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Vacation Policy for *k*-out-of-*n* Redundant System with Reboot Delay

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Abstract
Redundancy is a well-known concept for system resilience; *k*-out-of-*n* redundancy stipulates that a minimum number of functional components must be present for the system to function. Introducing a reboot delay acknowledges the temporal complexities of system recovery after a failure. A dynamic component is added by the vacation policy, which introduces strategic downtime for system components. This work predicts the performance measures of a multi-state system consisting of two subsystems *A* and *B*. Subsystem A follows the *k*-out-of-*n*: *F* policy and subsystem *B* has *m* working units and *s* warm standby units. The system is under the consideration of a single unreliable repairman who may go allow for vacation. There are two possibilities for a repairman's vacation: if the system failed and the repairman is on vacation, in that case, if the repairman immediately returns from vacation, he/ she repairs the system but if the repairman does not return immediately from vacation, then the system takes a reboot action and when the repairman available, he/ she repairs the system. Failure and repair time of the units are expected to pursue an exponential distribution. In addition, the vacation time and reboot time regarding the failure of the units also pursue an exponential distribution. The concept of reboot and repairman's vacation are incorporated to make the model more practical and versatile. The expressions for several performance measures such as availability, reliability, and MTTF are obtained with the help of the Markov process. Likewise, sensitivity analysis is done to study the impact of various parameters on system performance measures. The results are explained by taking numerical illustrations.

Keywords- Reliability, MTTF, Sensitivity, Reboot, Vacation, Markov process, Warm standby, *k*-out-of-*n* system.

1. Introduction

Redundancy involves the replication of critical components to mitigate the impact of failures. The *k*-out-of-*n* redundancy model extends this concept by specifying that at least *k* out of *n* replicated components must be operational for the system to function as intended (Ram and Dohi, 2019). This approach aims to provide a robust safeguard against individual component failures. Incorporating a reboot delay introduces a temporal dimension to the redundancy model. Traditional redundancy models often assume an immediate reboot or switchover in the event of a failure. However, the reboot delay acknowledges that certain scenarios may warrant a strategic pause before initiating recovery actions (Wang and Chen, 2009). This delay could be influenced by factors such as system dynamics, operational constraints, or optimization objectives. Further, the term "Vacation Policy" introduces a dynamic element to system management. In the context of a redundant system, a vacation policy could imply a deliberate decision-making process regarding when and how to temporarily deactivate components for maintenance, upgrades, or other purposes (Kumar and Gupta, 2022). This strategic downtime, similar to a vacation for components, may be a key factor in optimizing the overall system performance and resource utilization.

Reliability models have drawn the attention of researchers for the performance evaluation of the system's various running functions. Since the 1960s, different repairable system models have been established and researched, for example, one-unit, series, parallel, series-parallel, redundant, *k*-out-of-*n*, and multi-state systems. The concept of *k*-out-of-*n* and redundancy has interest for both practitioners and researchers. The performance of any fault-tolerant system is exceptionally affected by the failure of its working units. High reliability and availability are commonly a basic necessity for a real-world repairable system (Ram, 2013). The provision of standbys is usually adopted for smooth functioning and to provide a satisfactory level of reliability and availability of the concerned system. In the recent past, numerous researchers have developed Markov models for the performance prediction of *k*-out-of-*n* and standby redundant systems (She and Pecht, 1992; Moustafa, 1996; Hsu et al., 2011).

In a repairable system, it is expected that a repairman is available all the time. But, in practice, it is not true i.e., the repairman might take a vacation. The vacation model originated in the theory of queening and has been well researched over the past three years and implemented effectively in many fields such as engineering, computer networks, infrastructures network systems, etc. Remarkable work on the repairman's vacation concept has been found (Doshi, 1986; Takagi, 1991; Tian and Zhang, 2006; Yu et al., 2013; Jain and Gupta, 2013). They proposed various server vacation policies such as single vacation, multiple vacation, and hybrid vacation. Over the last decade, motivated by the principle of vacation queening, several researchers incorporated the vacation model into a repairable system (Ke and Wu, 2012; Jain and Meena, 2017; Yang and Tsao, 2019). Kadi et al. (2020) studied a queueing system with the vacation behaviours of the customers by using techniques based on probability-generating functions. Kalyanaraman and Sundaramoorthy (2019) Investigated a Markovian queueing system with a single server with the consideration of three states namely, busy, repaired, and working vacation. Chakravarthy et al. (2020) studied a queueing system with a backup served. In this research, a phase-type distribution is used to represent the service times. Ahuja et al. (2021) investigated a single server queueing model that is unreliable, as well as multiple-stage service and functioning vacation. Bhagat et al. (2021) studied a machine repair problem (MRP) with M identical operating machines and control arrival policy. In this research, the authors assumed that the server may go for a working vacation in case all the customers are served and used the *R-K* method to calculate the reliability measures of the queueing model. Thakur et al. (2021) investigated the $M/M/1/N$ single server Markovian queueing model with working vacation and unreliable servers. In this research (Thakur et al., 2021), the authors used the M-threshold recovery policy to recover the server broken down.

The implementation of a repairman's vacation makes the simulation of the repairable system more practical and versatile from the viewpoint of the fair use of human resources. To calculate the repairable system's indices by incorporating the concept of repairman's vacation, Jain and Singh (2004) considered a machine system with two repairmen which may allow for vacation when many failed units are less than a threshold level. Ke and Lin (2005) calculated and discussed the reliability availability, MTTF, and sensitivity of *k*-out-of- M+m systems consisting of M operating units and m spares with an unreliable server that takes vacations. In the study of Hu et al. (2010), a single repair with a single vacation is considered to investigate the three-unit system's performance measures by using supplementary variables and vector Markov process theory. Guo et al. (2011) considered a series system consisting of n units with vacation and replacement policies and calculated some reliability indices and optimal profit of the system using the eigenfunction. Yuan (2012) and Yuan and Cui (2013) discussed the reliability indices of *k*-out-of-*n*: *G* and consecutive *k*-out-of-*n*: *F* systems respectively with *R* repairmen who may allow for going on multiple vacations. Ke and Wang (2012) studied a repair system having *M* operating, two types of standby, and *R* repairman by incorporating the concept of repairman's vacation. In addition, some steady-state probabilities are calculated by matrix geometric theory and discussed. Wu et al. (2021) calculated the optimal replacement policy for a deteriorating repairable system with the consideration of multiple vacations with a single repairman.

There has been a great deal of research on vacation policy. Jia and Wu (2009) studied a repairable system with multiple vacations of repairmen with the assumption that the system waits for repair when the repairman is on vacation until the repairman is available. In real life, one can see that the stoppage of a system due to any error generally takes some time to recover/restart, of course with some recovery rate. The time between the system failure and restart is known as reboot delay. When we take a reboot action, that means the repairman is not available the system takes some time to recover. Many research papers in the area of reliability theory have incorporated the concept of reboot while analyzing the repairable redundant systems in a different context (Wang and Chen, 2009; Jain et al., 2014). Hsu et al. (2009) used the Bayesian approach when dealing with a redundant repairable system with reboot action and imperfect coverage. Hsu et al. (2011) proposed a redundant system with an unreliable server and reboot delay. Fundamentals of reboot can be found in the book written by Trivedi (2008). Goyal and Ram (2023) investigated a (*KM*+*IS*) system using a reboot process and considered the coverage probability only for switching failure.

