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Alternative Transient and Steady State Analysis of Some Gaver's Parallel Systems

Ramaz Khurodze
Georgian National Academy of Sciences,
Rustaveli Avenue 52, Tbilisi, Georgia.
E-mail: khurodzeramaz@gtu.ge

Revaz Kakubava
Faculty of Informatics and Control Systems,
Georgian Technical University, Kostava Street 77, Tbilisi, Georgia.
Corresponding author: r.kakubava@gmail.com, r.kakubava@gtu.ge

Murman Kublashvili
Muskhelishvili Institute of Computational Mathematics,
4 Grigol Peradze Street, Tbilisi, Georgia.
E-mail: murman.kublashvili@gtu.ge

Teimuraz Saghinadze
Muskhelishvili Institute of Computational Mathematics,
4 Grigol Peradze Street, Tbilisi, Georgia.
E-mail: saghinadze.teimuraz@gtu.ge

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Abstract
This paper presents one of the most interesting generalizations of Gaver's basic two-unit parallel system sustained by a cold standby unit and attended by a repairman with multiple vacations. The system was studied using the supplementary variables technique, like many other similar semi-Markov systems. D.P. Gaver, Jr. was the first to apply this method for constructing and studying reliability models, and since then it has been then widely used by other researchers to study various reliability problems. As a result, non-classical boundary value problem of mathematical physics with nonlocal boundary conditions has been obtained. Until now, a solution to this problem was obtained in terms of Laplace transforms. Naturally, the most significant part of the problem is a system of partial differential equations (Kolmogorov forward equations). In this study, we demonstrate that Kolmogorov equations are redundant and we can solve the problem by avoiding the necessity of using them. We present here a novel, purely probabilistic approach. The results are formulated as rigorous mathematical statements, offering a significant simplification in the reliability analysis of stochastic systems. Our findings show that this novel approach can be applied to study both semi-Markov and some non semi-Markov models where the supplementary variables technique is used.

Keywords- Semi-Markov process, Probabilistic reasoning, Laplace transform, Failure, Repair.

1. Introduction

This paper has two main purposes: first, to demonstrate the advantages of our novel approach to a well-known reliability problem and second, to draw the attention of leading experts to the existence of this surprisingly simple and highly effective method for studying semi-Markov (SM) processes on the whole.

It is generally acknowledged that semi-Markov models, where the supplementary variables technique (SVT) is used, result in non-classical boundary value problem (BVP) of mathematical physics. These models always contain partial differential equation(s) (PDEs) (Kolmogorov forward or backward equation(s)) and non-local boundary conditions. Gaver's parallel systems are no exception to this rule. Traditionally, the solution to this problem is obtained in terms of Laplace transforms which is often a very complicated process. However, as shown in this paper, deep probabilistic reasoning of the considered systems resolves this problem.

This new probabilistic approach was initially used by Kakubava (2020) in the context of queuing theory (QT), followed by its applications to the Gaver's parallel systems (Khurodze et al., 2020; Khurodze et al., 2022; Khurodze et al., 2023; Khurodze et al., 2024). Continuing the analysis of Gaver's parallel systems, we further apply this method to a more complicated variant of the system, the one with cold standby and multiple vacations. This work aims to show, once again, the efficiency and conceptual simplicity of a novel probabilistic method that will undoubtedly be useful to practitioners in solving the applied problems they encounter. Our approach reveals the inherent probabilistic nature of the SM process.

Recall that outstanding experts in reliability theory in their works figuratively call SM process a marriage of renewal theory and Markov chain theory. Without a thorough understanding of the systems in probabilistic terms, it is impossible to discern their nature simply by studying the BVP. As this approach borrows ideas from the supplementary variable's technique, it is appropriate to discuss the origin and application of this method.

