1,b_2],\ z\in[c_1,c_2]\}$ is the cubic domain and $\partial\Omega$ is the boundary of Ω.

The terms u, v and w are velocity components and have its usual meaning in two and three dimensions, Re is the "Reynolds number", uu_x+vu_y and $uu_x+vu_y+wu_z$ are the "nonlinear convection terms" in one, two and three dimensions respectively. The terms $\left(u_{xx}+u_{yy}\right)/\mathrm{Re}$ and $\left(u_{xx}+u_{yy}+u_{zz}\right)/\mathrm{Re}$ are the diffusion terms.

Various analytical and numerical methods have been applied for the computational study of nonlinear CBEs. An exact solution of 2D CBEs is generated by Fletcher (1983). Radwan (1999) used both 4th order accurate compact ADI and Du Fort Frankel methods to solve it. Bahadir (2003) used a fully implicit FDM to solve 2D CBEs while Srivastava et al. (2013b), Srivastava et al. (2013c) and Shukla et al. (2014) used implicit, implicit logarithm FD methods and modified cubic B-spline DQM respectively. Zhu et al. (2010) and Zhao et al. (2011) intended a discrete ADM and LDG finite element method respectively while a lattice Boltzmann method is used to solve it by Liu and Shi (2011). Srivastava et al. (2013a) generated an exact solution of 3D CBEs while Shukla et al. (2016) employed MCB-DQM to solve it. Recently, a hybrid trigonometric DQM is used for Burgers' equation by Arora and Joshi (2018).

Baishya (2019) reproduced the advantage of a new technique using Hermite orthogonal basis elements to solve DEs. Vaid and Arora (2019) presented a collocation method based on trigonometric cubic B-spline functions to approximate a singular perturbed delay DE while

International Journal of Mathematical, Engineering and Management Sciences
Vol. 4, No. 4, 1051–1067, 2019
https://dx.doi.org/10.33889/IJMEMS.2019.4.4-084

Chauhan and Srivastava (2019) presented a review report on latest computational techniques to solve DEs using various orders Runge-Kutta algorithms.

In this work, the solutions of CBEs in 2D and 3D are obtained via DQM based on a modified form of CTB shape functions. We modify the shape function and use for the integration of space derivatives. Therefore, the CBEs are transformed into the form of integral equations. Then, we use SSP-RK54 to solve these integral equations.

2. The DQM

The DQM was set up by Bellman et al. (1972). Its simplicity and computationally effortlessness makes more suitable to solve PDEs. To apply this method, the problem domains are uniformly distributed at the grid points x_i $(i=1,2,...,M)$, y_j $(j=1,2,...,N)$, and z_k $(k=1,2,...,L)$ with space step sizes $h_1=(a_2-a_1)/(M-1)$, $h_2=(b_2-b_1)/(N-1)$ and $h_3=(c_2-c_1)/(L-1)$ respectively.

In 2-dimensional problem, the partial derivatives (p.d.) of r^{th} order of u are estimated at x_i, y_j as

$$\frac{\partial^r u_{ij}}{\partial x^r}=\sum_{p=1}^{M} a_{ip}^{(r)} u_{pj}\,, \tag{7}$$

$$\frac{\partial^r u_{ij}}{\partial y^r}=\sum_{p=1}^{N} a_{jp}^{(r)} u_{ip}\,. \tag{8}$$

Similarly, the approximation of p.d. of r^{th} order of v are estimated s

$$\frac{\partial^r v_{ij}}{\partial x^r}=\sum_{p=1}^{M} a_{ip}^{(r)} v_{pj}\,, \tag{9}$$

$$\frac{\partial^r v_{ij}}{\partial y^r}=\sum_{p=1}^{N} a_{jp}^{(r)} v_{ip}\,. \tag{10}$$

where $u_{ij}=u\left(x_i,y_j,t\right)$ and $v_{ij}=v\left(x_i,y_j,,t\right)$.

In 3-dimensional problem, the p.d. of r^{th} order of u are estimated at x_i, y_j and z_k, which are

$$\frac{\partial^r u_{ijk}}{\partial x^r}=\sum_{p=1}^{M} a_{ip}^{(r)} u_{pjk}\,, \tag{11}$$

$$\frac{\partial^r u_{ijk}}{\partial y^r}=\sum_{p=1}^{N} b_{jp}^{(r)} u_{ipk}\,, \tag{12}$$

$$\frac{\partial^r u_{ijk}}{\partial z^r}\equiv\sum_{p=1}^{L} c_{kp}^{(r)} u_{ijp}\,. \tag{13}$$

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Similarly, p.d. of v and w are estimated at x_i, y_j and z_k as:

$$\frac{\partial^r v_{ijk}}{\partial x^r} = \sum_{p=1}^{M} a_{ip}^{(r)} v_{pjk}, \tag{14}$$

$$\frac{\partial^r v_{ijk}}{\partial y^r} = \sum_{p=1}^{N} b_{jp}^{(r)} v_{ipk}, \tag{15}$$

$$\frac{\partial^r v_{ijk}}{\partial z^r} \equiv \sum_{p=1}^{L} c_{kp}^{(r)} v_{ijp}, \tag{16}$$

$$\frac{\partial^r w_{ijk}}{\partial x^r} = \sum_{p=1}^{M} a_{ip}^{(r)} w_{pjk}, \tag{17}$$

$$\frac{\partial^r w_{ijk}}{\partial y^r} = \sum_{p=1}^{N} b_{jp}^{(r)} w_{ipk}, \tag{18}$$

$$\frac{\partial^r w_{ijk}}{\partial z^r} \equiv \sum_{p=1}^{L} c_{kp}^{(r)} w_{ijp}, \tag{19}$$

where $u_{ijk} = u(x_i, y_j, z_k, t)$, $v_{ijk} = v(x_i, y_j, z_k, t)$, $w_{ijk} = w(x_i, y_j, z_k, t)$ and $a_{ij}^{(r)}$, $b_{ij}^{(r)}$, $c_{ij}^{(r)}$ are weighting coefficients.