In the present research, a repairable system with two dissimilar systems with unit failure, reboot, and repairman vacation is considered. The considered multi-unit repairable system has two subsystems namely *A* and *B*, in which subsystem *A* follows *k*-out-of-*n*: *F* policy and subsystem *B* has *m* working units and *s* warm standby units. The repairman in the system follows a single vacation policy. Under regular circumference, when the system fails, the repairman immediately returns from vacation and repairs it. But if the repairman does not immediately return from vacation, in that case, reboot action is taken into account and the system takes a reboot delay action. This research is a unique combination of *k*-out-of-*n* redundancy, reboot delay, and a vacation policy. By the mathematical modeling of this model, authors derive some differential equations by the Markov process and study how these elements interact to enhance the reliability and performance of a system.

The present research work is organized into six sections with subsections. Section 1 highlights the introduction of different approaches such as *k*-out-of-*n* model, redundancy and vacation policy with an extensive literature. Section 2 begins with the assumption related to the model, and notations used throughout the study and ends with the system model description. In next section 3, a set of differential equations using the Markov process is formulated. Also, this section gives the probability of each state obtained by using the Laplace transform. Section 4 is dedicated to deriving some specific expressions for the system performance measures. Some numerical illustration to explore the practical situation and sensitivity analysis of reliability concerning different parameters is also a part of this section. In section 5, results discussion regarding each reliability measure are discussed with the help of graphs. Some concluding remarks with a summary are given at length in section 6.

2. Model Details

2.1 Assumptions and Notations

The assumptions made for the formulation of the Markov model are as follows:

- At time zero, the system is in new condition and starts to work.
- The considered system has two subsystems *A* and *B*. Subsystem *A* follows *k*-out-of-*n*: *F* policy whereas subsystem *B* has *m* operating unit and *s* warm standby units.
- On failure of the working units, the available warm standby units are used one by one with negligible switching time.
- The repairman is not always available to repair the system i.e., the repairman may go allowed for vacation. When the system fails and the repairman is on vacation there are two possibilities either the repairman's immediate return from vacation with probability *a* or not immediate return from vacation with probability *b*.
- When the repairman immediately returns from vacation, it starts to repair the system but when the repairman is not immediately returned from vacation, the system takes a reboot delay action in that state.
- The working units that are likely to be failures have an exponential distribution.
- After being repaired, the failed unit is considered as good as a new one.

Some notations (see Table 1) which have been used in model development are as follows:

Table 1. Notations.

t	Time scale
s	Laplace transform variable
$\lambda_{k-1}/\ \lambda_k$	Failure rate of *k*-1 units/ failure rate of *k* units.
$\lambda_{m-1}/\ \lambda_m$	Failure rate of *m*-1 working units/ failure rate of *m* working units
$\lambda_{s-1}/\ \lambda_s$	failure rate of *s*-1 standby units/ failure rate of *s* standby units
a	Probability that the repairman is immediately returned from vacation
b	Probability that the repairman will not immediately return from vacation
x	Elapsed vacation time
y	Elapsed reboot time
z	Elapsed repair time
η	Vacation rate
β	Reboot rate
μ	Repair rate

2.2 Model Description

Consider a multi-unit system having two subsystems *A* and *B* where subsystem *A* has *k*-out-of-*n*: *F* policy and subsystem *B* comprises *m* operating unit and *s* warm standby units. For proper functioning of subsystem *A*, at least (*n*-*k*+1) units should work properly. For subsystem *B*, if *m* units fail, standby units take place one by one, and the system is worked until *s* standby units fail. The system is maintained by a single unreliable repairman who may allow for going on vacation and repairing the failed unit according to an exponential distribution with repair rate μ. When the system has failed, there are two possibilities i.e., either the repairman is in the system or may go on vacation. If the repairman is available then he/she repairs the system but, if the repairman goes on vacation and (i) can immediately return from vacation with probability *a* then he/she repairs the system and if (ii) the repairman did not immediately return from vacation with probability *b* then a reboot delay action is taken by the system with rate β and when the repairman goes back from vacation, he/she repairs the system. The lifetime distribution of working and standby units follows an exponential distribution. In addition, the reboot time of the failed unit and the vacation time of the repairman also follow an exponential distribution. The state transition diagram of the considered system is shown in Figure 1.

For mathematical formulation, the system state probabilities are defined as:

$P_0(t)$ Probability of the state that all units are good.
$P_1(t)$ Probability of the state that k-1 units of subsystem A have failed.
$P_2(x, t)$ Probability of the completely failed state when k units of subsystem A have failed.
$P_3(y, t)$ Probability of the completely failed state when k units of subsystem A have failed and the repairman is not returned immediately from vacation.
$P_4(z, t)$ Probability of the completely failed state when k units of subsystem A have failed and the repairman is returning immediately from vacation.
$P_5(t)$ Probability of the degraded state that m-1 operating units of subsystem B have failed.
$P_6(t)$ Probability of the degraded state that m operating units of subsystem B have failed.
$P_7(t)$ Probability of the degraded state that s-1 operating units of subsystem B have failed.
$P_8(x, t)$ Probability of the completely failed state when s units of subsystem B have failed.
$P_9(y, t)$ Probability of the completely failed state when s units of subsystem B have failed and the repairman is not returning immediately from vacation.
$P_{10}(z, t)$ Probability of the completely failed state when s units of subsystem B have failed and the repairman is returning immediately from vacation.

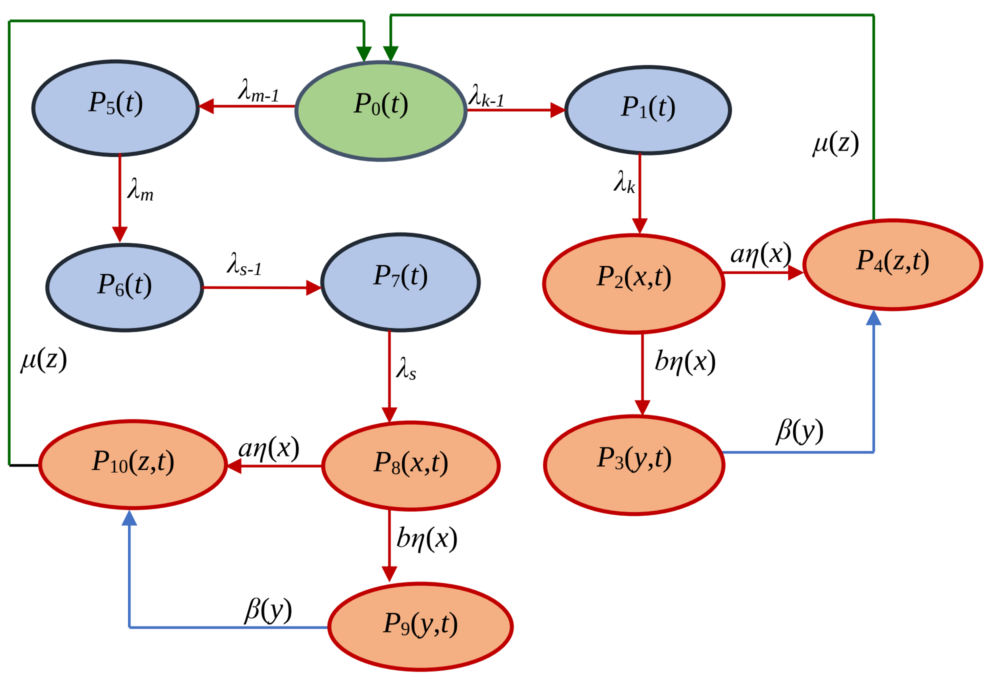


Figure 1. System states diagram.

3. Formulation of the Model

This section gives the state's transition probabilities associated with the model, these probabilities are further enjoyed to estimate the reliability characteristics such as availability, reliability, MTTF, and sensitivity.

