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A Lot Sizing Model for a Deteriorating Product with Shifting Production Rates, Freshness, Price, and Stock-Dependent Demand with Price Discounting

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Abstract
Many production systems need to be able to change the rate at which they manufacture products for various reasons, hence, the need to find the optimal lot size under these multiple levels of production. This research addresses the need for optimizing inventory in a system with a shifting production rate and other challenging product characteristics such as product deterioration with limited life span, and product demand that is dependent on the stock level, the state of freshness of the product, and the selling price. The product also needs to be discounted as it gets close to the expiry date in order to boost demand and prevent wastage beyond its life span. Our objective is to maximize profit by determining the optimal selling price and inventory cycle time by deriving the relevant equations for these decision variables. The Newton-Raphson method was used to numerically solve for the optimal values of these variables. Sensitivity analyses were performed to derive useful insights for managerial decision-making.

Keywords- Economic production quantity, Inventory management, Deteriorating product, Product freshness, Variable production rate.

1. Introduction

The manufacturing industry is constantly striving to enhance its profitability and operational efficiency, and inventory management is an important practice for such. However, traditional approaches often fall short in promptly adapting to market dynamics and optimizing decision-making. In the complex world of supply chain, managing production systems has become increasingly crucial for industries aiming to enhance their flexibility, adaptability and agility. The classical Economic Order Quantity (EPQ) model assumes that the demand rate is constant, the production rate is constant, and the product is of perfect quality. Companies, however, frequently stimulate demand by offering attractive prices to customers. Also, the state of the machine may affect the production rate, the quality of the products, and the efficiency of the manufacturing process. Unlike traditional production models that assume fixed production rates, flexibility in production systems is essential for improving responsiveness, particularly in industries where adaptability and efficiency are paramount. As industries continue to transition towards more agile and adaptable production systems, this research recognizes the growing need to accommodate dynamic production rates. It acknowledges the specific challenges presented by perishable products that require a responsive approach to production. Companies must, therefore, have a well-crafted production and inventory strategy that maximizes the overall profit by balancing the need to sell the products in volumes against the risk of having unsold inventory that may need to be discarded. Various characteristics of the production-inventory system have been investigated by many authors, and this is discussed next, starting with recent works in product deterioration.

Effective management of deteriorating inventory items is critical for manufacturers and retailers. Due to the perishable nature of these items, stock holders monitor the levels of inventory to prevent losses due to spoilage. Careful evaluation of costs associated with deterioration stands out as a defining characteristic of these inventory systems. Managing such products has resulted in significant research within the area of inventory control. Pal et al. (2015) presented an inventory model for deteriorating items experiencing fluctuating demand in a fuzzy environment and incorporated the impact of inflation and shortages in the model. Wu et al. (2016) developed models for deteriorating items having a lifespan within a supplier-retailer-customer chain. Viji and Karthikeyan (2018) proposed a production-inventory model for deteriorating items with three levels of production where the rate of deterioration follows a two-parameter Weibull distribution. Sepehri et al. (2021) proposed a sustainable inventory model for deteriorating products with both quality and environmental concerns. Duary et al. (2022) discussed an inventory problem for deteriorating items that integrates the concepts of advance and delay payment. The model also incorporated the impact of advertisements on product and partial backlog. Lu et al. (2022) examined the implications of various carbon emission policies on the optimal production-inventory decisions for deteriorating items. Salas-Navarro et al. (2023) proposed a vendor managed inventory model for deteriorating items with a three-layer supply chain. Tiwari et al. (2022) explored how the inventory of imperfect quality items is affected by deterioration and trade credit policy. Mahlangu et al. (2023) presented a model for two deteriorating mutually complementary items with time dependent demand.

Apart from deterioration, price is another key factor that affects lot sizing decisions as it significantly influences a consumer's willingness to consume a product or service, hence the demand, and several researchers have investigated the dependence of demand on price. Chen et al. (2019) presented an inventory model for a product with short life cycle that deteriorates in a finite horizon, multi-period setting, stock-level-dependence, with time- and price-dependent deterministic demand. Khan et al. (2020) presented two inventory models for perishable items with linearly time-dependent increasing holding costs, and demand that depends on its selling price and the frequency of advertisement. Halim et al. (2021) discussed an EPQ model with nonlinear price structure and stock dependent demand, while also accounting for the possibility of overtime production opportunities. Torkaman et al. (2022) introduced a Mixed-Integer Nonlinear Program (MINLP) to address the Production-Routing Problem with Price-Dependent demand (PRP-PD). Akhtar et al. (2023) presented an inventory model with price dependent demand that maximizes the retailer's total profit over a finite time horizon.

The freshness condition of products is another factor that significantly impacts the economic prospects of manufacturers and retailers. Freshness of products such as meat, tomatoes, or vegetables, is a crucial factor in determining their quality and safety for consumption. Consumers prefer fresh products due to their better taste, texture, and nutritional value. People typically do not buy products when they are close to their expiry date for several reasons. Firstly, expired products are potential health hazards due to the growth of harmful bacteria or the breakdown of active ingredients. This can be particularly concerning in the food industry. Secondly, some expired products like pharmaceutical products may lose their effectiveness or become unsafe to consume, leading to health risks and waste of money. Thirdly, the perception of expiring products being of lower quality or value may also deter consumers from purchasing them. This can be particularly true for products that have a short shelf-life. As a result, manufacturers and retailers may be forced to discount or dispose of them, leading to financial losses and waste. Food and pharmaceutical safety regulations are in place to ensure that products are safe for consumption, and freshness is a key factor in determining the safety of the products consumed. Fujiwara and Perera (1993) were the first to investigate the impact of utility deterioration, specifically declining freshness, on inventory management for perishable products. They used an exponential penalty cost function to model the deterioration. Cardello and Schutz (2002) conducted an analysis of the various factors associated with the freshness condition of food products and its significance in the food industry. Bai and Kendall (2008) presented a model that manages a deteriorating inventory and shelf space of fresh produce in a single period, assuming that the demand rate is deterministic and dependent on the level of inventory displayed and the freshness condition of the item. Piramuthu and Zhou (2013) extended Bai and Kendall's by linking the demand directly to the amount of shelf-space allotted to the specific item and its current quality, using auto-identification technology like Radio Frequency Identification (RFID), which includes the necessary sensors that generate information on an item. Sebatjane and Adetunji (2020) presented a multi-echelon lot sizing model of a growing item, where the demand is dependent on the price and freshness condition of the item. Banerjee and Agrawal (2017) proposed an inventory model that considers the demand for deteriorating items, which initially depends only on its selling price but later also on its freshness condition.