The SVT in reliability was initially applied by D.P. Gaver, Jr., in constructing and studying the reliability models (Gaver, 1963). This concept was then widely used by other researchers to investigate various reliability problems. For more than 60 years, the SVT has been extensively used for SM systems in mathematical theory of reliability (MTR) and QT. As a result, the problem is reduced to the study of a two-dimensional Markov process. Given the Markov process, it was natural for Cox (1955) and other researchers to write down the Kolmogorov equations (forward or backward) and then obtain a non-classical BVP of mathematical physics. Many high-class specialists apply this technique to solve these problems. Among them, Gaver stands out for his excellence in both reliability and queuing. Since its implementation, this technique has become one of the primary tools for MTR analysis. We highly appreciate Gaver's fundamental work and expect that by using our unique probabilistic method to achieve a transient solution of the system, we can simplify the derivation of his results and those of obtained here and clarify their meaning. This work captured the interest of researchers in MTR. Only a few of the numerous research papers that drew inspiration from the Gaver's original parallel system are mentioned here: Dhillon and Anude (1993), Haji and Yunus (2015), Vanderperre and Makhanov (2002, 2018, 2019, 2020), Yue et al. (2006). They attest to this influence. The study of repairable systems is a topic of primary importance in MTR. Gaver's parallel system serves as a classic example of such a system that clarifies lots of practical issues.

In this paper, we adopt terms and definitions from the following sources: Haji and Yunus (2015), Yue et al. (2006) and for convenience, introduce several new notations. Due to the strong practical background of the Gaver's parallel system, many researchers have thoroughly studied it under various failure and repair assumptions (see Dhillon and Anude, 1993; Gupur, 2011; Vanderperre and Makhanov, 2002; 2018; 2019; Yue et al., 2006). Frequently it is taken into account that the repairman leaves for a vacation or performs other work when there are no malfunctioning units in the system to be repaired, which can significantly affect the performance of a system.

In Yue et al. (2006), the authors studied Gaver's parallel system attended by a cold standby unit and a repairman with multiple vacations. They derived several reliability expressions, such as the Laplace transform of the reliability, mean time to the first failure, availability and failure rate of the system. The authors use a dynamic solution to calculate availability and reliability; the existence and well-posedness of a positive dynamic solution are not explored.

In Haji and Yunus (2015), the authors study the well-posedness and existence of a unique positive dynamic solution of the system using C_0-semigroup theory of linear operators. For background reading on semigroup theory, the reference is done to Greiner (1987), Engel and Nagel (2000), Casarino et al. (2003). They first formulate the system model as an abstract Cauchy problem within a Banach space, then show that the system operator generates a positive contraction of the C_0-semi group, and finally prove the system's well-posedness and the existence of a unique positive dynamic solution.

Our novel method has significantly changed the mathematical statement of the problem by substituting (R) with Equations (2)-(8). This study again raises the question of well-posedness, but in a different situation. Currently, the problem has not yet been resolved.

The rest of this paper is organized as follows: in Section 2, based on the studies by Haji and Yumus (2015) and Yue et al. (2006), we consider Gaver's parallel system attended by a cold standby unit and a repairman with multiple vacations. All equations in the section are taken from Haji and Yunus (2015) and Yue et al. (2006) without modification.

The main body of our research is presented in Section 3. Here, we demonstrate the needlessness of the set (R) of PDEs, which is the most important component of the mathematical model described in Section 2. There, simple theorems are proved that allow us to express the unknown functions by BCs and find a solution to the problem without PDEs. In Section 4, steady state explicit expressions for unknown functions in closed form are obtained.

Section 5 contains our comments and discussion of the results. Finally, in Section 6 we provide a brief conclusion and discuss the implications of the current study that may be interesting to consider in the future.

2. Description of the Considered System and Mathematical Modeling

Recall that, in general, redundant units can operate actively in parallel and hence be susceptible to failure. Alternatively, they can function as spares to be used sequentially to replace failed units; in this case, the redundant units are not susceptible to failure being in a standby condition. The first type of redundancy is sometimes called parallel redundancy, while the second one is referred to as standby redundancy, or cold standby in Haji and Yunus (2015), Vanderperre and Makhanov (2002, 2018, 2019, 2020), Yue et al. (2006).

In this section, following Haji and Yunus (2015) and Yue et al. (2006), we consider Gaver's parallel system attended by a cold standby unit and a repairman with multiple vacations. Only minor modifications have been made to the original notations to bring them in line with modern style. Each parallel unit fails at a constant rate λ, and the cold standby unit does not fail.

Let σ denote a random variable representing the repair time, with G and g being its cumulative distribution and probability density functions, respectively $g(x) = G'(x)$. Hence, the repair rate function can be expressed as $\mu(x) = g(x)/\big(1 - G(x)\big)$.