3.1 Governing Equations

From the state transition diagram, the following differential equations have been derived with the help of the Markov process.

$$\left(\frac{\partial}{\partial t}+\lambda_{k-1}+\lambda_{m-1}\right)P_0(t)=\int_0^\infty \mu(z)P_4(z,t)dz+\int_0^\infty \mu(z)P_{10}(z,t)dz \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\lambda_k\right)P_1(t)=\lambda_{k-1}P_0(t) \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+a\eta(x)+b\eta(x)\right)P_2(x,t)=0 \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta(y)\right)P_3(y,t)=0 \tag{4}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+\mu(z)\right)P_4(z,t)=0 \tag{5}$$

$$\left(\frac{\partial}{\partial t}+\lambda_m\right)P_5(t)=\lambda_{m-1}P_0(t) \tag{6}$$

$$\left(\frac{\partial}{\partial t}+\lambda_{s-1}\right)P_6(t)=\lambda_m P_5(t) \tag{7}$$

$$\left(\frac{\partial}{\partial t}+\lambda_s\right)P_7(t)=\lambda_{s-1}P_6(t) \tag{8}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+a\eta(x)+\mathrm{b}\,\eta(x)\right)P_8(x,t)=0 \tag{9}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta(y)\right)P_9(y,t)=0 \tag{10}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+\mu(z)\right)P_{10}(z,t)=0 \tag{11}$$

Boundary conditions

$$P_2(0,t)=\lambda_k P_1(t) \tag{12}$$

$$P_3(0,t)=\int_0^\infty b\eta(x)P_2(x,t)dx \tag{13}$$

$$P_4(0,t)=\int_0^\infty a\eta(x)P_2(x,t)dx+\int_0^\infty \beta(y)P_3(y,t)dy \tag{14}$$

$$P_8(0,t)=\lambda_s P_7(t) \tag{15}$$

$$P_9(0,t)=\int_0^\infty b\eta(x)P_8(x,t)dx \tag{16}$$

$$P_{10}(0,t)=\int_0^\infty a\eta(x)P_8(x,t)dx+\int_0^\infty \beta(y)P_9(y,t)dy \tag{17}$$

Let initially, all units of the system be operating so that,

$P_0(0) = 1$ and $P_i(0) = 0$, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

Probability of each transition state in terms of Laplace transform,

$$\bar{P}_0(\mathrm{s})=\frac{1}{D(\mathrm{s})} \tag{18}$$

$$\bar{P}_1(\mathrm{s}) = \frac{\lambda_{k-1}}{\mathrm{s}+\lambda_k}\bar{P}_0(\mathrm{s}) \tag{19}$$

$$\bar{P}_2(\mathrm{s}) = \frac{\lambda_{k-1}\lambda_k}{\mathrm{s}+\lambda_k}F(\mathrm{s})\bar{P}_0(\mathrm{s}) \tag{20}$$

$$\bar{P}_3(\mathrm{s}) = \frac{b\bar{\mathrm{S}}_\eta(\mathrm{s})\lambda_{k-1}\lambda_k}{\mathrm{s}+\lambda_k}G(\mathrm{s})\bar{P}_0(\mathrm{s}) \tag{21}$$

$$\bar{P}_4(\mathrm{s}) = \left[\frac{a\lambda_{k-1}\lambda_k\bar{\mathrm{S}}_\eta(\mathrm{s})}{\mathrm{s}+\lambda_k} + \frac{b\lambda_{k-1}\lambda_k\bar{\mathrm{S}}_\eta(\mathrm{s})\bar{S}_\beta(\mathrm{s})}{\mathrm{s}+\lambda_k}\right]H(\mathrm{s})\bar{P}_0(\mathrm{s}) \tag{22}$$

$$\bar{P}_5(\mathrm{s}) = \frac{\lambda_{m-1}}{\mathrm{s}+\lambda_m}\bar{P}_0(\mathrm{s}) \tag{23}$$

$$\bar{P}_6(\mathrm{s}) = \frac{\lambda_m\lambda_{m-1}}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_m)}\bar{P}_0(\mathrm{s}) \tag{24}$$

$$\bar{P}_7(\mathrm{s}) = \frac{\lambda_{m-1}\lambda_m\lambda_{s-1}}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_m)}\bar{P}_0(\mathrm{s}) \tag{25}$$

$$\bar{P}_8(\mathrm{s}) = \frac{\lambda_{m-1}\lambda_{s-1}\lambda_m\lambda_s}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_m)}F(\mathrm{s})\bar{P}_0(\mathrm{s}) \tag{26}$$

$$\bar{P}_9(\mathrm{s}) = \frac{b\bar{\mathrm{S}}_\eta(\mathrm{s})\lambda_{m-1}\lambda_{s-1}\lambda_m\lambda_s}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_m)}G(\mathrm{s})\bar{P}_0(\mathrm{s}) \tag{27}$$

$$\bar{P}_{10}(\mathrm{s}) = \left[\frac{a\lambda_{m-1}\lambda_{s-1}\lambda_m\lambda_s\bar{\mathrm{S}}_\eta(\mathrm{s})}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_m)} + \frac{b\lambda_{m-1}\lambda_{s-1}\lambda_m\lambda_s\bar{\mathrm{S}}_\eta(\mathrm{s})\bar{\mathrm{S}}_\beta(\mathrm{s})}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_m)}\right]H(\mathrm{s})\bar{P}_0(\mathrm{s}) \tag{28}$$

Up and down states probabilities are:

$$\begin{aligned}\bar{P}_{up}(\mathrm{s}) &= \bar{P}_0(\mathrm{s}) + \bar{P}_1(\mathrm{s}) + \bar{P}_5(\mathrm{s}) + \bar{P}_6(\mathrm{s}) + \bar{P}_7(\mathrm{s}) \\ &= \bar{P}_0(\mathrm{s})\left[1 + \frac{\lambda_{k-1}}{\mathrm{s}+\lambda_k} + \frac{\lambda_{m-1}}{\mathrm{s}+\lambda_m} + \frac{\lambda_{m-1}\lambda_m}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_m)} + \frac{\lambda_{m-1}\lambda_m\lambda_{s-1}}{(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_m)}\right]\end{aligned} \tag{29}$$

$$\begin{aligned}\bar{P}_{down}(\mathrm{s}) &= \bar{P}_2(\mathrm{s}) + \bar{P}_3(\mathrm{s}) + \bar{P}_4(\mathrm{s}) + \bar{P}_8(\mathrm{s}) + \bar{P}_9(\mathrm{s}) + \bar{P}_{10}(\mathrm{s}) \\ &= U(\mathrm{s})\left\{F(\mathrm{s}) + a\bar{\mathrm{S}}_\eta(\mathrm{s})H(\mathrm{s}) + b\bar{\mathrm{S}}_\eta(\mathrm{s})G(\mathrm{s}) + b\bar{\mathrm{S}}_\eta(\mathrm{s})\bar{\mathrm{S}}_\beta(\mathrm{s})H(\mathrm{s})\right\} \\ &\quad + V(\mathrm{s})\left\{F(\mathrm{s}) + aH(\mathrm{s}) + bG(\mathrm{s}) + b\bar{\mathrm{S}}_\eta(\mathrm{s})\bar{\mathrm{S}}_\beta(\mathrm{s})H(\mathrm{s})\right\}\end{aligned} \tag{30}$$

where,

$D(\mathrm{s}) = T(\mathrm{s}) - \bar{\mathrm{S}}_\mu(\mathrm{s})\bar{\mathrm{S}}_\eta(\mathrm{s})\left[\left\{a + b\bar{\mathrm{S}}_\beta(\mathrm{s})\right\}\left\{U(\mathrm{s}) + V(\mathrm{s})\right\}\right]$, $T(\mathrm{s}) = \mathrm{s}+\lambda_{k-1} + \lambda_{m-1}$, $U(\mathrm{s}) = \dfrac{\lambda_k\lambda_{k-1}}{\mathrm{s}+\lambda_k}$,