3. CTB Functions

The CTB shape functions are twice continuous differentiable piecewise functions in the domain under consideration. These functions are defined as

$$T_k(x) = \frac{1}{\chi}\begin{cases} \beta^3(x_{k-2}), & x \in [x_{k-2}, x_{k-1}) \\ \left(\alpha(x_k)\beta(x_{k-2}) + \alpha(x_{k+1})\beta(x_{k-1})\right)\beta(x_{k-2}) + \alpha(x_{k+2})\beta^2(x_{k-1}), & x \in [x_{k-1}, x_k) \\ \alpha^2(x_{k+1})\beta(x_{k-2}) + \alpha(x_{k+2})\left(\alpha(x_{k+1})\beta(x_{k-1}) + \alpha(x_{k+2})\beta(x_k)\right), & x \in [x_k, x_{k+1}) \\ \alpha^3(x_{k+2}), & x \in [x_{k+1}, x_{k+2}) \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

where $\alpha = s\,\text{in}\left(\frac{x_k - x}{2}\right)$, $\beta = \sin\left(\frac{x - x_k}{2}\right)$, $\chi = \sin(0.5)\sin(h)\sin(1.5h)$.

The vector $\{T_k(x)\}$ forms a shape in concern domain. The values of $T_k(x)$, $T_k'(x)$ and $T_k''(x)$ at x_j are given by Lemma 1.

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Lemma 1.

$$T_k\left(x_j\right)=\begin{cases} p_2, & \text{if } k=j \\ p_1, & \text{if } k=j\pm 1; \\ 0, & \text{else}\end{cases} \; T_k'\left(x_j\right)=\begin{cases} p_3, & \text{if } k=j+1 \\ -p_3, & \text{if } k=j-1; \\ 0, & \text{else}\end{cases} \; T_k''\left(x_j\right)=\begin{cases} p_5, & \text{if } k=j \\ p_4, & \text{if } k-j=\pm 1, \\ 0, & \text{else}\end{cases}$$

where, $p_1=\sin^2(0.5h)\csc(h)\csc(1.5h)$, $p_2=\dfrac{2}{1+2\cos(h)}$, $p_3=\dfrac{3}{4}\csc(1.5h)$,

$$p_4=\frac{3\left(1+3\cos(h)\right)\csc^2(0.5h)}{16+\left(2\cos(0.5h)+\cos(1.5h)\right)},\; p_5=\frac{-3\cot^2(1.5h)}{2+4\cos(h)}.$$

Now we modify the CTB shape functions with preserving the property of diagonally dominance for resulting matrix as:

$$\left.\begin{array}{l} \hat{T}_1=2T_0+T_1,\ \hat{T}_2=-T_0+T_2, \\ \hat{T}_m=T_m \text{ for } m=3,\ldots,N-2, \\ \hat{T}_{N-1}=-T_{N+1}+T_{N-1},\ \hat{T}_N=2T_{N+1}(x)+T_N, \end{array}\right\} \tag{21}$$

where $\left\{\hat{T}_1,\ \hat{T}_2,\ \hat{T}_3,\ldots,\hat{T}_N\right\}$ again forms a basis in aforesaid domain.

The substitution of the values of $\hat{T}_j,\ j=1,2,\ldots,M$ in Equation (7), gives

$$\hat{T}_m'\left(x_i\right)=\sum_{p=1}^{M}a_{ip}^{(1)}T_m\left(x_p\right), \tag{22}$$

by Equation (21) and Lemma 1, Equation (22) reduces into

$$A\vec{c}^{(1)}[i]=\vec{B}[i], \tag{23}$$

where $\vec{c}^{(1)}[i]=\left[a_{i1}^{(1)},a_{i2}^{(1)},\ldots,a_{iM}^{(1)}\right]^T$, and A is the coefficient matrix which is given by

$$A=\begin{bmatrix} 2p_1+p_2 & p_2 & & & & & \\ 0 & p_2 & p_1 & & & & \\ & p_1 & p_2 & p_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & p_1 & p_2 & p_1 & \\ & & & & p_1 & p_2 & 0 \\ & & & & & p_1 & 2p_1+p_2 \end{bmatrix}_{N\times N}.$$

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https://dx.doi.org/10.33889/IJMEMS.2019.4.4-084

The vectors $\vec{B}[i]$ are given by

$$\vec{B}[1]=[-2p_3 \quad 2p_3 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0]^T,$$
$$\vec{B}[2]=[-p_3 \quad 0 \quad p_3 \quad 0 \quad \cdots \quad 0 \quad 0]^T,$$
$$\vdots$$
$$\vec{B}[N-1]=[0 \quad 0 \quad \cdots \quad 0 \quad -p_3 \quad 0 \quad p_3]^T,$$
$$\vec{B}[N]=[0 \quad 0 \quad \cdots \quad 0 \quad 0 \quad -p_3 \quad p_3]^T.$$

The solution of system (23) is done by "Thomas algorithm".

Weighting coefficients $a_{ij}^{(2)},\ 1\le i, j \le M$ are computed by the recurrence relations given by Shu (2000) which is defined below

$$a_{ij}^{(2)}=\begin{cases} 2\left(a_{ij}^{(1)}a_{ii}^{1}-\dfrac{a_{ij}^{(1)}}{x_i-x_j}\right) & \text{if } j\neq i \\ -\displaystyle\sum_{j=1,\, j\neq i}^{M} a_{ij}^{(2)} & \text{else} \end{cases}. \tag{24}$$

4. Implementation of the Method to CDEs

Now we substitute the approximated spatial derivatives values in the CBEs (1) and (4). By substituting in Equation (1), we get

$$\begin{cases} \dfrac{\partial u_{ij}}{\partial t}=-u_{ij}\displaystyle\sum_{p=1}^{M}a_{ip}^{(1)}u_{pj}-v_{ij}\sum_{p=1}^{N}b_{jp}^{(1)}u_{ip}+\upsilon\left[\sum_{p=1}^{M}a_{ip}^{(2)}u_{pj}+\sum_{p=1}^{N}b_{jp}^{(2)}u_{ip}\right], \\ \dfrac{\partial v_{ij}}{\partial t}=-u_{ij}\displaystyle\sum_{p=1}^{M}a_{ip}^{(1)}u_{pj}-v_{ij}\sum_{p=1}^{N}b_{jp}^{(1)}u_{ip}+\upsilon\left[\sum_{p=1}^{M}a_{ip}^{(2)}u_{pj}+\sum_{p=1}^{N}b_{jp}^{(2)}u_{ip}\right], \end{cases} \tag{25}$$