Also, let δ denote a random variable representing vacation time, with V and v being its cumulative distribution and probability density functions, respectively $v(x) = V'(x)$. Hence, vacation rate function $\alpha(x) = v(x)/\big(1 - V(x)\big)$.

$p_0(x,t)h+o(h)$ is the probability that two units at time t are operating, one unit is under standby, the repairman is in vacation, the system is good, the elapsed vacation time lies in $[x,x+h)$;

$p_1(x,t)h+o(h)$: the probability that at time t two units are operating, one unit is waiting for repair, the repairman is in vacation, the system is good, the elapsed vacation time lies in $[x,x+h)$;

$p_2(x,t)h+o(h)$: the probability that at time t one unit is operating, two units are waiting for repair, the repairman is in vacation, the system is good, the elapsed vacation time lies in $[x,x+h)$;

$p_3(x,t)h+o(h)$: the probability that at time t two units are operating, one unit being repaired, the system is good, the elapsed repair time $[x, x+h)$;

$p_4(x,t)h+o(h)$: the probability that at time t one unit is operating, one unit being repaired, one unit is waiting for repair, the system is good, the elapsed repair time $[x, x+h)$;

$p_5(x,t)h+o(h)$ represents the probability that at time t three units are waiting for repair, the repairman is in vacation, the system is down, the elapsed vacation time lies in $[x, x+h)$;

$p_6(x,t)h+o(h)$: the probability that at time t one unit being repaired, two units are waiting for repair, the system is down, the elapsed repair time lies in $[x, x+h)$.

The following set of equations describes the above system developed in Haji and Yunus (2015), Yue et al. (2006),

$$(R)\begin{cases}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_o(t,x)=-[2\lambda+\alpha(x)]p_o(t,x),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_1(t,x)=2\lambda p_o(t,x)-[2\lambda+\alpha(x)]p_1(t,x),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_2(t,x)=2\lambda p_1(t,x)-[\lambda+\alpha(x)]p_2(t,x),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_3(t,x)=-[2\lambda+\mu(x)]p_3(t,x),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_4(t,x)=2\lambda p_3(t,x)-[\lambda+\mu(x)]p_4(t,x),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_5(t,x)=\lambda p_2(t,x)-\alpha(x)p_5(t,x),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)p_6(t,x)=\lambda p_4(t,x)-\mu(x)p_6(t,x),\end{cases}$$

with the boundary conditions

$$(\mathrm{BC})\begin{cases}p_o(t,0)=\int_0^t\alpha(x)\,p_o(t,x)dx+\int_0^t\mu(x)\,p_3(t,x)dx+\delta(t),\\ p_1(t,0)=p_2(t,0)=p_5(t,0)=0,\\ p_3(t,0)=\int_0^t\alpha(x)p_1(t,x)dx+\int_0^t\mu(x)p_4(t,x)dx,\\ p_4(t,0)=\int_0^t\alpha(x)p_2(t,x)dx+\int_0^t\mu(x)p_6(t,x)dx,\\ p_6(t,0)=\int_0^t\alpha(x)\,p_5(t,x)dx,\end{cases},$$

and the initial conditions

$$(IC)\begin{cases} p_o(0,x) = \delta(x), \\ p_1(0,x) = 0, i = 1,2,3,4,5,6, \end{cases} \text{where } \delta(x) = \begin{cases} 1, x = 0, \\ 0, x \neq 0. \end{cases}$$

3. New Alternative Solution / A Modified Supplementary Variable Method

We need to remember a few simple facts about the birth process. In particular, the solution to the following simple pure birth process is needed to describe the probabilities we use in the subsequent proofs.

$$\begin{cases} P'(i,0,t) = -2\lambda P(i,0,t) \\ P'(i,1,t) = -2\lambda P(i,1,t) + 2\lambda P(i,0,t) \\ P'(i,2,t) = -\lambda P(i,2,t) + 2\lambda P(i,1,t) \\ P'(i,3,t) = \lambda P(i,2,t) \end{cases} \quad (1)$$

Solutions to the system with different initial conditions are presented below. Here, $i \in \{1,2,3\}$ corresponds to the number of failed units at time $t = 0$. Hence, one can read $P(i,j,t)$ as probability of having $j (j \in \{0,1,2,3\})$ failed units at time t, given that at $t = 0$ the system started off with i failed units.