$V(\mathrm{s}) = \dfrac{\lambda_s\lambda_{s-1}\lambda_m\lambda_{m-1}}{(\mathrm{s}+\lambda_s)(\mathrm{s}+\lambda_{s-1})(\mathrm{s}+\lambda_m)}$, $F(\mathrm{s}) = \left\{\dfrac{1-\bar{\mathrm{S}}_\eta(\mathrm{s})}{\mathrm{s}}\right\}$, $G(\mathrm{s}) = \left\{\dfrac{1-\bar{\mathrm{S}}_\beta(\mathrm{s})}{\mathrm{s}}\right\}$, $H(\mathrm{s}) = \left\{\dfrac{1-\bar{\mathrm{S}}_\mu(\mathrm{s})}{\mathrm{s}}\right\}$,

$\bar{\mathrm{S}}_\eta(\mathrm{s}) = \dfrac{\eta}{\mathrm{s}+\eta(a+b)}$, $\bar{\mathrm{S}}_\mu(\mathrm{s}) = \dfrac{\mu}{\mathrm{s}+\mu}$, $\bar{\mathrm{S}}_\beta(\mathrm{s}) = \dfrac{\beta}{\mathrm{s}+\beta}$.

4. Performance Measures and Numerical Examples

This section gives the reliability Characteristics like availability, reliability, MTTF, sensitivity, and relative sensitivity of reliability obtained from transient states of the system. Also, the numerical evaluation to replicate the behaviour of the system by considering a numerical example is present. The numerical evaluation accomplished here presents the validation of the proposed model and allows decision-makers to analyze the numerical effects of variation of specific input parameters on the system's overall performance measures which can be required for enhancing the efficacy of the redundant system.

4.1 Availability Analysis

The availability of the considered system is the sum of the probabilities of the good and degraded states and is given by,

$$\bar{A}(s)=\bar{P}_0(s)\left[1+\frac{\lambda_{k-1}}{s+\lambda_k}+\frac{\lambda_{m-1}}{s+\lambda_m}+\frac{\lambda_{m-1}\lambda_m}{(s+\lambda_{s-1})(s+\lambda_m)}+\frac{\lambda_{m-1}\lambda_m\lambda_{s-1}}{(s+\lambda_{s-1})(s+\lambda_s)(s+\lambda_m)}\right] \tag{31}$$

For computation purposes, we compute the availability of the system in two cases:

***Case* 1**: when a = 0.8, b = 0.2, and other input parameter values are set as λ_{k-1} = 0.4, λ_k = 0.5, λ_{m-1} = 0.2, λ_m = 0.3, λ_{s-1} = 0.09, λ_s = 0.1 (Arora et al., 2020; Tyagi et al., 2019), x = 1, y = 1, $\mu(z)$ = 1, $\beta(y)$ = 2 and $\eta(x)$=0.1 in Equation (31) and then take the inverse Laplace to obtain the availability of the system in terms of time t.

$$\begin{aligned}A(t)=&-0.00092650e^{(-1.9980t)}+0.017516e^{(-1.0877t)}+0.58187e^{(-0.5452t)}\cos(0.3t)+1.1557e^{(-0.5452t)}\\&\times\sin(0.27365t)+0.016089e^{(-0.3082t)}-0.11563e^{(-0.1028t)}\cos(0.062310e^{t})+0.18072e^{(-0.1028t)}\\&\times\sin(0.062310e^{t})+0.50108.\end{aligned}$$

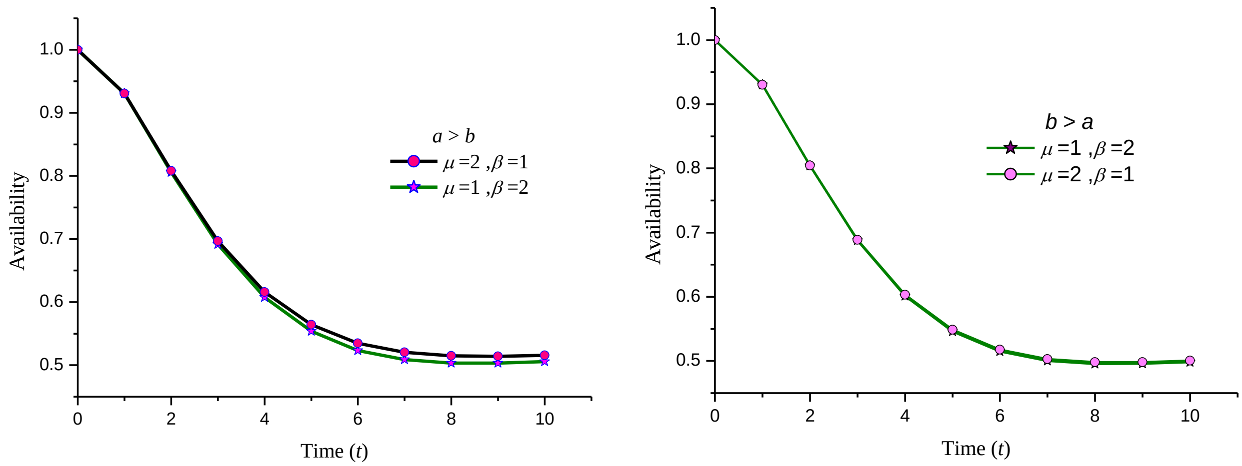
***Case* 2**: When a = 0.2 and b = 0.8 and all other parameters are fixed as described above, take the inverse Laplace to transform to get the availability of the system as follows,

$$\begin{aligned}A(t)=&-0.0038293e^{(-1.9918t)}+0.029253e^{(-1.1221t)}+0.57741e^{(-0.5314t)}\cos(0.29258t)+1.0729e^{(-0.5314t)}\\&\times\sin(0.29258t)+0.014420e^{(-0.30841t)}-0.11167e^{(-0.1025t)}\cos(0.062451e^{t})+0.17808e^{(-0.1025t)}\\&\times\sin(0.062451e^{t})+0.49442.\end{aligned}$$

After varying time t = 0 to 10, the availability of the system is obtained as summarized in Table 2 and visualized in Figure 2.

Table 2. Availability vs time at different values of repair and reboot rate.

Time (t)	Availability $P_{up}(t)$			
	$a > b$		$b > a$	
	μ = 1, β = 2	μ = 2, β = 1	μ = 1, β = 2	μ = 2, β = 1
0	1.00000	1.00000	1.00000	1.00000
1	0.93053	0.93083	0.93031	0.93038
2	0.80572	0.80791	0.80408	0.80463
3	0.69137	0.69670	0.68738	0.68871
4	0.60740	0.61576	0.60115	0.60324
5	0.55369	0.56412	0.54592	0.54851
6	0.52336	0.53476	0.51489	0.51771
7	0.50871	0.52024	0.50019	0.50301
8	0.50350	0.51468	0.49529	0.49802
9	0.50339	0.51401	0.49563	0.49819
10	0.50561	0.51569	0.49829	0.50071



(i) (ii)

Figure 2. Availability vs time with different values of repair and reboot rate when (a) a = 0.8, b = 0.2 and (b) a = 0.2, b = 0.8.