In addition to acknowledging the critical role of price and freshness, it is equally important for companies to consider the impact of stock levels on demand. The interplay between these factors, along with the shelf life of products, can significantly influence the demand patterns and the overall inventory management strategies of companies. Önal et al. (2016) presented an EOQ model that incorporates product assortment, pricing, and storage capacity (shelf-space and backroom) for perishable products, where the demand rate is dependent on the selling price and the level of stock displayed. Li and Teng (2018) presented an integrated pricing and lot-sizing model for deteriorating products. The model considers various factors such as the selling price, reference price, freshness, and the level of stock displayed, all of which influence the demand for the products. Sebatjane and Adetunji (2022) also considered a coordinated multi-echelon system for a growing item inventory system with product expiration. Alakan et al. (2019) proposed a three-level supply chain model to analyse the dynamics and interactions within an interconnected supply chain structure under finite production rates and stock dependent demand. Agi and Soni (2020) proposed a model that jointly optimizes the price and lot size for perishable products considering both physical deterioration and freshness degradation, where the demand is influenced by factors such as the product's age, the price and the stock level. Priyan and Mala (2020) presented a game theoretic approach for managing an inventory system for materials having varying quality characteristics both as a raw material and finished products in a pharmaceutical supply chain. The study also assumed that the finished product's demand is age and freshness dependent. Vahdani and Sazvar (2022) examined the coordinated dynamic pricing and inventory control for online retailers while considering social learning and the Expiration Date-Based Pricing (EDBP) policy.

Thus far, the discussion has been around the product characteristics related to its demand, price, stock level and deterioration as key factors in managing the production-inventory system. However, it is crucial to acknowledge that the nature of the manufacturing process itself is important. The manufacturing process plays a significant role in shaping the overall efficiency and effectiveness of the production-inventory system. In many production systems, different machines or processes within the system can operate at varying rates, usually called the multi-state mode of production. This capability provides the system with the flexibility needed to adjust production levels, optimize resource utilization, and improve responsiveness to changing market conditions. Manufacturing plants often make the decision to scale production up or down due to a variety of factors. One significant reason for scaling the production and operating in a multistage mode is the fluctuation in the demand for their products; when the demand changes, like seasonal variations, it becomes necessary to align the production output with the market requirements. By doing so, businesses can prevent the negative consequences of high holding costs due to a rapid inventory buildup. Another key factor that may influence many organizations to scale the production is the need to flexibly allocate their resources effectively, leading to a more efficient utilization of operational expenses, including labor costs, energy consumption, and raw material usage. This allows them to better manage their inventory levels and avoid unnecessary expenses. Power consumption may be a good reason to scale the production rate in some other production system and shift between production levels. In situations where there is a shortage or limited availability of power supply, companies may need to scale down their production rate to ensure they operate within the existing energy constraint imposed by the power utilities. Another motivation for scaling production is the focus on quality control and process improvements. Gupta and Arora (2010) examined a production inventory system that considers alternating production rates caused by market fluctuations and other relevant constraints to satisfy various demand patterns. Bhowmick and Samanta (2011) presented a continuous production control inventory model with variable production rates, allowing for a switch between production rates during the production process to maintain a certain level of manufactured items at the initial stages. Sivashankari and Panayappan (2015) considered a production inventory model that considers deteriorating items and utilizes two different production rates to avoid excessive inventory and enhance consumer satisfaction while maximizing profit. Mishra (2018) presented a production-inventory model that incorporates three different rates and focuses on deteriorating products. In this model, the demand rate is influenced by both the advertising cost and selling price. The production rates, which are assumed to be finite, are directly proportional to the demand rate. To address production challenges similar to those experienced during the Covid-19 pandemic, Malumfashi et al. (2022) proposed a model for delayed deteriorating items having a two-phase production period, and a holding cost function that is linearly increasing with time.

While fluctuating demand, seasonality, or power consumption have been presented as common reasons for companies to scale the production output, another factor that can influence the decision to shift between production levels is the state of the manufacturing equipment, especially for systems that operate with degrading or deteriorating machines. This is because even when demand remains stable and power supply is consistent, a variety of issues can arise with the equipment used in the production process. During manufacturing, a production process may deteriorate when a machine fails to function properly or stops working altogether, and this may be caused by many factors including a lack of maintenance, wear and tear on equipment, human errors, overuse, aging and condition of the equipment, improper use, poor quality materials, environmental factors, and unforeseen circumstances or randomness. This is usually referred to as process deterioration. Deterioration of a process can be a major challenge for manufacturing operations, causing production delays and downtime that can lead to missed deadlines, decrease in product quality, increased waste, lost business opportunities, increased costs, and decreased efficiency. Numerous researchers have explored the impacts of manufacturing efficiency, reliability, process availability and preventive maintenance in recent years (Rahim and Ben-Daya, 2001; Llaurens, 2011). While there are tools such as predictive maintenance techniques that can help to identify potential issues before they occur, there is no foolproof method to predict when a breakdown will happen. When machines are not functioning optimally, it may become necessary to shift to a lower production rate and avoid further damage or unplanned repairs. By shifting to lower production rates, manufacturing plants can dedicate more attention and resources to identifying and rectifying any issues that may arise within their production lines, thereby ensuring efficient use of resources. Khouja and Mehrez (1994) presented an extension of the EPQ model that incorporates situations where the production rate becomes a decision variable, and also accounts for degradation in the quality of the production process that occurs as the production rate increases. Kenne and Nkeungoue (2008) proposed a homogenous Markov process that incorporates the hedging point policy for machines experiencing both failures and repairs. This model takes into account the age-dependent nature of the machine, which affects the occurrence of failures and subsequent repairs. Ben-Daya et al. (2008) studied an EPQ model with a shifting production rate under stoppages due to speed losses. They demonstrated that process deterioration could be the result of minor stoppages and speed losses, which in practice may affect the efficiency of the process. Tshinangi et al. (2022) conducted a study focusing on a degrading production system that incorporates shifting production rates, imperfect quality, and partial backlogging of demand along with lost sales to understand the impact of these factors on system performance, such as inventory levels, cycle time, shortage levels and overall cost incurred.