Initial conditions (1):
$P(0,0,0) = 1; P(0,1,0) = 0; P(0,2,0) = 0; P(0,3,0) = 0.$

Solution 1:

$$\begin{cases} P(0,0,t) = \exp(-2\lambda t) \\ P(0,1,t) = 2\lambda t \exp(-2\lambda t) \\ P(0,2,t) = 4\exp(-\lambda t)\,(1 - (1 + \lambda t)\exp(-\lambda t)) \\ P(0,3,t) = 1 - 4\exp(-\lambda t) + (3 + 2\lambda t)\exp(-2\lambda t). \end{cases}$$

Initial conditions (2):
$P(1,0,0) = 0; P(1,1,0) = 1; P(1,2,0) = 0; P(1,3,0) = 0.$

Solution 2:

$$\begin{cases} P(2,0,t) = 0 \\ P(2,1,t) = 0 \\ P(2,2,t) = \exp(-\lambda t) \\ P(2,3,t) = 1 - \exp(-\lambda t). \end{cases}$$

Initial conditions (3):
$P(3,0,0) = 0; P(3,1,0) = 0; P(3,2,0) = 0; P(3,3,0) = 1.$

Solution 3:

$$\begin{cases} P(3,0,t) = 0 \\ P(3,1,t) = 0 \\ P(3,2,t) = 0 \\ P(3,3,t) = 1. \end{cases}$$

If we apply new probabilistic method as presented in Kakubava (2020), Khurodze et al. (2020), Khurodze et al. (2022), Khurodze et al. (2023), Khurodze et al. (2024) we get,

$$p_0(t,x) = \big(1 - V(x)\big)\exp(-2\lambda x)\, p_0(t - x, 0) \quad (2)$$

$$p_1(t,x) = \big(1 - V(x)\big)(2\lambda x \exp(-2\lambda x))p_0(t-x,0) \tag{3}$$

$$p_2(t,x) = \big(1 - V(x)\big) \cdot \big(4\exp(-\lambda x)\,(1-(1+\lambda x)\exp(-\lambda x))\big)p_0(t-x,0) \tag{4}$$

$$p_3(t,x) = \big(1 - G(x)\big)\exp(-2\lambda x)\,p_3(t-x,0) \tag{5}$$

$$p_4(t,x) = \big(1 - G(x)\big)(2\exp(-\lambda x)\,(1-\exp(-\lambda x)) \cdot p_3(t-x,0) + \exp(-\lambda x)\,p_4(t-x,0) \tag{6}$$

$$p_5(t,x) = \big(1 - V(x)\big) \cdot (1 - 4\exp(-\lambda x) + (3+2\lambda x)\exp(-2\lambda x))p_0(t-x,0) \tag{7}$$

$$p_6(t,x) = \big(1 - G(x)\big)((1-\exp(-2\lambda x))^2 p_3(t-x,0) + (1-\exp(-\lambda x))p_4(t-x,0) + p_6(t-x,0)) \tag{8}$$

where, $\xi(t)$ is elapsed repair time from the start of the repair to the time instant t; $\eta(t)$ is elapsed vacation time from the start of the vacation to the time epoch t; $A_0(t) = \{$at the time epoch t, the system is in the state $0\}$. $A_3(t) = \{$at the time epoch t, the system is in the state $3\}$. $B(x) = \{$for time interval of length x none of two parallel units fail$\}$.
$C(x) = \{\delta > x\}$,

$D(x) = \{\sigma > x\}$,

$A_0(t,x,h) = \{A_0(t);\ x < \eta(t) < x + h\};\ B(x);\ D(x)\}$,

$A_3(t,x,h) = \{A_3(t);\ x < \xi(t) < x + h\};\ B(x);\ C(x)\}$.

We discuss here how to obtain the expressions (2)-(8). To illustrate our approach, let us consider expressions (2) and (5).

Theorem 1. $p_o(t,x)$ can be expressed as
$p_0(t,x) = \big(1 - V(x)\big)e^{-2\lambda x}p_0(t-x,0)$.

Proof:
It is easy to understand that
$A_0(t,x,h) = \{A_0(t);\ x < \eta(t) < x + h\};\ B(x);\ D(x)\} = p_0(t,x) + o(h)$

$A_0(t,x,h) = A_0(t-x) \cap B(x) \cap D(x)$.