4.2 Reliability Analysis

The reliability function of the system in terms of Laplace transform is given by,

$$\bar{R}(s)=\frac{1}{s+\lambda_{k-1}+\lambda_{m-1}}\left[1+\frac{\lambda_{k-1}}{s+\lambda_k}+\frac{\lambda_{m-1}}{s+\lambda_m}+\frac{\lambda_{m-1}\lambda_m}{(s+\lambda_{s-1})(s+\lambda_m)}+\frac{\lambda_{m-1}\lambda_m\lambda_{s-1}}{(s+\lambda_{s-1})(s+\lambda_s)(s+\lambda_m)}\right] \tag{32}$$

To obtain the reliability behavior of the system, set default parameter λ_{k-1} = 0.4, λ_k = 0.5, λ_{m-1} = 0.2, λ_m = 0.3, λ_{s-1} = 0.09, λ_s = 0.1, x = 1, y = 1, $\mu(z)$ = 1, $\beta(y)$ = 2, $\eta(x)$=0.1, a = 0.8 and b = 0.2 in Equation (32), and then after taking inverse Laplace, we get Equation (33) as follow:

$$R(t)=\text{-}3.3451e^{(-0.6t)}+4e^{(-0.5t)}+0.14286e^{(-0.3t)}+5.6022e^{(-0.09t)}\text{-}5.4000e^{(-0.1t)} \tag{33}$$

The reliability of the system per unit of time is calculated by varying time t from 0 to 10, the numerical results for the same are summarized in Table 3(a) and (b).

Table 3. Reliability of the system w.r.t time at different values of failure rates.

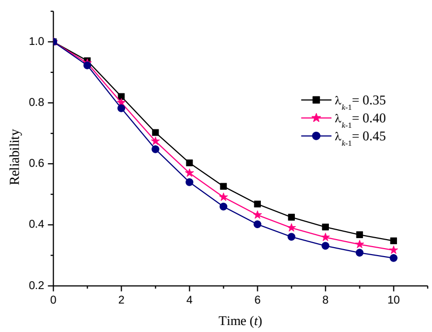
(a) Reliability of the system with different values of λ_{k-1}, λ_k and λ_{m-1}.

Time(t)	Reliability								
	λ_{k-1} = 0.35	λ_{k-1} = 0.40	λ_{k-1} = 0.50	λ_k = 0.45	λ_k = 0.50	λ_k = 0.55	λ_{m-1} = 0.15	λ_{m-1} = 0.20	λ_{m-1} = 0.25
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.93765	0.92999	0.92250	0.93612	0.93019	0.92455	0.92443	0.93001	0.93560
2	0.82011	0.80056	0.78214	0.81497	0.80075	0.78755	0.78370	0.80059	0.81749
3	0.70225	0.67381	0.64777	0.69314	0.67399	0.65679	0.64475	0.67385	0.70294
4	0.60310	0.56967	0.53979	0.59021	0.56984	0.55215	0.52965	0.56970	0.60975
5	0.52593	0.49050	0.45946	0.50973	0.49065	0.47468	0.44153	0.49053	0.53953
6	0.46800	0.43243	0.40175	0.44903	0.43256	0.41928	0.37663	0.43245	0.48828
7	0.42496	0.39022	0.36062	0.40380	0.39035	0.37990	0.32955	0.39025	0.45095
8	0.39263	0.35918	0.33089	0.36984	0.35930	0.35141	0.29529	0.35920	0.42311
9	0.36766	0.33562	0.30866	0.34375	0.33572	0.32996	0.26990	0.33564	0.40138
10	0.34753	0.31689	0.29119	0.32294	0.31699	0.31699	0.25045	0.31691	0.38337

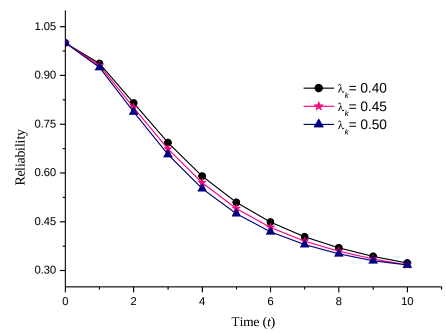
Table 3 continued…

(b) Reliability of the system with different values of λ_m, $\lambda_{s\text{-}1}$ and λ_s.

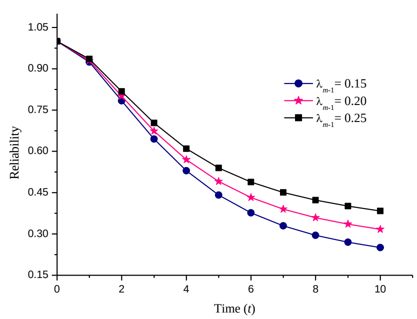
t	Reliability								
	$\lambda_m = 0.25$	$\lambda_m = 0.30$	$\lambda_m = 0.35$	$\lambda_{s\text{-}1} = 0.04$	$\lambda_{s\text{-}1} = 0.09$	$\lambda_{s\text{-}1} = 0.14$	$\lambda_s = 0.05$	$\lambda_s = 0.10$	$\lambda_s = 0.15$
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1	0.92972	0.92973	0.92973	0.93008	0.93028	0.92987	0.93013	0.92921	0.92966
2	0.80016	0.80015	0.80015	0.80077	0.80075	0.80042	0.80078	0.80015	0.80030
3	0.67344	0.67334	0.67334	0.67441	0.67394	0.67333	0.67437	0.67361	0.67331
4	0.56949	0.56918	0.56918	0.57110	0.56976	0.56846	0.57096	0.56958	0.56854
5	0.49064	0.49002	0.49002	0.49329	0.49057	0.48809	0.49300	0.49046	0.48833
6	0.43301	0.43197	0.43197	0.43716	0.43248	0.42838	0.43663	0.43242	0.42882
7	0.39134	0.38980	0.38980	0.39746	0.39027	0.38414	0.39664	0.39023	0.38483
8	0.36088	0.35879	0.35879	0.36944	0.35922	0.35074	0.36824	0.35920	0.35171
9	0.33793	0.33525	0.33525	0.34935	0.33565	0.32459	0.34771	0.33564	0.32588
10	0.34753	0.31689	0.29119	0.33446	0.31692	0.30313	0.33231	0.31691	0.30478



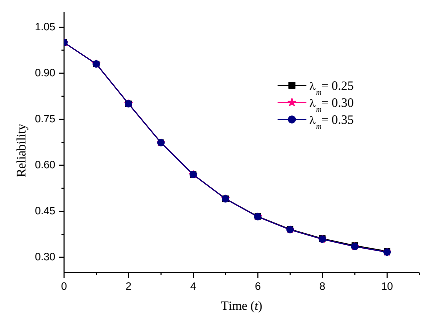
(i) Variation in $\lambda_{k\text{-}1}$.



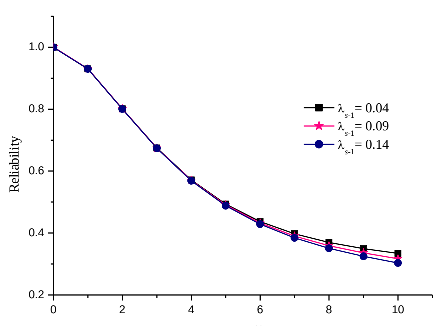
(ii) Variation in λ_k.



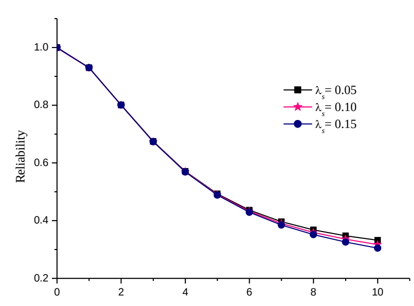
(iii) variation in $\lambda_{m\text{-}1}$.



(iv) Variation in λ_m.



(v) Variation in $\lambda_{s\text{-}1}$.



(vi) Variation in λ_s.

Figure 3. Reliability vs time with variation in failure rates.