1.1 Research Contribution

The research gap addressed in this study is presented in Table 1. The table lists inventory models from various previous authors, highlighting the different factors considered and what this paper contributes to the research on lot sizing models with varying production rates. Current literature review indicates the need for a model for deteriorating inventory items that considered imperfect systems with shifting production rates, freshness, price and stock-dependent demand, as well as price discounting.

Table 1. Analysis of related literature.

Characteristics of the inventory models							
Authors	**EOQ/EPQ models**	**Production**	**Imperfect quality**	**Deterioration**	**Demand**	**Discount**	**Freshness**
Ben-Daya et al. (2008)	EPQ	Variable	No	No	Constant	No	No
Bhowmich & Samanta (2011)	EPQ	Variable	No	Exponential	Constant	No	No
Banerjee and Agrawal (2017)	EOQ	No	No	Weibull and Exponential functions	Price dependent	Yes	Yes
Viji and Karthikeyan (2018)	EPQ	Variable	No	Weibull	Constant	No	No
Agi and Soni (2020)	EOQ	No	No	Constant	Age, stock, and price dependent	No	Yes
Tshinangi et al. (2022)	EPQ	Variable	Yes	Exponential	Constant	No	No
Salas-Navarro et al. (2023)	EPQ	Constant	No	Constant	Time dependent	No	No
This paper	EPQ	Variable	Yes	Exponential	Price and stock dependent	Yes	Yes

This paper introduces an extended inventory model for deteriorating items that specifically incorporates the influence of product freshness on demand. The model builds upon the work of Tshinangi et al. (2022) and extends this research by considering the dynamic relationship between the concept of shifting production rates in a deteriorating process and both freshness condition and deterioration of products, guided by Banerjee and Agrawal (2017). Initially, the demand for the product (e.g., meat) is determined solely by its selling price and its displayed stock level when it is fresh. As freshness declines, demand then depends on the product's freshness condition. Also, there is a shift in production rate as production continues. Since fresher products are more attractive to buyers than those that appear stale, discounts are applied once the freshness has declined over a certain period. Our contribution to the existing literature is the development of an EPQ model for a multi-state production system with imperfect quality of manufactured product, that accounts for the impact of the freshness of products on its demand, where deterioration of both the product and the process are allowed. The model also considers using sales discounts to drive demand as the product gets close to its expiry date in such manner that the net profit is maximized over the entire cycle length. It can be seen from Table 1 that currently, there is no model that has been proposed, that considers all these characteristics of a product over its life cycle, and where the items could be manufactured in more than one state of operation of the resource. The model can find application in the food processing environment, like food canning.

The structure of the remainder of this paper is now discussed. In section 2, the notations necessary for the development of the inventory model are outlined. Section 3 focuses on the formulation of the inventory model, considering the shift in production, freshness-, stock- and price-dependent demand, and the inclusion of deterioration, discount and imperfect production. In section 4, numerical examples are solved to provide practical illustrations. A sensitivity analysis is conducted and observations are discussed in section 5. Finally, sections 6 and 7 offer managerial insights and conclusions respectively, followed by the discussion of potential opportunities for future research.

2. Notations and Assumptions

The following notations are utilized in this paper as shown in Table 2.

Table 2. Notations adopted in the model.

Symbols	Description
C_d	Deterioration cost per unit item
C_{dp}	Disposal cost per unit item
$d_{1,2}$	Proportion of defective units produced
$D(S_p, I(t), t)$	The demand for the product at time t
h_p	The inventory carrying cost per unit item per time
$I(t)$	Instantaneous inventory level
$k_{1,2}$	Initial production rate at the start of the cycle, and production rate following the shift respectively
$p_{c1,}p_{c2}$	The unit production cost at the start and after the machine's production rate has been scaled down respectively
Q^*	Optimal batch size
QD_P	Quantity of deteriorated products
S_C	The fixed setup cost
S_p	The market selling price of the product.
T	Duration of the cycle
TC	Total cost
t_1	The time during the cycle when the production rate shifts from k_1 to k_2
t_2	The total production time per cycle
t_3	The time from which the deterioration of the product begins
t_4	The time from which price discount is offered
$\theta(t)$	Deterioration rate
γ	The parameter indicating the sensitivity of demand to the instantaneous level of inventory
ρ_1, ρ_2	Aggregation parameters for some known variables
A	Demand parameter
b	The elasticity of the unit selling price
β	The freshness parameter
α	Discount percentage offered on selling price
n	The shelf-life of the product

Our model is based on the following assumptions:

- The inventory procedure is for a single product.
- At the start of the process, a production rate of k_1 is employed. After a time, t_1, the decision maker switches to a lower production rate of k_2.
- Some manufactured products are accidentally damaged (or contaminated) and have to be discarded as scrap during each of the two production phases at constant rates d_i with $i \in \{1,2\}$.
- The manufactured product is subject to deterioration. The deterioration function is of the form:

$$\theta(t) = \begin{cases} \theta e^{-\theta t}, & \text{for } t \geq t_3 \\ 0 & , \text{otherwise} \end{cases} \tag{1}$$

with $\theta \geq 0$.

- The product has a maximum shelf-life, n, beyond which its perceived value is lost.

The freshness decreases linearly from a particular time, $t_3 < n$, following the function, $\varphi(t)$, similar to the model proposed by Banerjee and Agrawal (2017), where,

$$\varphi(t)\begin{cases} = 1 & \text{if } t < t_3 \\ = \frac{[n-\beta(t-t_2)]}{n} & \text{if } t \geq t_3 \end{cases} \tag{2}$$

This implies that while production ends at t_2, products are still considered fresh for a length of time until time t_3, from when its freshness starts to decline until when it is considered unacceptable after the shelf life is reached. There has been no decline in quality up until t_3, and the freshness function $\varphi(t)$ will be equal to 1 for $t \in [0, t_3]$.