On applying B.C.'s (3) into Equation (25), we get

$$\begin{cases} \dfrac{\partial u_{ij}}{\partial t}=-u_{ij}\displaystyle\sum_{p=2}^{M-1}a_{ip}^{(1)}u_{pj}-v_{ij}\sum_{p=2}^{N-1}b_{jp}^{(1)}u_{ip}+\upsilon\left[\sum_{p=2}^{M-1}a_{ip}^{(2)}u_{pj}+\sum_{p=2}^{N-1}b_{jp}^{(2)}u_{ip}\right]+F_{ij}, \\ \dfrac{\partial v_{ij}}{\partial t}=-u_{ij}\displaystyle\sum_{p=2}^{M-1}a_{ip}^{(1)}u_{pj}-v_{ij}\sum_{p=2}^{N-1}b_{jp}^{(1)}u_{ip}+\upsilon\left[\sum_{p=2}^{M-1}a_{ip}^{(2)}u_{pj}+\sum_{p=2}^{N-1}b_{jp}^{(2)}u_{ip}\right]+G_{ij}, \end{cases} \tag{26}$$

where

$$F_{ij}=-u_{ij}\left(a_{i1}^{(1)}u_{1j}+a_{iM}^{(1)}u_{Mj}\right)-v_{ij}\left(b_{i1}^{(1)}u_{i1}+b_{iN}^{(1)}u_{iN}\right)+\upsilon\left[\left(a_{i1}^{(2)}u_{1j}+a_{iM}^{(2)}u_{Mj}\right)+\left(b_{i1}^{(2)}u_{i1}+b_{iN}^{(2)}u_{iN}\right)\right],$$

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$$G_{ij} = -u_{ij}\left(a_{i1}^{(1)}v_{1j} + a_{iM}^{(1)}v_{Mj}\right) - v_{ij}\left(b_{i1}^{(1)}v_{i1} + b_{iN}^{(1)}v_{iN}\right) + \upsilon\left[\left(a_{i1}^{(2)}v_{1j} + a_{iM}^{(2)}v_{Mj}\right) + \left(b_{i1}^{(2)}v_{i1} + b_{iN}^{(2)}v_{iN}\right)\right],$$

Equation (25) can also be written as

$$\frac{du_{ij}}{dt} = L_1\left(u_{ij}\right) \text{ and } \frac{dv_{ij}}{dt} = L_2\left(u_{ij}\right), \tag{27}$$

where L_1 and L_2 denote spatial nonlinear differential operator.

Now, by substituting in Equation (4), we get

$$\begin{cases} \dfrac{du_{ijk}}{dt} = -u_{ijk}\displaystyle\sum_{p=1}^{M} a_{ip}^{(1)}u_{pjk} - v_{ijk}\sum_{p=1}^{N} b_{jp}^{(1)}u_{ipk} - w_{ijk}\sum_{p=1}^{L} c_{kp}^{(1)}u_{ijp} + \upsilon\left[\sum_{p=1}^{M} a_{ip}^{(2)}u_{pjk} + \sum_{p=1}^{N} b_{jp}^{(2)}u_{ipk}\right. \\ \qquad\qquad \left. + \displaystyle\sum_{p=1}^{L} c_{kp}^{(2)}u_{ijp}\right], \\ \dfrac{dv_{ijk}}{dt} = -u_{ijk}\displaystyle\sum_{p=1}^{M} a_{ip}^{(1)}v_{pjk} - v_{ijk}\sum_{p=1}^{N} b_{jp}^{(1)}v_{ipk} - w_{ijk}\sum_{p=1}^{L} c_{kp}^{(1)}v_{ijp} + \upsilon\left[\sum_{p=1}^{M} a_{ip}^{(2)}v_{pjk} + \sum_{p=1}^{N} b_{jp}^{(2)}v_{ipk}\right. \\ \qquad\qquad \left. + \displaystyle\sum_{p=1}^{L} c_{kp}^{(2)}v_{ijp}\right], \\ \dfrac{dw_{ijk}}{dt} = -u_{ijk}\displaystyle\sum_{p=1}^{M} a_{ip}^{(1)}w_{pjk} - v_{ijk}\sum_{p=1}^{N} b_{jp}^{(1)}w_{ipk} - w_{ijk}\sum_{p=1}^{L} c_{kp}^{(1)}w_{ijp} + \upsilon\left[\sum_{p=1}^{M} a_{ip}^{(2)}w_{pjk} + \sum_{p=1}^{N} b_{jp}^{(2)}w_{ipk}\right. \\ \qquad\qquad \left. + \displaystyle\sum_{p=1}^{L} c_{kp}^{(2)}w_{ijp}\right]. \end{cases} \tag{28}$$

On applying B.C.'s (6) into Equation (28), we have

$$\begin{cases} \dfrac{du_{ijk}}{dt} = -u_{ijk}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}u_{pjk} - v_{ijk}\sum_{p=2}^{N-1} b_{jp}^{(1)}u_{ipk} - w_{ijk}\sum_{p=2}^{L-1} c_{kp}^{(1)}u_{ijp} + \upsilon\left[\sum_{p=2}^{M-1} a_{ip}^{(2)}u_{pjk} + \sum_{p=2}^{N-1} b_{jp}^{(2)}u_{ipk}\right. \\ \qquad\qquad \left. + \displaystyle\sum_{p=2}^{L-1} c_{kp}^{(2)}u_{ijp}\right] + F_{ijk}, \\ \dfrac{dv_{ijk}}{dt} = -u_{ijk}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}v_{pjk} - v_{ijk}\sum_{p=2}^{N-1} b_{jp}^{(1)}v_{ipk} - w_{ijk}\sum_{p=2}^{L-1} c_{kp}^{(1)}v_{ijp} + \upsilon\left[\sum_{p=2}^{M-1} a_{ip}^{(2)}v_{pjk} + \sum_{p=2}^{N-1} b_{jp}^{(2)}v_{ipk}\right. \\ \qquad\qquad \left. + \displaystyle\sum_{p=2}^{L-1} c_{kp}^{(2)}v_{ijp}\right] + G_{ijk}, \\ \dfrac{dw_{ijk}}{dt} = -u_{ijk}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}w_{pjk} - v_{ijk}\sum_{p=2}^{N-1} b_{jp}^{(1)}w_{ipk} - w_{ijk}\sum_{p=2}^{L-1} c_{kp}^{(1)}w_{ijp} + \upsilon\left[\sum_{p=2}^{M-1} a_{ip}^{(2)}w_{pjk} + \sum_{p=2}^{N-1} b_{jp}^{(2)}w_{ipk}\right. \\ \qquad\qquad \left. + \displaystyle\sum_{p=2}^{L-1} c_{kp}^{(2)}w_{ijp}\right] + H_{ijk}, \end{cases} \tag{29}$$