4.3 Mean Time to Failure (MTTF)

Mean time to system failure is defined as follows,

$$MTTF = \int_0^\infty R(t)dt = \lim_{s\to 0}\left[\int_0^\infty R(t)\,e^{-st}\,dt\right].$$

So, the MTTF of the considered system is obtained by taking $s\to 0$ in Equation (32) and given by,

$$MTTF = \frac{1}{\lambda_{k-1}+\lambda_{m-1}}\left[1+\frac{\lambda_{k-1}}{\lambda_k}+\frac{\lambda_{m-1}}{\lambda_m}+\frac{\lambda_{m-1}\lambda_m}{\lambda_{s-1}\lambda_m}+\frac{\lambda_{m-1}\lambda_m\lambda_{s-1}}{\lambda_{s-1}\lambda_s\lambda_m}\right] \tag{34}$$

To get the numerical results of MTTF with respect to variation in failure rates, set failure rate values $\lambda_{k\text{-}1} = 0.4$, $\lambda_k = 0.5$, $\lambda_{m\text{-}1} = 0.2$, $\lambda_m = 0.3$, $\lambda_{s\text{-}1} = 0.09$, $\lambda_s = 0.1$ and vary one by one from 0.1 to 0.9 i.e., to get MTTF with respect to $\lambda_{k\text{-}1}$, set $\lambda_k = 0.5$, $\lambda_{m\text{-}1} = 0.2$, $\lambda_m = 0.3$, $\lambda_{s\text{-}1} = 0.09$, $\lambda_s = 0.1$ in Equation (34) and then vary $\lambda_{k\text{-}1}$ from 0.1 to 0.9. Similarly, all parameters vary one by one while other parameters are fixed at their values. The computed results are given in Table 4 and graphically shown in Figure 4.

Table 4. Effects of failure rates on MTTF of the system.

Variation in failure rates	Mean time to failure (MTTF)					
	$\lambda_{k\text{-}1}$	λ_k	$\lambda_{m\text{-}1}$	λ_m	$\lambda_{s\text{-}1}$	λ_s
0.1	16.4814	20.2963	13.3704	8.4888	11.148	10.7778
0.2	13.1482	15.7222	11.7037	11.1482	9.4814	9.1111
0.3	12.0370	12.9778	11.1482	13.0476	8.9259	8.5555
0.4	11.4815	11.1482	10.8704	14.4722	8.6481	8.2777
0.5	11.1482	9.8412	10.7037	15.5803	8.4814	8.1111
0.6	10.9259	8.8611	10.5926	16.4667	8.3703	8.0000
0.7	10.7672	8.0987	10.5132	17.1919	8.2910	7.9206
0.8	10.6482	7.4888	10.4537	17.7963	8.2314	7.8611
0.9	10.5556	6.9899	10.4074	18.3077	8.1851	7.8148

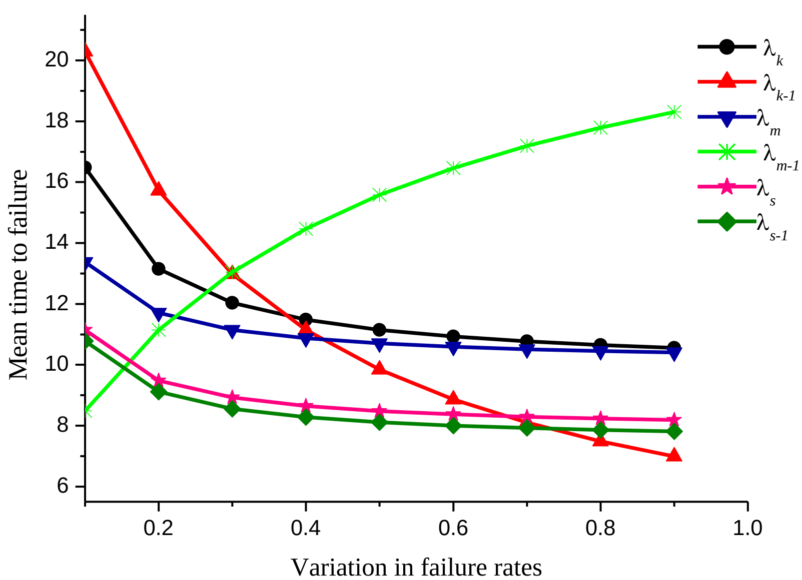


Figure 4. Mean time to system failure vs variation in failure rates.

4.4 Sensitivity Analysis

Sensitivity analysis is done to determine the effect of the system input (independent) variable on the system performance measures under a given set of assumptions.

4.4.1 Sensitivity of Reliability

Reliability sensitivity is obtained by partial differentiation of reliability expression regarding failure rates *i.e.* if it is represented by Δ_θ then,

$$\Delta_\theta = \frac{\partial R(t)}{\partial \theta},$$

where, $\theta = \lambda_{k\text{-}1}, \lambda_k, \lambda_m, \lambda_{m\text{-}1}, \lambda_{s\text{-}1}, \lambda_s$.

Table 5. Sensitivity of reliability with respect to various parameters.

Time (t)	Sensitivity of reliability					
	$\frac{\partial R(t)}{\partial \lambda_{k-1}}$	$\frac{\partial R(t)}{\partial \lambda_k}$	$\frac{\partial R(t)}{\partial \lambda_{m-1}}$	$\frac{\partial R(t)}{\partial \lambda_m}$	$\frac{\partial R(t)}{\partial \lambda_{s-1}}$	$\frac{\partial R(t)}{\partial \lambda_s}$
0	0	0	0	0	0	0
1	-0.15286	-0.11736	0.02184	-5.6E-5	-1.78E-4	-1.9E-4
2	-0.38074	-0.27563	0.11591	-6.9E-4	-0.00226	-0.00253
3	-0.54518	-0.36431	0.26277	-0.0027	-0.00917	-0.01026
4	-0.63251	-0.38067	0.42404	-0.0066	-0.02333	-0.02615
5	-0.66325	-0.34978	0.57165	-0.01249	-0.04605	-0.05174
6	-0.66035	-0.29636	0.69151	-0.02018	-0.07759	-0.08739
7	-0.64061	-0.23745	0.77970	-0.02926	-0.11733	-0.13247
8	-0.61422	-0.18266	0.83816	-0.03925	-0.16405	-0.18570
9	-0.58653	-0.13623	0.87138	-0.04964	-0.21624	-0.24541
10	-0.55991	-0.09915	0.88454	-0.05999	-0.27222	-0.30977

4.4.2 Relative Sensitivity of Reliability

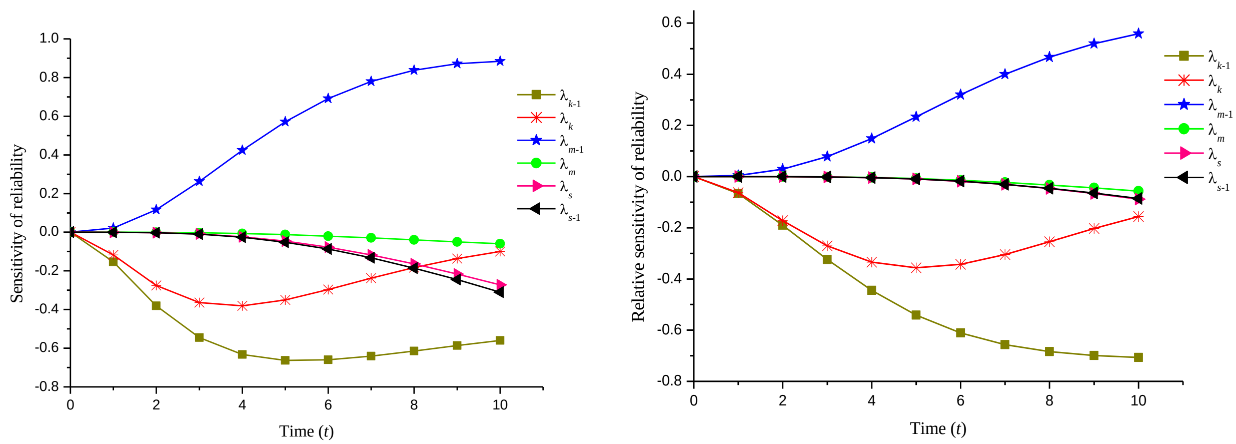
Relative sensitivity is related to a unit percentage change in parameter value (Shekhar et al., 2017). In the study of relative sensitivity, one can compare the relative effect of different parameters.

$$R\Delta_\theta = \frac{\partial R(t) / R(t)}{\partial \theta / \theta},$$

where, $\theta = \lambda_{k\text{-}1}, \lambda_k, \lambda_m, \lambda_{m\text{-}1}, \lambda_{s\text{-}1}, \lambda_s$.