Demand for the product is price-, stock-, and freshness-dependent and is represented as follows:

$$D\left(S_p, I(t), \varphi(t)\right)\begin{cases} A - bS_p + \gamma I(t) & 0 < t \leq t_3 \\ \left(A - bS_p\right)\varphi(t) & t_3 < t \leq t_4 \\ \left[A - b(1-\alpha)S_p\right]\varphi(t) & t_4 < t \leq T \end{cases} \tag{3}$$

with $A, b,$ and $\gamma \neq 0$.

3. Process Description and Model Formulation

Figure 1 illustrates the changes in inventory level throughout the cycle.

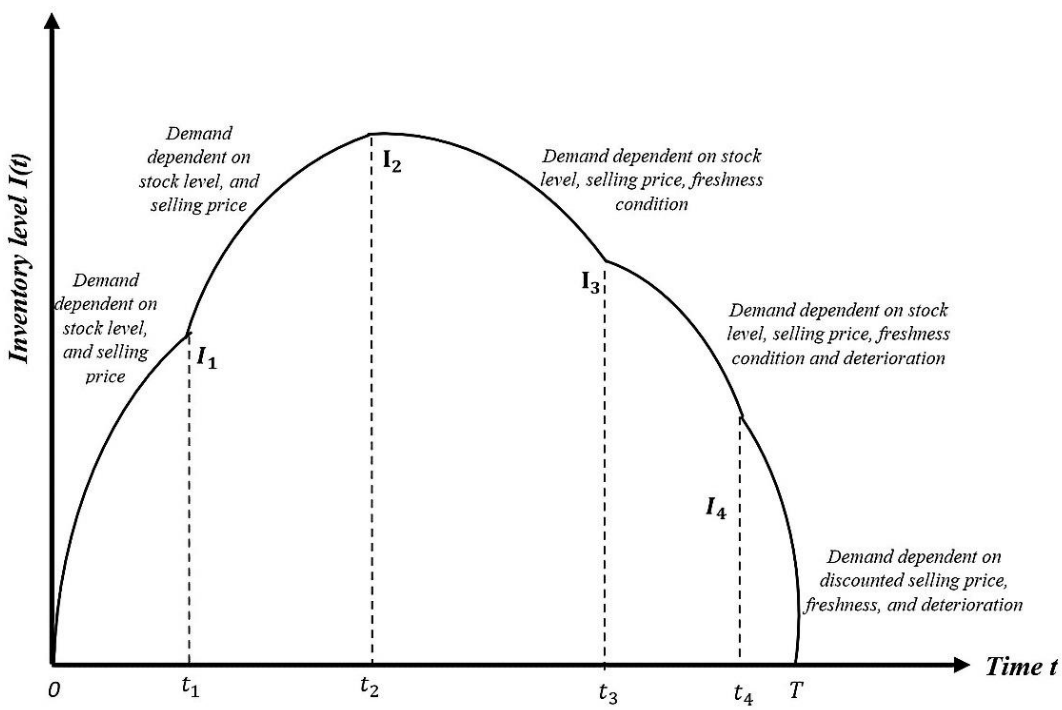

Figure 1. Inventory profile with a shift in production rate.

At the beginning of the production cycle, the product is manufactured at a production rate k_1, and inventory is built up until when it reaches the level I_1 at time t_1. During the interval $[0, t_1]$, the product is considered completely fresh, and the inventory is withdrawn solely due to demand. The demand itself is influenced by two main factors: the level of stock displayed to customers and the selling price of the product. In the interval $[0, t_1]$, it is assumed that some manufactured products are damaged and taken away at a rate d_1. At time t_1, the operator scales down the machine and continues the production at a rate k_2 until time t_2, during which the inventory reaches its maximum level, I_2. In the interval $[t_1, t_2]$, inventory continues to be withdrawn due to demand, which still depends on both the stock level and the selling price of the product, while the damaged are taken away at the rate d_2. At time t_2, the system stops production.

During the interval $[t_2, t_3]$, the inventory continues to deplete due to demand. After t_3, the freshness of the product begins to decline, product deterioration starts, and inventory depletion occurs due to both demand and deterioration. The demand function now depends on both the selling price and the freshness of the product, but no longer on the level of stock displayed. To stimulate demand, a fixed discount of $\alpha\%$ is offered on the selling price starting from time t_4. The inventory level hits zero at time T. The differential equations that govern the inventory situations in the interval $[0, T]$ are as follows:

$$\frac{dI(t)}{dt} = (1 - d_1)k_1 - \left[A - bS_p + \gamma I(t)\right] \qquad 0 \le t \le t_1 \tag{4}$$

$$\frac{dI(t)}{dt} = (1 - d_2)k_2 - \left[A - bS_p + \gamma I(t)\right] \qquad t_1 \le t \le t_2 \tag{5}$$

$$\frac{dI(t)}{dt} = -\left[A - bS_p + \gamma I(t)\right] \qquad t_2 \le t \le t_3 \tag{6}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -\left[A - bS_p\right]\varphi(t) \qquad t_3 \le t \le t_4 \tag{7}$$

$$\frac{dI(t)}{dt} + \theta I(t) = -\left[A - b(1-\alpha)S_p\right]\varphi(t) \qquad t_4 \le t \le T \tag{8}$$

Solving Equation (4), we obtain:

$$I(t) = \frac{\left[(1-d_1)k_1 - A + bS_p\right]}{\gamma} + L_1 e^{-\gamma t} \tag{9}$$