All three events on the right-hand side are independent, hence
$p_0(t,x)h + o(h) = \big(1 - V(x)\big)e^{-2\lambda x}p_0(t-x,0)h + \mathrm{o}(h)$.

Dividing both sides by h and taking the limit $h \to 0$, we obtain Equation (2).

Theorem 2. $p_3(t,x)$ can be expressed as:
$p_3(t,x) = \big(1 - G(x)\big)e^{-2\lambda x}p_3(t-x,0)$

Proof:
It is easy to understand that
$A_3(t,x,h) = \{A_3(t);\ x < \xi(t) < x + h\};\ B(x);\ C(x)\} = p_3(t,x) + o(h)$

$A_3(t,x,h) = A_3(t-x) \cap B(x) \cap C(x)$.
All three events on the righthand side are independent, hence
$p_3(t,x)h + o(h) = \big(1 - G(x)\big)\exp(-2\lambda x)\,p_0(t-x,0)h + o(h)$.

Dividing both sides by h and taking the limit as $h \to 0$, results in Equation (5).

The remaining equations in Equations (2)-(8) can be obtained in an analogous way.

On the other hand, one can simply check that each of these expressions satisfy partial differential equations presented in (R).

We denote Laplace transform of a function by placing a bar over it.

The system of linear equations that we get after applying Laplace transform to the BC is:

$$\begin{cases} \bar{p}_0(s,0) = \bar{p}_0(s,0)\bar{v}(s+2\lambda) + \bar{p}_3(s,0)\bar{g}(s+2\lambda) + 1 \\ \bar{p}_3(s,0) = -2\lambda\bar{p}_0(s,0)\bar{v}'(s+2\lambda) + \\ + 2\bar{p}_3(s,0)\big(\bar{g}(s+\lambda) - \bar{g}(s+2\lambda)\big) + \bar{p}_4(s,0)\bar{g}(s+\lambda) \\ \bar{p}_4(s,0) = 4p_0(s,0)\big(\bar{v}(s+\lambda) - \bar{v}(s+2\lambda) + \lambda\bar{v}'(s+2\lambda)\big) + \\ + 2\bar{p}_3(s,0)\big(\bar{g}(s) - 2\bar{g}(s+\lambda) + \bar{g}(s+2\lambda)\big) + \\ + \bar{p}_4(s,0)\big(\bar{g}(s) - \bar{g}(s+\lambda)\big) + \bar{p}_6(s,0)\bar{g}(s) \\ \bar{p}_6(s,0) = 4p_0(s,0)\big(\bar{v}(s) - 4\bar{v}(s+\lambda) + 3\bar{v}(s+2\lambda) - 2\lambda\bar{v}'^{(s+2\lambda)}\big). \end{cases}$$

One can solve the linear system for $\bar{p}_0(s,0), \bar{p}_1(s,0)$, $\bar{p}_2(s,0)$, $\bar{p}_3(s,0)$, $\bar{p}_4(s,0)$, $\bar{p}_5(s,0)$ and $\bar{p}_6(s,0)$. Resulting expressions are quite long, so we leave them implicit.

Taking the Laplace transform of both sides of Equations (2-8) is quite simple to perform. On the right hand side one can insert the expressions $\bar{p}_0(s,0)$,$\bar{p}_1(s,0)$, $\bar{p}_2(s,0)$, $\bar{p}_3(s,0)$, $\bar{p}_4(s,0)$, $\bar{p}_5(s,0)$ and $\bar{p}_6(s,0)$, and obtain explicit expressions for $\bar{p}_0(s,x)$,$\bar{p}_1(s,x)$, $\bar{p}_2(s,x)$, $\bar{p}_3(s,x)$, $\bar{p}_4(s,x)$, $\bar{p}_5(s,x)$ and $\bar{p}_6(s,x)$.