Table 6. Relative sensitivity of reliability with respect to various parameters.

Time (t)	Relative sensitivity of reliability					
	$R\Delta_{\lambda_{k-1}}$	$R\Delta_{\lambda_k}$	$R\Delta_{\lambda_{m-1}}$	$R\Delta_{\lambda_m}$	$R\Delta_{\lambda_{s-1}}$	$R\Delta_{\lambda_s}$
0	0	0	0	0	0	0
1	-0.06574	-0.06310	0.00470	-2E-5	-2E-5	-2E-5
2	-0.19023	-0.17214	0.02895	-2.6E-4	-2.8E-4	-2.8E-4
3	-0.32362	-0.27032	0.07799	-0.0012	-0.00137	-0.00136
4	-0.4441	-0.3341	0.14886	-0.00347	-0.00413	-0.00409
5	-0.54084	-0.35654	0.23308	-0.00764	-0.00949	-0.00939
6	-0.61080	-0.34265	0.31981	-0.01400	-0.01819	-0.01794
7	-0.65662	-0.30424	0.39959	-0.02250	-0.03055	-0.03006
8	-0.68398	-0.25427	0.46668	-0.03278	-0.04653	-0.04567
9	-0.69901	-0.20294	0.51924	-0.04437	-0.06581	-0.06443
10	-0.70672	-0.15643	0.55823	-0.05679	-0.08797	-0.08590



(i) (ii)

Figure 5. (i) Sensitivity and (ii) relative sensitivity of reliability of the system with respect to various failures.

5. Results and Discussion

In Table 2 as well as Figure 2, one can observe the effects of failures, repair, and reboot in two cases i.e., when $a > b$ and $b > a$. As Figure 2 clearly shows that the availability of the considered system decreases with respect to an increase in time and it is greater in case of $a > b$ in comparison to case $b > a$. Further, from Figure 2 (i), it is noticed that the maximum achievable availability of the considered system is 0.9308 at time equal to 1 while holding $\mu = 2$, $\beta = 1$ and 0.93053 while holding $\mu = 1$, $\beta = 2$. Hence, this graph clearly shows that the maximum availability has been achieved when the repair rate is taken to be greater than the reboot rate. Similarly, in Figure 2 (ii), maximum availability is achieved when the repair rate has a greater value than the reboot rate.

Figure 3 represents the variation in the reliability of the system regarding time for various system failure rates. Figure 3 (i)-(vi) shows the decrement in reliability as time extends. To empathize with failure effects on system reliability, take a change in the value of each failure while other parameters are fixed at their value. From Figure 3, it can be observed that $\lambda_{k\text{-}1}$, $\lambda_{m\text{-}1}$ significantly influence system reliability, λ_k, $\lambda_{s\text{-}1}$ and λ_s affects system reliability moderately while λ_m rarely influences system reliability.

Figure 4 reveals that the MTTF of the system decreases with the increment in the value of failure rates in the time interval of [0,10]. With respect to λ_k, MTTF of the system is decreased in a uniform manner but with respect to $\lambda_{k\text{-}1}$, it decreased rapidly from 20.2963 to 6.98990. With respect to λ_m, MTTF decreases uniformly but regarding $\lambda_{m\text{-}1}$, it increases rapidly. Also, From the graph, it is seen that MTTF gradually decreases with increment in the value of both $\lambda_{s\text{-}1}$ and λ_s. However, for a higher value of $\lambda_{s\text{-}1}$ and λ_s, it ultimately becomes almost constant.

Tables 5 and 6 give the value of sensitivity and relative sensitivity of reliability and corresponding Figure 5 reveals the trend of reliability sensitivity with respect to different failure rates. From the graph, it is seen that the parameter λ_m causes a weak change in reliability sensitivity resulting in the reliability of the system is less sensitive regarding λ_m. Whereas, the parameter $\lambda_{m\text{-}1}$ causes a stronger change in reliability sensitivity, results, the reliability of the system is more sensitive regarding $\lambda_{m\text{-}1}$.

6. Conclusion

This work analyses the repair system supported by two subsystems A and B follow *k*-out-of-*n*: *F* policy and have warm standbys respectively. Some practical concepts such as reboot, unreliable repairman, and vacation are combined to enlarge and analyze the reliability measures of fault-tolerant systems. The differential equations of the probabilities for each state are derived with the help of a state transition diagram and the performance measures like availability, reliability, and MTTF are obtained by using the Laplace transformation. Likewise, sensitivity and relative sensitivity of reliability are done to determine how the reliability of the system is affected by each parameter. Some important results are below:

Availability of the system is greater when the repair rate has a greater value than the reboot rate. λ_{m-1} is most significant important to influence system reliability and λ_m is rarely important. Furthermore, the MTTF of the system is decreased rapidly with respect to λ_{k-1} and decreased uniformly with respect to λ_k. Also, for a higher value of λ_{s-1} and λ_s, it ultimately becomes almost constant. In the context of sensitivity of reliability, the sensitivity and relative sensitivity of reliability are quite close to each other with respect to λ_{s-1} and λ_s. Relative sensitivity of reliability affected by each parameter can be ranked as $\lambda_{k-1} > \lambda_{m-1} > \lambda_k > \lambda_{s-1} > \lambda_s > \lambda_m$.

The sensitivity analysis is carried out for the validation of the system performance measures. The sensitivity analysis conducted may be useful to the decision-makers for advance changes to the system being taken into consideration. The concepts of vacation and the unreliable server incorporated in our model can be observed frequently in daily life including in banking sectors, communication centers, power plants, hospitals, call centers, etc. In the future, we proceed to modify our model by considering the multi-server vacation policy with reboot delay action. Also, future research could focus on the development of optimization algorithms to determine the most effective vacation policies and reboot delay durations for specific system configurations.

Conflicts of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgments

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Appendix

Probability of the state that all units are good in the interval $(t, t+\Delta t)$ is given by (on the basis Figure 1).

$$P_0(t+\Delta t) = (1-\lambda_{k-1}\,\Delta t)(1-\lambda_{m-1}\,\Delta t)P_0(t) + \int_0^\infty \mu(z)\ P_4(z,t)\Delta t dz + \int_0^\infty \mu(z)\ P_{10}(z,t)\Delta t dz,$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} + (\lambda_{k-1}+\lambda_{m-1})P_0(t) = \int_0^\infty \mu(z)\ P_4(z,t)dz + \int_0^\infty \mu(z)\ P_{10}(z,t)dz.$$

Now taking $\lim_{\Delta t \to 0}$ we get

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} + (\lambda_{k-1}+\lambda_{m-1})P_0(t) = \int_0^\infty \mu(z)\ P_4(z,t)dz + \int_0^\infty \mu(z)\ P_{10}(z,t)dz,$$

$$\left[\frac{\partial}{\partial t} + \lambda_{k-1} + \lambda_{m-1}\right] P_0(t) = \int_0^\infty \mu(z)\ P_4(z,t)dz + \int_0^\infty \mu(z)\ P_{10}(z,t)dz \tag{1}$$

Probability of the state that k-1 units of subsystem A have failed is given by
$P_1(t+\Delta t) = (1-\lambda_k\, \Delta t)P_1(t) + \lambda_{k-1}\Delta t\, P_0(t)$,
$\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} + (\lambda_k)P_1(t) = \lambda_{k-1}\, P_0(t)$.