From Equation (9) under the boundary condition, $I(0) = 0$ we obtain:

$$L_1 = -\frac{\left[(1-d_1)k_1 - A + bS_p\right]}{\gamma} \tag{10}$$

Substituting Equation (10) into Equation (9) results in:

$$I(t) = \frac{\rho_1}{\gamma}\left[1 - e^{-\gamma t}\right] \qquad 0 \le t \le t_1 \tag{11}$$

with

$$(1 - d_1)k_1 - A + bS_p = \rho_1 \tag{12}$$

Linearizing the exponential terms containing t in Equation (11) by using Taylor's series expansion for $e^{-\gamma t}$ leads to the following:

$$e^{-\gamma t} = \sum_{m=1}^{\infty} \frac{(-1)^m \gamma^m t^m}{m!} = 1 - \frac{\gamma t}{1} + \frac{\gamma^2 t^2}{2!} - \frac{\gamma^3 t^3}{3!} + \frac{\gamma^4 t^4}{4!} \approx 1 - \gamma t \tag{13}$$

Substituting Equation (13) into Equation (11) yields,

$$I(t) = \rho_1 t \qquad 0 \le t \le t_1 \tag{14}$$

Solving differential Equation (5) yields:

$$I(t) = \frac{[(1-d_2)k_2 - A + bS_p]}{\gamma} + L_2 e^{-\gamma t} \tag{15}$$

From Equation (15) under the boundary condition, $I(t_1) = I_1$, we obtain:

$$L_2 = \left[I_1 - \frac{(1-d_2)k_2 - A + bS_p}{\gamma}\right] e^{\gamma t_1} \tag{16}$$

By substituting Equation (16) into Equation (15), we derive:

$$I(t) = \frac{\rho_2}{\gamma} + \left[I_1 - \frac{\rho_2}{\gamma}\right] e^{-\gamma(t-t_1)} \qquad t_1 \le t \le t_2 \tag{17}$$

with

$$(1-d_2)k_2 - (A - bS_p) = \rho_2 \tag{18}$$

Utilizing Taylor's series expansion for $e^{-\gamma(t-t_1)}$ to linearize the exponential terms near $t = t_1$ in Equation (17) yields:

$$I(t) = I_1 - (\gamma I_1 - \rho_2)(t - t_1) \qquad t_1 \le t \le t_2 \tag{19}$$

The solution of the differential Equation (6) is:

$$I(t) = -\frac{(A - bS_p)}{\gamma} + L_3 e^{-\gamma t} \tag{20}$$

From Equation (20) under the boundary condition, $I(t_2) = I_2$, we obtain:

$$L_3 = \left[I_2 + \frac{A - bS_p}{\gamma}\right] e^{-\gamma t_2} \tag{21}$$

Substituting Equation (21) into Equation (20) leads to:

$$I(t) = -\frac{A - bS_p}{\gamma} + \left[I_2 + \frac{A - bS_p}{\gamma}\right] e^{-\gamma(t-t_2)} \qquad t_2 \le t \le t_3 \tag{22}$$

Again, linearizing the exponential term $e^{-\gamma(t-t_2)}$ to near $t = t_2$ in Equation (22) utilizing Taylor's series expansion yields:

$$I(t) = I_2 - (\gamma I_2 + A - bS_p)(t - t_2) \qquad t_2 \le t \le t_3 \tag{23}$$

The solution to differential Equation (7) is obtained as follows:

$$I(t) = -(A - bS_p)\left[\frac{[n - \beta(t-t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right] + L_4 e^{-\theta t} \tag{24}$$

On solving Equation (24) under the boundary condition $I(t_3) = I_3$, we obtain the following:

$$L_4 = \left\{I_3 + (A - bS_p)\left[\frac{[n - \beta(t_3-t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right]\right\} e^{\theta t_3} \tag{25}$$

Substituting Equation (25) back into Equation (24) leads to:

$$I(t) = -(A - bS_p)\left[\frac{\varphi(t)}{\theta} + \frac{\beta}{n\theta^2}\right] + \left[I_3 + (A - bS_p)\left[\frac{\varphi(t_3)}{\theta} + \frac{\beta}{n\theta^2}\right]\right] e^{-\theta(t-t_3)} \qquad t_3 \le t \le t_4 \tag{26}$$

with

$$\frac{[n - \beta(t_3-t_2)]}{n} = \varphi(t_3) \tag{27}$$

Linearizing the exponential terms $e^{-\theta(t-t_3)}$ in Equation (27) by applying Taylor's series expansion near $t = t_3$, results in the following:

$$I(t) = I_3 - \theta\{I_3 + (A - bS_p)\varphi(t_3)\}(t - t_3) \qquad t_3 \leq t \leq t_4 \tag{28}$$

The solution to differential Equation (8) is:

$$I(t) = -[A - b(1-\alpha)S_p]\left[\frac{[n-\beta(t-t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right] + L_5 e^{-\theta t} \qquad t_4 \leq t \leq T \tag{29}$$

Using the boundary condition, $I(t_4) = I_4$ in Equation (29), we obtain L_5 as follows:

$$L_5 = \left[I_4 + [A - b(1-\alpha)S_p]\left[\frac{\varphi(t_4)}{\theta} + \frac{\beta}{n\theta^2}\right]\right]e^{\theta t_4} \tag{30}$$

Substituting Equation (30) back into Equation (29) leads to:

$$I(t) = -[A - b(1-\alpha)S_p]\left[\frac{[n-\beta(t-t_2)]}{n\theta} + \frac{\beta}{n\theta^2}\right] + \left[I_4 + [A - b(1-\alpha)S_p]\left[\frac{\varphi(t_4)}{\theta} + \frac{\beta}{n\theta^2}\right]\right]e^{-\theta(t-t_4)} \qquad t_4 \leq t \leq T \tag{31}$$

Linearizing the exponential term $e^{-\gamma(t-t_4)}$ to near $t = t_4$ in equation (31) utilizing Taylor's series expansion yields:

$$I(t) = I_4 - \theta\{I_4 + [A - b(1-\alpha)S_p]\,\varphi(t_4)\}(t - t_4) \qquad t_4 \leq t \leq T \tag{32}$$

3.1 Set Up Cost (TSC)

The total set up cost is given by the following function

$$SC = \sum_{i=1}^{2} S_{c_i} \tag{33}$$

3.2 Holding Cost (THC)

We can express the total holding cost using the following function:

$$THC = h_P\left[\int_0^{t_1} I(t)dt + \int_{t_1}^{t_2} I(t)dt + \int_{t_2}^{t_3} I(t)dt + \int_{t_3}^{t_4} I(t)dt + \int_{t_4}^{T} I(t)dt\right] \tag{34}$$

$$THC = \frac{1}{2}h_P[\,I_1t_2 + I_2(t_3 - t_1) + I_3(t_4 - t_2) + I_4(T - t_3)] \tag{35}$$