4. Steady State Analysis

Taking limits of Equations (2-8) $as\ t \to \infty$ results in following equations:

$$p_0(x) = \big(1 - V(x)\big)\exp(-2\lambda x)\, p_0(0) \tag{9}$$

$$p_1(x) = \big(1 - V(x)\big)(2\lambda x \exp(-2\lambda x))p_0(0) \tag{10}$$

$$p_2(x) = \big(1 - V(x)\big) \cdot \big(4\exp(-\lambda x)\,(1 - (1 + \lambda x)\exp(-\lambda x))\big)p_0(0) \tag{11}$$

$$p_3(x) = \big(1 - G(x)\big)\exp(-2\lambda x)\, p_3(0) \tag{12}$$

$$p_4(x) = \big(1 - G(x)\big)(2\exp(-\lambda x)\,(1 - \exp(-\lambda x)) \cdot p_3(0) + \exp(-\lambda x)\, p_4(0) \tag{13}$$

$$p_5(x) = \big(1 - V(x)\big) \cdot (1 - 4\exp(-\lambda x) + (3 + 2\lambda x)\exp(-2\lambda x))p_0(0) \tag{14}$$

$$p_6(x) = \big(1 - G(x)\big)((1 - \exp(-2\lambda x))^2 p_3(0) + (1 - \exp(-\lambda x))p_4(0) + p_6(0)) \tag{15}$$

where, $p_i(x) = \lim_{t\to\infty} p_i(t,x)$.

Taking limit of boundary conditions for $p_0(t,x), p_3(t,x), p_4(t,x)$ and $p_6(t,x)$ as $t \to \infty$ results in the following equations:

$$p_o(0) = \int_0^\infty \alpha(x)\, p_o(x)dx + \int_0^\infty \mu(x)\, p_3(x)dx \tag{16}$$

$$p_3(0) = \int_0^\infty \alpha(x)p_1(x)dx + \int_0^\infty \mu(x)p_4(x)dx \tag{17}$$

$$p_4(0) = \int_0^\infty \alpha(x)p_2(x)dx + \int_0^\infty \mu(x)p_6(x)dx \tag{18}$$

$$p_6(0) = \int_0^\infty \alpha(x)\, p_5(x)dx \tag{19}$$

Combining Equations (9)-(15) and (16)-(19) results in the following linear system of equations:

$$p_o(0) = a_1 p_0(0) + a_2 p_3(0) \tag{20}$$

$$p_3(0) = a_3 p_0(0) + a_4 p_3(0) + a_5 p_4(0) \tag{21}$$

$$p_4(0) = a_6 p_0(0) + a_7 p_3(0) + a_8 p_4(0) + p_6(0) \tag{22}$$

$$p_6(0) = a_9 p_0(0) \tag{23}$$

where,

$a_1 \coloneqq \int_0^\infty V'(x)e^{-2\lambda x}dx,$

$a_2 \coloneqq \int_0^\infty G'(x)e^{-2\lambda x}dx,$

$a_3 \coloneqq \int_0^\infty V'(x)2\lambda x e^{-2\lambda x}dx,$

$a_4 \coloneqq \int_0^\infty G'(x)2e^{-\lambda x}\left(1 - e^{-\lambda x}\right)dx,$

$a_5 \coloneqq \int_0^\infty G'(x)e^{-\lambda x}dx,$

$a_6 \coloneqq \int_0^\infty V'(x)4e^{-\lambda x}\left(1 - (1 + \lambda x)e^{-\lambda x}\right)dx,$

$a_7 \coloneqq \int_0^\infty G'(x)\left(1 - e^{-\lambda x}\right)^2 dx,$

$a_8 \coloneqq \int_0^\infty G'(x)\left(1 - e^{-\lambda x}\right)dx,$ and

$a_9 \coloneqq \int_0^\infty V'(x)\left(1 - 4e^{-\lambda x} + (3 + 2\lambda x)e^{-2\lambda x}\right)dx.$

Also, we have the following normalization condition:

$\sum_{i=0}^{6} \int_0^\infty p_i(x,t)dx = 1.$

Once again, taking limit $as\ t \to \infty$ results in

$\int_0^\infty \sum_{i=0}^{6} p_i(x)\, dx = 1.$

Using results of Equations (9)-(15), we get

$$p_0(0) \int_0^\infty \left(1 - V(x)\right)dx + \left(p_3(0) + p_4(0) + p_6(0)\right) \int_0^\infty \left(1 - G(x)\right)dx = 1 \tag{24}$$

The system of equations consisting of Equations (20)-(24) has a unique solution with respect to $p_0(0)$, $p_3(0)$, $p_4(0)$, and $p_6(0)$. After substituting these values back into Equations (9)-(15), the solution to the steady state system is obtained in an explicit and closed form.