Now taking $\lim_{\Delta t \to 0}$ we get

$\lim_{\Delta t \to 0} \frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} + \lambda_k P_1(t) = \lambda_{k-1}\, P_0(t)$,

$$\left[\frac{\partial}{\partial t} + \lambda_k\right] P_1(t) = \lambda_{k-1}\, P_0(t) \tag{2}$$

Probability of the completely failed state when k units of subsystem A have been failed is given by
$P_2(x+\Delta x, t+\Delta t) = (1-a\eta(x)\,\Delta t)(1-b\eta(x)\,\Delta t)P_2(x,t)$,
$\frac{P_2(x+\Delta x,t+\Delta t)-P_2(x,t)}{\Delta t} + [a\eta(x)+b\eta(x)]P_2(x,t) = 0$.

Now taking $\lim_{\Delta x \to 0}$ *and* $\lim_{\Delta t \to 0}$, we get,

$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right] P_2(x,t) + [a\eta(x)+b\eta(x)]P_2(x,t) = 0$,

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + a\eta(x) + b\eta(x)\right] P_2(x,t) = 0 \tag{3}$$

Probability of the completely failed state when k units of subsystem A have been failed and the repairman is not returned immediately from vacation is given by
$P_3(y+\Delta y, t+\Delta t) = (1-\beta(y)\Delta t)P_3(y,t)$,

$$\frac{P_3(y+\Delta y, t+\Delta t) - P_3(y,t)}{\Delta t} + [\beta(y)]P_3(y,t) = 0.$$

Now taking $\lim_{\Delta y \to 0}$ *and* $\lim_{\Delta t \to 0}$, we get,
$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right] P_3(y,t) + \beta(y)P_3(y,t) = 0$,

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \beta(y)\right] P_3(y,t) = 0 \tag{4}$$

Probability of the completely failed state when k units of subsystem A have been failed and the repairman is returning immediately from vacation is given by
$P_4(z+\Delta z, t+\Delta t) = (1-\mu(z)\Delta t)P_4(z,t)$,

$\frac{P_4(z+\Delta z,t+\Delta t)-P_4(z,t)}{\Delta t} + \mu(z)P_4(z,t) = 0$.

Now taking $\lim_{\Delta z \to 0}$ *and* $\lim_{\Delta t \to 0}$, we get,

$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right] P_4(z,t) + \mu(z)P_4(z,t) = 0$,

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial t}+\mu(z)\right]P_4(z,t)=0 \tag{5}$$

Probability of the degraded state that *m*-1 operating units of subsystem B have failed is given by

$P_5(t+\Delta t)=(1-\lambda_m\,\Delta t)P_5(t)+\lambda_{m-1}\Delta t\,P_0(t)$,

$\frac{P_5(t+\Delta t)-P_5(t)}{\Delta t}+(\lambda_m)P_5(t)=\lambda_{m-1}\,P_0(t)$.

Now taking $\lim_{\Delta t\to 0}$, we get,

$\lim_{\Delta t\to 0}\frac{P_5(t+\Delta t)-P_5(t)}{\Delta t}+\lambda_m P_5(t)=\lambda_{m-1}\,P_0(t)$,

$$\left[\frac{\partial}{\partial t}+\lambda_m\right]P_5(t)=\lambda_{m-1}\,P_0(t) \tag{6}$$

Probability of the degraded state that *m* operating units of subsystem B have failed is given by

$P_6(t+\Delta t)=(1-\lambda_{s-1}\,\Delta t)P_6(t)+\lambda_m\Delta t\,P_5(t)$,

$\frac{P_6(t+\Delta t)-P_6(t)}{\Delta t}+(\lambda_{s-1})P_6(t)=\lambda_m\,P_5(t)$.

Now taking $\lim_{\Delta t\to 0}$, we get,

$\lim_{\Delta t\to 0}\frac{P_6(t+\Delta t)-P_6(t)}{\Delta t}+\lambda_{s-1}P_6(t)=\lambda_m\,P_5(t)$,

$$\left[\frac{\partial}{\partial t}+\lambda_{s-1}\right]P_6(t)=\lambda_m\,P_5(t) \tag{7}$$

Probability of the degraded state that *s*-1 operating units of subsystem B have failed is given by

$P_7(t+\Delta t)=(1-\lambda_s\,\Delta t)P_7(t)+\lambda_{s-1}\Delta t\,P_6(t)$,

$\frac{P_7(t+\Delta t)-P_7(t)}{\Delta t}+(\lambda_s)P_7(t)=\lambda_{s-1}\,P_6(t)$.

Now taking $\lim_{\Delta t\to 0}$, we get,

$\lim_{\Delta t\to 0}\frac{P_7(t+\Delta t)-P_7(t)}{\Delta t}+\lambda_s P_7(t)=\lambda_{s-1}\,P_6(t)$,

$$\left[\frac{\partial}{\partial t}+\lambda_s\right]P_7(t)=\lambda_{s-1}\,P_6(t) \tag{8}$$

Probability of the completely failed state when *s* units of subsystem B have been failed is given by

$P_8(x+\Delta x,t+\Delta t)=(1-a\eta(x)\,\Delta t)(1-b\eta(x)\,\Delta t)P_8(x,t)$,

$\frac{P_8(x+\Delta x,t+\Delta t)-P_8(x,t)}{\Delta t}+[a\eta(x)+b\eta(x)]P_8(x,t)=0$.

Now taking $\lim_{\Delta x\to 0}$ *and* $\lim_{\Delta t\to 0}$, we get,

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right]P_8(x,t)+[a\eta(x)+b\eta(x)]P_8(x,t)=0,$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+a\eta(x)+b\eta(x)\right]P_8(x,t)=0 \quad (9)$$

Probability of the completely failed state when *s* units of subsystem B have been failed and the repairman is not returning immediately from vacation is given by
$P_9(y+\Delta y,t+\Delta t)=(1-\beta(y)\Delta t)P_9(y,t)$,

$$\frac{P_9(y+\Delta y,t+\Delta t)-P_9(y,t)}{\Delta t}+[\beta(y)]P_9(y,t)=0.$$

Now taking $\lim_{\Delta y\to 0}$ *and* $\lim_{\Delta t\to 0}$, we get,

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}\right]P_9(y,t)+\beta(y)P_9(y,t)=0,$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta(y)\right]P_9(y,t)=0 \quad (10)$$

Probability of the completely failed state when *s* units of subsystem B have been failed and the repairman is returning immediately from vacation is given by
$P_{10}(z+\Delta z,t+\Delta t)=(1-\mu(z)\Delta t)P_{10}(z,t)$,

$$\frac{P_{10}(z+\Delta z,t+\Delta t)-P_{10}(z,t)}{\Delta t}+\mu(z)P_{10}(z,t)=0.$$

Now taking $\lim_{\Delta z\to 0}$ *and* $\lim_{\Delta t\to 0}$, we get,

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right]P_{10}(z,t)+\mu(z)P_{10}(z,t)=0,$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+\mu(z)\right]P_{10}(z,t)=0 \quad (11)$$

Initial condition

$$P_i(0)=\begin{cases}1 & i=0\\ 0 & i\neq 0\end{cases} \quad (12)$$

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