3.3 Cost of Deterioration (TCD)

The cost of deterioration during the entire cycle is

$$TCD = C_d\theta\left[\int_{t_3}^{t_4} I(t)dt + \int_{t_4}^{T} I(t)dt\right] \tag{36}$$

$$TCD = C_d\theta\left[\frac{1}{2}[I_3 + I_3 - \theta[I_3 + (A - bS_p)\varphi(t_3)\,](t_4 - t_3)](t_4 - t_3) + \frac{1}{2}I_4(T - t_4) + \frac{1}{2}[I_4 - \theta[I_4 + [A - b(1-\alpha)S_p]\,\varphi(t_4)](T - t_4)](T - t_4)\right] \tag{37}$$

$$TCD \approx C_d\theta\left[\frac{1}{2}I_3(t_4 - t_3) + \frac{1}{2}(T - t_3)I_4\right] \tag{38}$$

3.4 Production Cost (PC)

The production cost incurred throughout the entire cycle [0, T] is:

$$PC = p_{c1}k_1t_1 + p_{c2}k_2(t_2 - t_1) \tag{39}$$

3.5 Disposal Cost (DSC)

The disposal cost is given by:

$$DSC = c_{dp}[d_1 k_1 t_1 + d_2 k_2 (t_2 - t_1)] \quad (40)$$

3.6 Total Cost (TC)

The total cost incurred per time is expressed as follows:

$$TC = \frac{1}{T}\Big\{\sum_{i=1}^{2} S_{c_i} + \frac{1}{2} h_P [\, I_1 t_2 + I_2(t_3 - t_1) + I_3(t_4 - t_2) + I_4(T - t_3)] + C_d\theta\left[\frac{1}{2} I_3(t_4 - t_3) + \frac{1}{2}(T - t_3) I_4\right] + [p_{c1} + d_1 c_{dp}]k_1 t_1 + [p_{c2} + d_2 c_{dp}]k_2(t_2 - t_1)\Big\} \quad (41)$$

3.7 Total Revenue (TR)

The average total revenue earned during the interval $[0, T]$ is:

$$TR = \frac{S_p}{T}\Big\{\int_0^{t_1} [A - bS_p + \gamma I(t)]dt + \int_{t_1}^{t_2} [A - bS_p + \gamma I(t)]dt + \int_{t_2}^{t_3} [A - bS_p + \gamma I(t)]dt + \int_{t_3}^{t_4} [A - bS_p]\varphi(t)dt + (1-\alpha)\int_{t_4}^{T} [A - b(1-\alpha)S_p]\varphi(t)dt\Big\} \quad (42)$$

$$TR = \frac{S_p}{T}\Big[\left[(A - bS_p) + \frac{1}{2}\gamma I_1\right] t_1 + \left[(A - bS_p) + \frac{1}{2}\gamma(I_1 + I_2)\right](t_2 - t_1) + \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right](A - bS_p + \gamma I_2)(t_3 - t_2) + \left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3)(A - bS_p) + (1-\alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)[A - b(1-\alpha)S_p]\Big] \quad (43)$$

Thus, the net profit earned per cycle is:

$$NP(S_p, T) = TR - TC \quad (44)$$

$$NP(S_p, T) = \frac{S_p}{T}\Big[\left[(A - bS_p) + \frac{1}{2}\gamma I_1\right] t_1 + \left[(A - bS_p) + \frac{1}{2}\gamma(I_1 + I_2)\right](t_2 - t_1) + \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right](A - bS_p + \gamma I_2)(t_3 - t_2) + \left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3)(A - bS_p) + (1-\alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)[A - b(1-\alpha)S_p]\Big] - \frac{1}{T}\Big[\sum_{i=1}^{2} S_{c_i} + \frac{1}{2} h_P [\, I_1 t_2 + I_2(t_3 - t_1) + I_3(t_4 - t_2) + I_4(T - t_3)] + C_d\theta\left[\frac{1}{2} I_3(t_4 - t_3) + \frac{1}{2}(T - t_3) I_4\right] + [p_{c1} + d_1 c_{dp}]k_1 t_1 + [p_{c2} + d_2 c_{dp}]k_2(t_2 - t_1)\Big] \quad (45)$$

The decision variables in this problem are the cycle time, T, and selling price, S_p, hence the optimization problem becomes:

$$Max_{S_p^*, T^*} NP(S_p, T) \quad (46)$$

This problem can be solved by solving the grad functions:

$$\frac{\partial NP(S_p, T)}{\partial T} = \frac{S_p}{T}\left[-\frac{\beta}{2n}(1-\alpha)(T - t_4)[A - b(1-\alpha)S_p] + (1-\alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right][A - b(1-\alpha)S_p]\right] - \frac{1}{T}\left[\frac{1}{2} h_P I_4 + \frac{1}{2} C_d \theta I_4\right] - \frac{1}{T} NP(S_p, T) \quad (47)$$

$$\frac{\partial NP(S_p,T)}{\partial S_p} = \frac{1}{T}\left[\left[\left(A - bS_p\right) + \frac{1}{2}\gamma I_1\right] t_1 + \left[\left(A - bS_p\right) + \frac{1}{2}\gamma(I_1 + I_2)\right](t_2 - t_1) + \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right]\left(A - bS_p + \gamma I_2\right)(t_3 - t_2) + \left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3)\left(A - bS_p\right) + (1 - \alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)\left[A - b(1 - \alpha)S_p\right]\right] + \frac{S_p}{T}\left[\left[-b + \frac{1}{2}\gamma\frac{\partial I_1}{\partial S_p}\right] t_1 + \left[-b + \frac{1}{2}\gamma\left(\frac{\partial I_1}{\partial S_p} + \frac{\partial I_2}{\partial S_p}\right)\right](t_2 - t_1) + \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right]\left(-b + \gamma\frac{\partial I_2}{\partial S_p}\right)(t_3 - t_2) - b\left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3) - b(1 - \alpha)(1 - \alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)\right] - \frac{1}{2T}\left[h_P\left[\frac{\partial I_1}{\partial S_p}t_2 + \frac{\partial I_2}{\partial S_p}(t_3 - t_1) + \frac{\partial I_3}{\partial S_p}(t_4 - t_2) + \frac{\partial I_4}{\partial S_p}(T - t_3)\right] + C_d\theta\left[\frac{\partial I_3}{\partial S_p}(t_4 - t_3) + \frac{\partial I_4}{\partial S_p}(T - t_3)\right]\right] \quad (48)$$