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where

$$F_{ijk} = -u_{ijk}\left(a_{i1}^{(1)}u_{1jk} + a_{iM}^{(1)}u_{Mjk}\right) - v_{ijk}\left(b_{i1}^{(1)}u_{i1k} + b_{iN}^{(1)}u_{iNk}\right) - w_{ijk}\left(c_{i1}^{(1)}u_{ij1} + c_{iL}^{(1)}u_{ijL}\right)$$
$$+\upsilon\left[\left(a_{i1}^{(2)}u_{1jk} + a_{iM}^{(2)}u_{Mjk}\right) + \left(b_{i1}^{(2)}u_{i1k} + b_{iN}^{(2)}u_{iNk}\right) + \left(c_{i1}^{(2)}u_{ij1} + c_{iL}^{(2)}u_{ijL}\right)\right],$$
$$G_{ijk} = -u_{ijk}\left(a_{i1}^{(1)}v_{1jk} + a_{iM}^{(1)}v_{Mjk}\right) - v_{ijk}\left(b_{i1}^{(1)}v_{i1k} + b_{iN}^{(1)}v_{iNk}\right) - w_{ijk}\left(c_{i1}^{(1)}v_{ij1} + c_{iL}^{(1)}v_{ijL}\right)$$
$$+\upsilon\left[\left(a_{i1}^{(2)}v_{1jk} + a_{iM}^{(2)}v_{Mjk}\right) + \left(b_{i1}^{(2)}v_{i1k} + b_{iN}^{(2)}v_{iNk}\right) + \left(c_{i1}^{(2)}v_{ij1} + c_{iL}^{(2)}v_{ijL}\right)\right],$$
$$H_{ijk} = -u_{ijk}\left(a_{i1}^{(1)}w_{1jk} + a_{iM}^{(1)}w_{Mjk}\right) - v_{ijk}\left(b_{i1}^{(1)}w_{i1k} + b_{iN}^{(1)}w_{iNk}\right) - w_{ijk}\left(c_{i1}^{(1)}w_{ij1} + c_{iL}^{(1)}w_{ijL}\right)$$
$$+\frac{1}{\mathrm{Re}}\left[\left(a_{i1}^{(2)}w_{1jk} + a_{iM}^{(2)}w_{Mjk}\right) + \left(b_{i1}^{(2)}w_{i1k} + b_{iN}^{(2)}w_{iNk}\right) + \left(c_{i1}^{(2)}w_{ij1} + c_{iL}^{(2)}w_{ijL}\right)\right].$$

The Equations (28) can also be written as

$$\frac{du_{ijk}}{dt} = L_1\left(u_{ijk}\right),\ \frac{dv_{ijk}}{dt} = L_2\left(u_{ijk}\right) \text{ and } \frac{dw_{ijk}}{dt} = L_3\left(u_{ijk}\right). \tag{30}$$

Finally "SSP-RK54" (Gottlieb et al., 2009) is applied to integrate Equations (27) and (30).

5. Stability Analysis

For linearizing the non-linear terms of CBEs, we assume u, v and w as locally constant (Saka et al., 2009). The discretized CBEs (26) are converted into

$$\begin{cases} \dfrac{\partial u_{ij}}{\partial t} = -U_{ij}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}u_{pj} - V_{ij}\sum_{p=2}^{N-1} b_{jp}^{(1)}u_{ip} + \upsilon\left[\sum_{p=2}^{M-1} a_{ip}^{(2)}u_{pj} + \sum_{p=2}^{N-1} b_{jp}^{(2)}u_{ip}\right] + F_{ij}, \\ \dfrac{\partial v_{ij}}{\partial t} = -U_{ij}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}u_{pj} - V_{ij}\sum_{p=2}^{N-1} b_{jp}^{(1)}u_{ip} + \upsilon\left[\sum_{p=2}^{M-1} a_{ip}^{(2)}u_{pj} + \sum_{p=2}^{N-1} b_{jp}^{(2)}u_{ip}\right] + G_{ij}. \end{cases} \tag{31}$$

The system of ODEs (31) can also be expressed as

$$\frac{d\{W\}}{dt} = \begin{bmatrix} A & O \\ O & B \end{bmatrix}\{W\} + \{K\}, \tag{32}$$

where

(i) The vector $\{K\} = (F, G)^T$ contains non-homogeneous part and B.C.'s.

(ii) The null matrices are represented by O.

(iii) $\{W\} = (U,V)^T$.

(iv) $\begin{cases} A = -U_{ij}A_1 - V_{ij}B_1 + \upsilon A_2 + \upsilon B_2, \\ B = -U_{ij}A_1' - V_{ij}B_1' + \upsilon A_2' + \upsilon B_2', \end{cases}$

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where A_r and B_r are $(N-2)(M-2)$ order square matrices as

$$A_r = \begin{bmatrix} a_{22}^{(r)}I & a_{23}^{(r)}I & \cdots & a_{2(N-1)}^{(r)}I \\ a_{32}^{(r)}I & a_{33}^{(r)}I & \cdots & a_{3,N-1}^{(r)}I \\ \vdots & \vdots & \ddots & \vdots \\ a_{(N-1)2}^{(r)}I & a_{(N-1)3}^{(r)}I & \cdots & a_{(N-1)(N-1)}^{(r)}I \end{bmatrix}, \ B_r = \begin{bmatrix} H_r & O & \cdots & O \\ O & H_r & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & H_r \end{bmatrix},$$

where

$$H_r = \begin{bmatrix} b_{22}^{(r)} & b_{23}^{(r)} & \cdots & b_{2(M-1)}^{(r)} \\ b_{32}^{(r)} & b_{33}^{(r)} & \cdots & b_{3(M-1)}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(M-1)2}^{(r)} & b_{(M-1)3}^{(r)} & \cdots & b_{(M-1)(M-1)}^{(r)} \end{bmatrix}.$$