Dear reader, please pay particular attention to this highly effective and extremely essential method presented in this work. Note that all semi-Markov models are reduced to a non-classical BVP such as ((R), (BC)).

To gain a better understanding of our approach and compare it with other methods currently in use, we recommend reviewing the references (Kakubava, 2020; Khurodze et al., 2020; Khurodze et al., 2022; Khurodze et al., 2023; Khurodze et al. 2024). You will find how simple and successful our method is in studying any semi-Markov system. Applying our method to any classical semi-Markov models and comparing it with previous ones will provide more convincing evidence of its advantages. One would take great pleasure in doing that.

5. Discussion of Results

There is oftentimes a "hidden", simple and pure probabilistic approach to the study of stochastic systems. This is frequently illustrated in the works of outstanding mathematicians in probability theory, as well as notable experts in QT and MTR. They strongly recommend exerting efforts to uncover this "hidden" approach.

It is worth reminding readers of a particularly intriguing and instructive example. The probabilistic approach can be quite effective in various areas of mathematics without being directly related to stochastic analysis. One nice example of this is Bernstein's probabilistic proof of the well-known Weierstrass approximation theorem in Levasseur (1984). In this paper (and earlier in Kakubava, 2020; Khurodze et al., 2020; Khurodze et al., 2022; Khurodze et al., 2023; Khurodze et al., 2024), we were fortunate to seize this opportunity and advance reliability theory in a new, unexpected, and very interesting direction.

Let us recall once again that SVT leads to the BVP of mathematical physics with non-local boundary conditions. The parts of the BVP include an integro-differential equation (or a system of such equations (finite or infinite)); PDE or a system of such equations (finite or infinite); Kolmogorov equations; initial conditions; and integral boundary equation (or a system of such equations (finite or infinite)). There exist several methods to solve the BVP using the Laplace transform. The PDEs are naturally assumed to be main components of the BVP. However, recent findings in our previous works and in this paper reveal that this component may actually be unnecessary. This is certainly a noteworthy finding, which has led some of our colleagues to question the viability of this approach. Our alternate approach suggests that it is possible to study all SM systems avoiding to solve the PDEs. We believe that our approach shows the intrinsic probabilistic nature of the SM process, and it is impossible to detect these phenomena by studying the BVP without fully understanding the systems under consideration in probabilistic terms. At the same time, the method is surprisingly simple and effective. This approach can be applied to any SM models that use SVT, including all models based on a so-called Cox process (Cox, 1955): M/G/1 with vacations, retrials, reneging, balking, a finite waiting room, feedback, as well as all semi-Markov models of MTR, etc.

The starting point of our approach in this work, like that of earlier developed by the authors of this study involves the examination of the considered system simultaneously in two time epochs: 1) the current time epoch t and 2) its previous time epoch $t - x$, where x denotes one of the possible values of the supplementary variable. This consideration helps us to prove the theorems 1 and 2 as well as all other expressions.

Regarding the systems of PDEs, we decided to keep them for four reasons. First, to show the complexities associated with solving such a system without using our novel method. Second, the reader can, if desired, check the consistency of the new results with previous works of other authors. Third, if one wishes to find unknown functions using numerical methods, sometimes this can be done using an appropriate system of

PDE, rather than finding the inverse Laplace transform of any expression obtained by analytical methods, including ours. Finally, our method gives a general form for solution of various PDEs that frequently arise in the problems of modern science and technology.

6. Conclusion

In this work, we have successfully combined two distinct research concepts: the well-known, purely analytical supplementary variable technique and purely probabilistic reasoning. As a result, a novel, simple and highly efficient approach for investigation of SM systems was obtained. Thus, the main purpose of the present study stated in the introduction has been achieved.

We expect that leading experts in the field will further develop this approach in various directions, including: 1) The use of functional analysis; 2) Comparative analysis; 3) Asymptotic analysis; 4) The well-posedness of mathematical models; 5) The application of numerical methods, etc. Concerning the well-posedness of (R), (BC) and (IC) as an abstract Cauchy problem, our novel method significantly modifies the problem by substituting (R) with Equations (2)-(8). Thus, the question of well-posedness arises anew under these revised conditions. Everyone interested in this area is welcome to participate in the study.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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