The functions in Equations (47) and (48) are highly nonlinear, making it challenging to derive a closed-form analytical proof. Nonetheless, it is possible to numerically demonstrate the concavity of the profit function by establishing its positive (semi)definiteness using the Hessian matrix presented in Equation (49) while also satisfying the conditions presented in Equation (50):

$$H_M = \begin{bmatrix} \frac{\partial^2 NP(S_p,T)}{\partial S_p^2} & \frac{\partial^2 NP(S_p,T)}{\partial S_p \partial T} \\ \frac{\partial^2 NP(S_p,T)}{\partial T \partial S_p} & \frac{\partial^2 NP(S_p,T)}{\partial T^2} \end{bmatrix} \geq 0 \quad (49)$$

$$\frac{\partial^2 NP(S_p,T)}{\partial S_p^2} \leq 0, \frac{\partial^2 NP(S_p,T)}{\partial T^2} \leq 0 \quad (50)$$

Taking the second-order derivatives of $NP(S_p, T)$ in (45) with respect to S_p and T, we found the expression of the Hessian as

$$\frac{\partial^2 NP(S_p,T)}{\partial S_p^2} = \frac{2}{T}\left[\left[-b + \frac{1}{2}\gamma\frac{\partial I_1}{\partial S_p}\right] t_1 + \left[-b + \frac{1}{2}\gamma\left(\frac{\partial I_1}{\partial S_p} + \frac{\partial I_2}{\partial S_p}\right)\right](t_2 - t_1) + \left[1 - \frac{1}{2}\gamma(t_3 - t_2)\right]\left(-b + \gamma\frac{\partial I_2}{\partial S_p}\right)(t_3 - t_2) - b\left[\frac{2n - \beta(t_4 + t_3) + 2\beta t_2}{2n}\right](t_4 - t_3) - b(1 - \alpha)^2\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right](T - t_4)\right] \quad (51)$$

$$\frac{\partial^2 NP(S_p,T)}{\partial S_p \partial T} = \frac{\partial^2 NP(S_p,T)}{\partial T \partial S_p} = \frac{1}{T}\left[-\frac{\beta}{2n}(1 - \alpha)(T - t_4)\left[A - b(1 - \alpha)S_p\right] + (1 - \alpha)\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right]\left[A - b(1 - \alpha)S_p\right]\right] + \frac{S_p}{T}\left[\frac{\beta}{2n}b(1 - \alpha)^2(T - t_4) - b(1 - \alpha)^2\left[\frac{2n - \beta(T + t_4) + 2\beta t_2}{2n}\right]\right] - \frac{1}{T}\left\{\frac{1}{2}h_P\frac{\partial I_4}{\partial S_p} + \frac{1}{2}C_d\theta\frac{\partial I_4}{\partial S_p}\right\} - \frac{1}{T}\frac{\partial NP(S_p,T)}{\partial S_p} \quad (52)$$

$$\frac{\partial^2 NP(S_p,T)}{\partial T^2} = -\frac{\beta}{nT}\left[S_p(1 - \alpha)\right]\left[A - b(1 - \alpha)S_p\right] - \frac{2}{T}\frac{\partial NP(S_p,T)}{\partial T} \quad (53)$$

4. Numerical Example

A numerical experiment was conducted to demonstrate the application of the proposed model. The parameter values chosen for this illustration were based on recommendations and values adopted from previous models and examples found in the existing literature.

$A = 40$ units/day, $b = 0.2$, $c_d = \$0.8$ /unit, $c_{dp} = \$0.5$ /unit, $d_1 = 8\%$, $d_2 = 10\%$, $h_p = \$0.092$ /unit/day $k_1 = 150$ units/day, $k_2 = 130$ units/day, $n = 40$ days, $p_{c1} = \$0.54$ /unit, $p_{c2} = \$0.7$/unit, $SC_1 = \$657$/setup, $SC_2 = \$947$/setup, $\alpha = 20\%$, $\beta = 0.6$, $\theta = 0.004$, $\gamma = 0.8$, $t_1 = 1$day, $t_2 = 2$days,$t_3 = 3$days, $t_4 = 6$days.

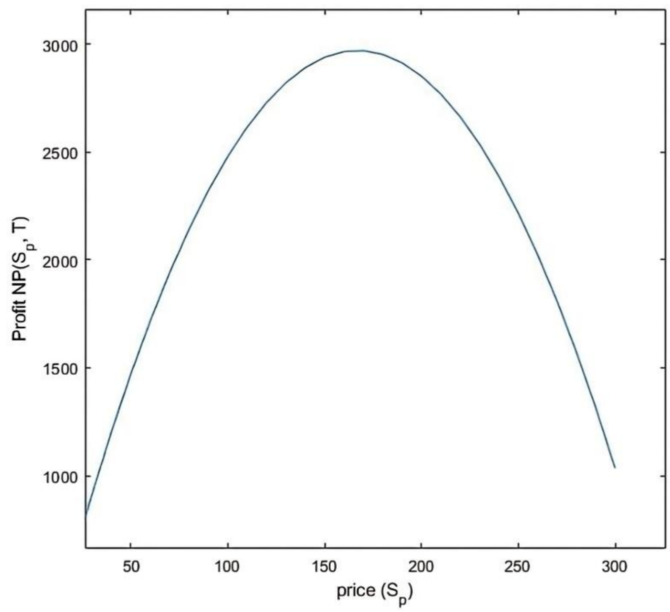

Figure 2. Graph representing a concave NP with respect to Sp.