The discretized CBEs (29) are converted into

$$\begin{cases} \dfrac{du_{ijk}}{dt} = -U_{ijk}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}u_{pjk} - V_{ijk}\sum_{p=2}^{N-1} b_{jp}^{(1)}u_{ipk} - W_{ijk}\sum_{p=2}^{L-1} c_{kp}^{(1)}u_{ijp} + \\ \qquad \upsilon\left[\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(2)}u_{pjk} + \sum_{p=2}^{N-1} b_{jp}^{(2)}u_{ipk} + \sum_{p=2}^{L-1} c_{kp}^{(2)}u_{ijp}\right] + F_{ijk}, \\ \dfrac{dv_{ijk}}{dt} = -U_{ijk}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}v_{pjk} - V_{ijk}\sum_{p=2}^{N-1} b_{jp}^{(1)}v_{ipk} - W_{ijk}\sum_{p=2}^{L-1} c_{kp}^{(1)}v_{ijp} + \\ \qquad \upsilon\left[\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(2)}v_{pjk} + \sum_{p=2}^{N-1} b_{jp}^{(2)}v_{ipk} + \sum_{p=2}^{L-1} c_{kp}^{(2)}v_{ijp}\right] + G_{ijk}, \\ \dfrac{dw_{ijk}}{dt} = -U_{ijk}\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(1)}w_{pjk} - V_{ijk}\sum_{p=2}^{N-1} b_{jp}^{(1)}w_{ipk} - W_{ijk}\sum_{p=2}^{L-1} c_{kp}^{(1)}w_{ijp} + \\ \qquad \upsilon\left[\displaystyle\sum_{p=2}^{M-1} a_{ip}^{(2)}w_{pjk} + \sum_{p=2}^{N-1} b_{jp}^{(2)}w_{ipk} + \sum_{p=2}^{L-1} c_{kp}^{(2)}w_{ijp}\right] + H_{ijk}. \end{cases} \tag{33}$$

The system of ODEs (33) can also be represented as

$$\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} A & O & O \\ O & B & O \\ O & O & C \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} F \\ G \\ H \end{bmatrix}, \tag{34}$$

where

(i) The $O's$ represents null matrices.

(ii) The vector $(F,G,H)^T$ contains non-homogeneous parts and B.C.'s.

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(iii) $$\begin{cases} A = -U_{ijk}A_1 - V_{ijk}A_2 - W_{ijk}A_3 + \frac{1}{\mathrm{Re}}A_4 + \frac{1}{\mathrm{Re}}A_5 + \frac{1}{\mathrm{Re}}A_6, \\ B = -U_{ijk}B_1 - V_{ijk}B_2 - W_{ijk}B_3 + \frac{1}{\mathrm{Re}}B_4 + \frac{1}{\mathrm{Re}}B_5 + \frac{1}{\mathrm{Re}}B_6, \\ C = -U_{ijk}C_1 - V_{ijk}C_2 - W_{ijk}C_3 + \frac{1}{\mathrm{Re}}C_4 + \frac{1}{\mathrm{Re}}C_5 + \frac{1}{\mathrm{Re}}C_6, \end{cases}$$

where A_i's ($i=1$ to 6) are $(M-2)(N-2)(L-2)$ order square matrices as given below:

$$A_1 = \begin{bmatrix} a_{22}^{(1)}I_x & a_{23}^{(1)}I_x & \cdots & a_{2(M-1)}^{(1)}I_x \\ a_{32}^{(1)}I_x & a_{33}^{(1)}I_x & \cdots & c_{3(L-1)}^{(2)}I_x \\ \vdots & \vdots & \ddots & \vdots \\ a_{(M-1)2}^{(1)}I_x & a_{(M-1)3}^{(1)}I_x & \cdots & a_{(M-1)(M-1)}^{(1)}I_x \end{bmatrix}, \quad A_2 = \begin{bmatrix} R_y & O & \cdots & O \\ O & R_y & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & R_y \end{bmatrix},$$

where

$$R_y = \begin{bmatrix} b_{22}^{(1)}I_y & b_{23}^{(1)}I_y & \cdots & b_{2(N-1)}^{(1)}I_y \\ b_{32}^{(1)}I_y & b_{33}^{(1)}I_y & \cdots & b_{3(L-1)}^{(2)}I_y \\ \vdots & \vdots & \ddots & \vdots \\ b_{(N-1)2}^{(1)}I_y & b_{(N-1)3}^{(1)}I_y & \cdots & b_{(N-1)(N-1)}^{(1)}I_y \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} R_z & O & \cdots & O \\ O & R_z & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & R_z \end{bmatrix}, \text{ where } R_z = \begin{bmatrix} b_{22}^{(1)} & b_{23}^{(1)} & \cdots & b_{2(L-1)}^{(1)} \\ b_{32}^{(1)} & b_{33}^{(1)} & \cdots & b_{3(L-1)}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(L-1)2}^{(1)} & b_{(L-1)3}^{(1)} & \cdots & b_{(L-1)(L-1)}^{(1)} \end{bmatrix}.$$

The matrices A_4, A_5, A_6 are similar to the matrices A_1, A_2, A_3 respectively. Similarly B_i and C_i ($i=1$ to 6) are defined. For stability analysis, the Eigenvalues (E.V.) of matrices are calculated and shown in Figure 1.

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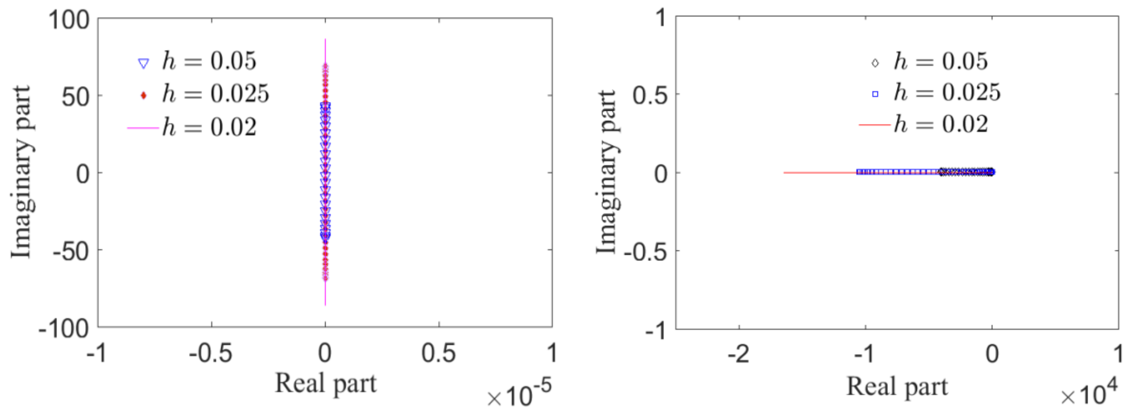

(a) (b)

Figure 1. Eigenvalues of (a) matrix A_1 and (b) matrix A_2

As the E.V. of $-(U_{ij}A_1 + V_{ij}B_1)$ are imaginary and of $(A_2 + B_2)$ are real and negative. So, we get negative real parts (r.p.) of all E.V. of A. Also, the matrices A_1', B_1', A_2', B_2' and A_1, B_1, A_2, B_2 are same. Therefore, the r.p. of all E.V. of the coefficient matrix of the system (32) are negative. Similarly, we can obtain the E.V. of the coefficient matrix of the system (34). As all the E.V. of the coefficient matrix of the system (32) satisfy the conditions of stability, hence the present method is unconditionally stable for CBEs.

6. Numerical Discussion

We consider three examples of CBEs in two and three dimensions. The analysis of errors is done by L_2 and L_∞ error norms, defined below:

$$L_2 = \sqrt{h\sum_{i=1}^{n} |U_i - u_i|^2},\ \mathrm{L}_\infty = \max_i |U_i - u_i|.$$

Example 1. Take the CBEs (1) in $[0,1]\times[0,1]$ with the exact solution

$$\begin{cases} u = 0.75 - \dfrac{1}{4\left[1+\exp\left(\left(-t-4x+4y\right)\mathrm{Re}\right)\right]}, \\ v = 0.75 + \dfrac{1}{4\left[1+\exp\left((1/32)\left(-t-4x+4y\right)\mathrm{Re}\right)\right]}. \end{cases} \tag{35}$$

The solution of Example 1 is evaluated with the parameters: $\Delta t = 0.0001$, $\upsilon = 10^{-2}$ at $t = 1.0$ for various grids and given in Tables 1 and 2 . The "rate of convergence" (ROC) is also calculated. As we can see from Tables 1 and 2 that the present method achieves superior results than obtained by Expo-FDM (Srivastava et al., 2013c) and almost similar results obtained by MCB-DQM (Shukla et al., 2014). Also we found that the ROC of the method is more than

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quadratic. Figure 2 shows the solutions (numerical and exact) comparison for $\upsilon = 10^{-2}$, $\Delta t =$ 0.0001 at $t = 1.0$.

Table 1. Error norms and ROC for component u

Grids	L_2			L_∞		
	Expo-FDM	Present method		Expo-FDM	Present method	
			ROC			ROC
4×4	8.570e-02	1.645e-02	-	9.704e-02	2.895e-03	-
8×8	4.942e-02	1.932e-03	3.09	4.688e-02	1.964e-04	3.88
16×16	1.919e-02	3.950e-04	2.29	2.046e-02	2.050e-05	3.26
32×32	8.681e-03	8.122e-05	2.28	9.074e-03	2.221e-06	3.20
64×64	-	1.540e-05	2.40	-	2.187e-07	3.34

Table 2. Error norms and ROC for component v

Grids	L_2			L_∞		
	Expo-FDM	Present method		Expo-FDM	Present method	
			ROC			ROC
4×4	8.570e-02	1.645e-02	-	9.704e-02	2.895e-03	-
8×8	4.943e-02	1.932e-03	3.09	4.688e-02	1.964e-04	3.88
16×16	1.919e-02	3.950e-04	2.29	2.047e-02	2.050e-05	3.26
32×32	8.687e-03	8.122e-05	2.28	9.081e-03	2.221e-06	3.20
64×64	-	1.540e-05	2.40	-	2.187e-07	3.34

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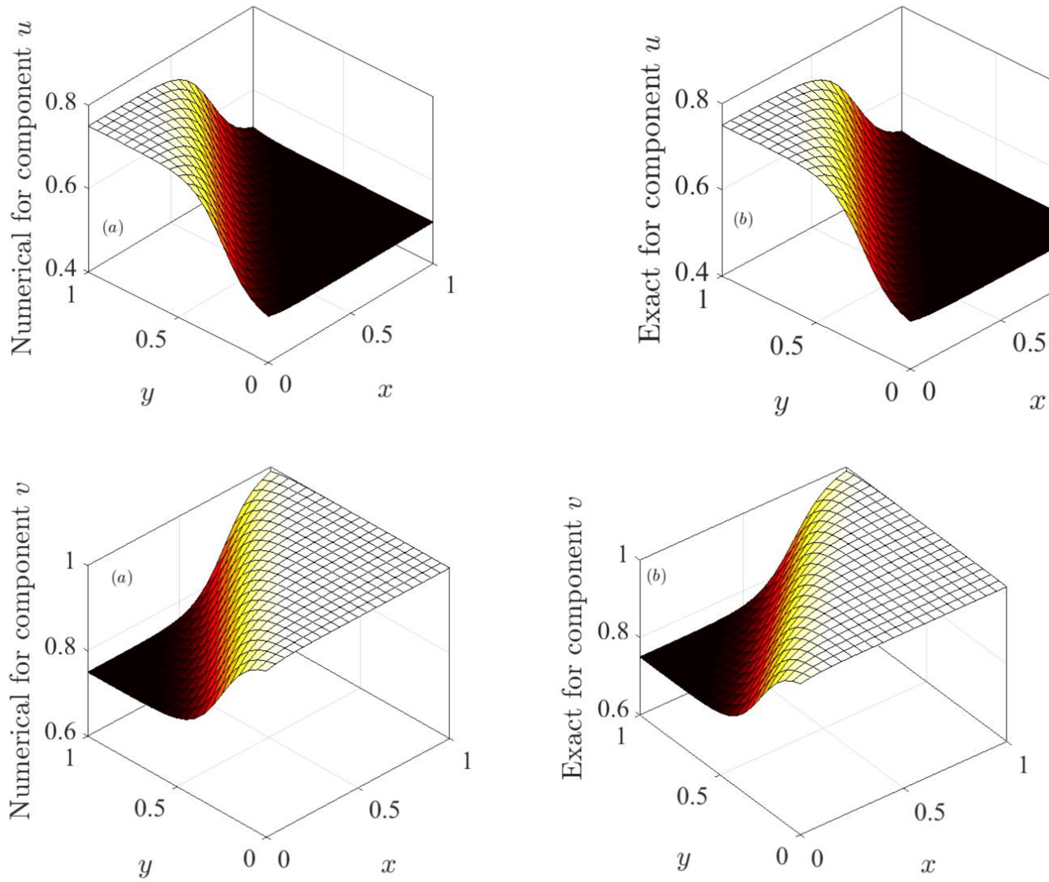

Figure 2. The solutions comparison of Example 1 for u and v respectively

Example 2. Now we consider CBEs (1) in the domain $[0, 0.5]\times[0, 0.5]$ with I.C.'s
$u = \sin \pi x + \cos \pi y,\ v = x + y$ at $t = 0$
and B.C.'s

$$\begin{cases} u = \cos(\pi y),\ v = y \text{ at } x = 0 \\ u = 1+\sin(\pi x),\ v = x \text{ at } y = 0 \\ u = 1+\cos(\pi y),\ v = 0.5+y \text{ at } x = 0.5 \\ u = \sin(\pi x),\ v = 0.5+x \text{ at } y = 0.5 \end{cases}. \tag{36}$$

The solutions of Example 2 are shown in Table 3 as a comparison with the solutions obtained by Expo-FDM (Srivastava et al., 2013c) and MCB-DQM (Shukla et al., 2014) for $\upsilon = 0.02$, grids 20×20 and $\Delta t = 0.0001$ at $t = 0.625$. We found a closed agreement between the results. The physical behaviours of solutions are represented in Figure 3 for $\upsilon = 0.01$, $\Delta t = 0.0001$ and $h = 0.05$ at time $t = 1$.

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Table 3. Comparison of the solutions at $t = 0.625$

Grid	u			v		
	Expo-FDM	MCB-DQM	Present	Expo-FDM	MCB-DQM	Present
(0.1, 0.1)	0.9715	0.9706	0.9705	0.0987	0.0984	0.0984
(0.3, 0.1)	1.1528	1.1515	1.1515	0.1416	0.1410	0.1411
(0.2, 0.2)	0.8631	0.8624	0.8624	0.1675	0.1673	0.1673
(0.4, 0.2)	0.9798	0.9807	0.9808	0.1711	0.1722	0.1722
(0.1, 0.3)	0.6632	0.6633	0.6634	0.2638	0.2638	0.2638
(0.3, 0.3)	0.7723	0.7722	0.7723	0.2265	0.2265	0.2265
(0.2, 0.4)	0.5818	0.5827	0.5827	0.3285	0.3293	0.3294
(0.4, 0.4)	0.7586	0.7617	0.7618	0.3250	0.3288	0.3288

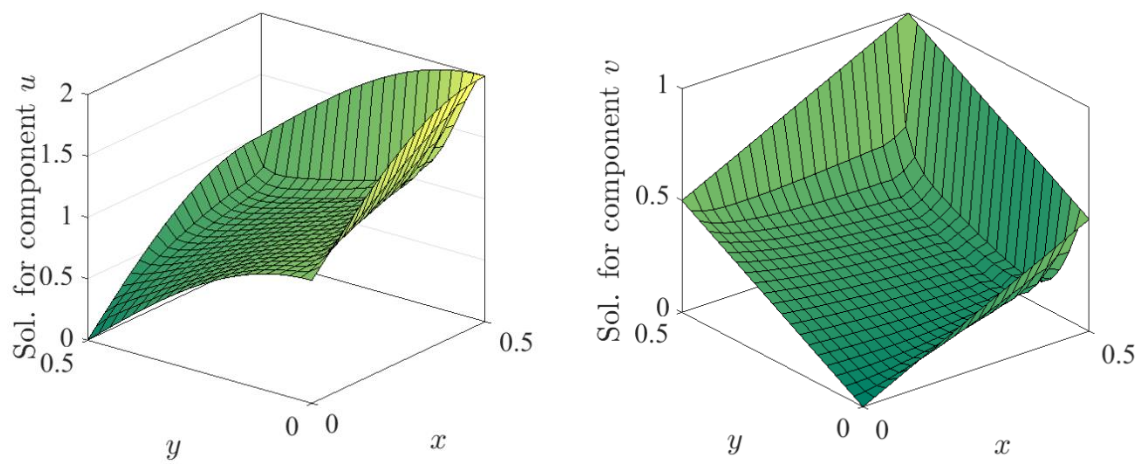

Figure 3. The physical behaviour of the numerical solutions for Example 2 at t=1

Example 3: Finally take the CBEs (4) with the exact solution

$$u = -\frac{2}{\mathrm{Re}}\left(f_x / f\right),\ v = -\frac{2}{\mathrm{Re}}\left(f_y / f\right),\ w = -\frac{2}{\mathrm{Re}}\left(f_z / f\right), \tag{37}$$

where $f = e^{-t}\sin x\ \sin y\ \sin z + x + 1.$

Table 4 shows the L_∞ and L_2 errors for $h = 0.04$, $\Delta t = 0.01$, $t = 1.0$ at different Re, where $h = \Delta x = \Delta y = \Delta z$. The obtained error norms are small which shows that the method presented in this paper gives accurate results. Figure 4 shows contour plots of u, v and w for $\mathrm{Re} = 100$ with $z = 0.5$.

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Table 4. The L_∞ and L_2 error norms

Re	Error norms	u	v	w
10^2	L_∞	2.1903e-06	3.5236e-05	3.5236e-05
	L_2	7.8066e-04	5.4508e-03	5.4508e-03
10^3	L_∞	3.4657e-07	7.0278e-07	7.0278e-07
	L_2	1.1998e-04	1.5859e-04	1.5859e-04
10^4	L_∞	3.8335e-08	4.2985e-08	4.2985e-08
	L_2	1.3185e-05	1.3894e-05	1.3894e-05

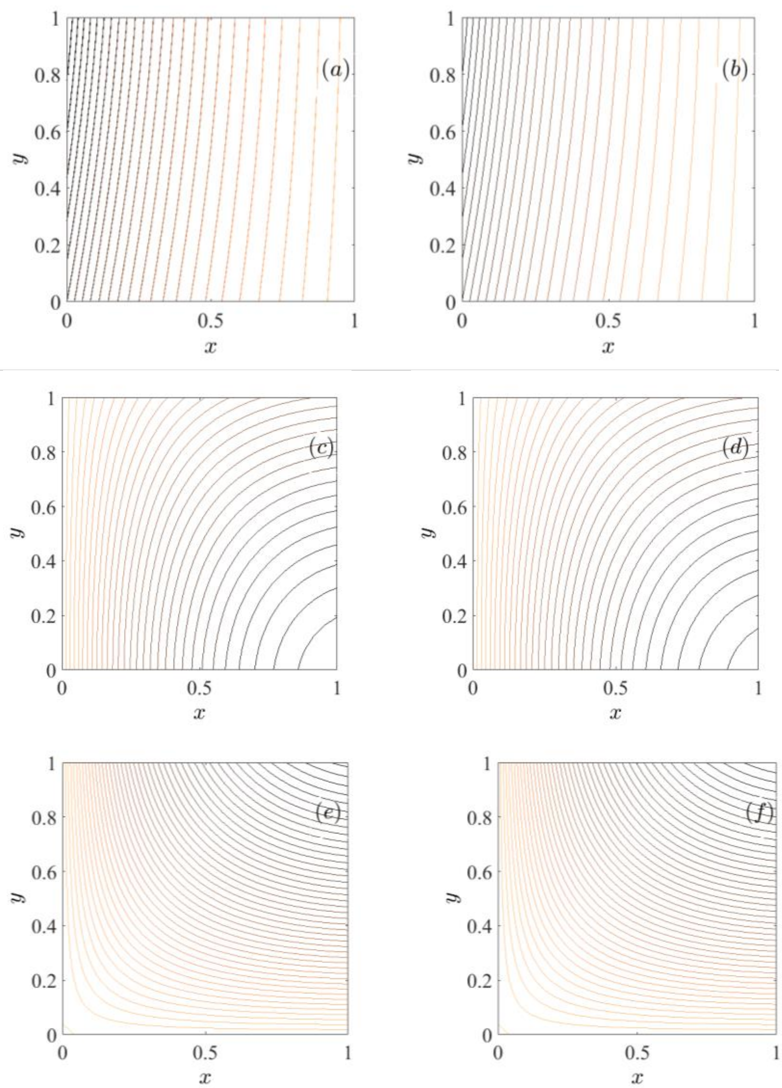

Figure 4. Contour plots (a), (c) and (e) numerical (b), (d) and (f) exact for u , v and w respectively with $z = 0.5$ for $\mathrm{Re} = 100$

7. Conclusion

In this work, the CBEs in two and three dimensions are solved via DQM based on modified form of TCB shape functions. The modified TCB shape functions are used for integration of space derivatives. Thus, the CBEs are transformed into the form of integral equations. These integral equations are solved by SSP-RK54 method. The analysis of the method is done through three examples. The different error norms are obtained and compared with the error norms of Expo-FDM (Srivastava et al., 2013c) and MCB-DQM (Shukla et al., 2014) for two dimensional problems. We found a good quality conformity between the solutions obtained by present and aforesaid methods. The ROC and stability analysis show that the method is quadratic convergent and stable unconditionally for CBEs.

Conflict of Interest

The authors confirm that there is no conflict of interest in the article contents.

Acknowledgement

The authors would like to express their sincere thanks to the referees for their precious suggestions.

References

Arora, G., & Joshi, V. (2018). A computational approach using modified trigonometric cubic B-spline for numerical solution of Burgers' equation in one and two dimensions. *Alexandria Engineering Journal, 57*(2), 1087-1098.

Bahadir, A.R. (2003). A fully implicit finite-difference scheme for two-dimensional Burgers' equation. *Applied Mathematics and Computation, 137*(1), 131-137.

Baishya, C. (2019). A new application of Hermite collocation method. *International Journal of Mathematical, Engineering and Management Sciences, 4*(1), 182-190.

Bateman, H. (1915). Some recent researches on the motion of fluids. *Monthly Weather Review, 43,* 163-170.

Bellman, R., Kashef, B.G., & Casti, J. (1972). Differential quadrature: a technique for the rapid solution of nonlinear differential equations. *Journal of Computational Physics, 10*(1), 40-52.

Burgers, J.M. (1948). A mathematical model illustrating the theory of turbulence. *Advances in Applied Mechanics, 1*, 171-199.

Chauhan, V., & Srivastava, P.K. (2019). Computational techniques based on Runge-Kutta method of various order and type for solving differential equations. *International Journal of Mathematical, Engineering and Management Sciences, 4*(2), 375-386.

Fletcher, C.A. (1983). Generating exact solutions of the two-dimensional Burgers' equations. *International Journal for Numerical Methods in Fluids, 3*(3), 213-216.

Gottlieb, S., Ketcheson, D.I., & Shu, C.W. (2009). High order strong stability preserving time discretizations. *Journal of Scientific Computing, 38*(3), 251-289.

Liu, F., & Shi, W. (2011). Numerical solutions of two-dimensional Burgers' equations by lattice Boltzmann method. *Communications in Nonlinear Science and Numerical Simulation, 16*(1), 150-157.

Radwan, S.F. (1999). On the fourth-order accurate compact ADI scheme for solving the unsteady nonlinear coupled Burgers' equations. *Journal Nonlinear Mathematical Physics*, *6*(1) 13-34.

Saka, B., Sahin, A., & Dağ, İ. (2009). B-spline collocation algorithms for numerical solutions of the RLW equation. *Numerical Methods for Partial Differential Equations, 27*(3), 581-607.

Shu, C., (2000). *Differential quadrature and its application in engineering*. Springer Verlag, London.

Shukla, H.S., Tamsir, M., Srivastava, V.K., & Kumar, J. (2014). Numerical solution of two dimensional coupled viscous Burger equation using modified cubic B-spline differential quadrature method. *AIP Advances*, *4*(11), 117134.

Shukla, H.S., Tamsir, M., Srivastava, V.K., & Rashidi, M.M. (2016). Modified cubic B-spline differential quadrature method for numerical solution of three-dimensional coupled viscous Burger equation. *Modern Physics Letters B*, *30*(11), 1650110.

Srivastava, V.K., Ashutosh, & Tamsir, M. (2013a). Generating exact solution of three-dimensional coupled unsteady nonlinear generalized viscous Burgers' equations. *International Journal of Modern Mathematical Sciences, 5*(1), 1-13.

Srivastava, V.K., Awasthi, M.K., & Singh, S. (2013b). An implicit logarithm finite difference technique for two dimensional coupled viscous Burgers' equation. *AIP Advances, 3*(12), 1-9.

Srivastava, V.K., Singh, S., & Awasthi, M.K. (2013c). Numerical solutions of coupled Burgers' equations by an implicit finite-difference scheme. *AIP Advances, 3*(8), 1-7.

Vaid, M.K., & Arora, G. (2019). Solution of second order singular perturbed delay differential equation using trigonometric B-spline. *International Journal of Mathematical, Engineering and Management Sciences*, *4*(2), 349-360.

Zhao, G., Yu, X., & Zhang, R. (2011). The new numerical method for solving the system of two-dimensional Burgers' equations. *Computers & Mathematics with Applications, 62*(8), 3279-3291.

Zhu, H., Shu, H., & Ding, M. (2010). Numerical solutions of two-dimensional Burgers' equations by discrete Adomian decomposition method, *Computers & Mathematics with Applications*, *60*(3), 840-848.