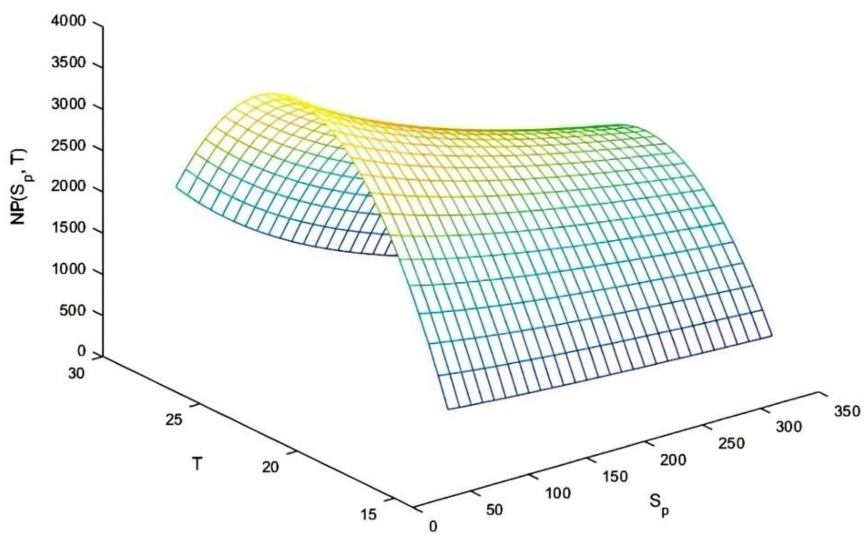

Figure 3. Graphical observation of NP against Sp and T.

Due to the complexity of the equations, the optimization is performed using the Newton-Raphson method, implemented using the MATLAB software. Solving Equations (47) and (48), we obtained the optimal values for the product's price $\mathrm{S_p^*} = \$167.6/\mathrm{unit}$, and the optimal length of the cycle $\mathrm{T^*} = 21.5$ days. Subsequently, we calculated the average total profit $\mathrm{NP}(\mathrm{S_p^*}, \mathrm{T^*}) = \$2977$ per day. Using numerical calculus and values of S_p ranging from 20 to 300, it is evident from Figure 2 that the profit function per unit time, $\mathrm{NP}(S_\mathrm{p}, \mathrm{T})$, is strictly concave versus the selling price. The concavity of the profit function can be observed in Figure 3 for the values of the selling price ranging between 30 and 310 and the cycle time ranging between 14 and 28 days.

5. Sensitivity Analysis

Sensitivity analysis was conducted to analyze the response of the model's objective (the profit function) and the key decision variables (the unit selling price, cycle time) to the model's parameters, by changing the values of the parameters one at a time, while keeping the other parameters constant. The parameter values were varied within the range of -20% to +60% of the values used in the previous numerical example at intervals of 20%, and the changes in the optimal values of the cycle time, selling price and net profit are summarized in Tables 3, 4 and 5, and Figures 4, 5 and 6 respectively. This helps to understand how the key response variables are robust to the parameter values, and understand which of the parameters should be monitored more closely by decision makers for possible review when the model is applied in practice.

Table 3. Influence of proposed model parameters on T^*.

	Change in T^* (%)																				
Change in parameter	**b**	C_d	C_{dp}	h_p	k_1	k_2	p_{c1}	p_{c2}	S_{C1}	S_{C2}	d_1	d_2	t_1	t_2	t_3	t_4	θ	γ	α	β	n
-20	41.7	0.0	0.1	0.5	-14.5	-10.6	1	1.1	9.6	14.8	0.9	1.4	-6.1	-21.9	21.7	4.3	-0.2	-13.44	-1.17	-2.38	7.9
+20	-16.6	0.0	-0.2	-0.6	14.9	10.6	-1.1	-1.2	-7.5	-10.3	-1.4	-1.4	10.5	11.9	4	-4	0.2	12.67	4.48	6.5	-1.8
+40	-27.9	0.0	-0.4	-1.2	30.3	21.2	-2.1	-2.3	-13.6	-18.1	-2.3	-2.9	24.4	14.3	-13.3	-7.7	0.2	19.58	7.38	12.75	-3.9
+60	-35.8	0.0	-0.5	-1.8	46.4	31.8	-3.0	-3.4	-18.7	*	-4.3	-4.2	40.8	*	-51.3	-11	0.3	22.54	10.28	21.45	-5.4

The cycle time T^* is highly sensitive to changes in the shape parameters b, γ and β; the production rates k_1 and k_2; the setup cost parameters S_{c1}, S_{c2}; the time parameters $t_1, t_{2,}\ t_3, t_4$; and the discount rate α. It is moderately sensitive to changes in the unit production costs $p_{c1,}\ p_{c2}$; the inventory holding cost h_p; the defective rates d_1, d_2; and the shelf-life period n. It seems rather insensitive to changes in the disposal cost C_{dp}; the deteriorating cost C_d and the deterioration rate θ. As the values of $b, h_p, p_{c1,}\ p_{c2}$, S_{c1}, S_{c2}, t_3, t_4 and n get higher, the optimal value of the cycle time, T^*, tends to decrease. On the other hand, for higher production rates k_1 and k_2, time parameter t_1, shape parameters γ and β and the discount rate α, the model suggests increasing values for the optimal cycle time, T^*.

The selling price S_p^* is highly sensitive with respect to the shape parameter b, the time parameter t_3 and the discount rate α. It is moderately sensitive to the production rates, k_1, k_2, the unit production costs p_{c1} and p_{c2} , the setup cost parameters S_{c1}, and S_{c2}, and the time parameters t_1, t_2 and t_4. Changes in C_d, C_{dp}, h_p, $\theta, \gamma, \beta, d_1, d_2$, and n, however, tend to have relatively minor impact on the unit selling price S_p^*, compared to the other response variables. As the values of the parameters $k_1 k_2, p_{c1}, p_{c2}, S_{c1}, S_{c2}, t_1, t_2, t_3$, and the discount rate α increase, the model suggests corresponding increase in the optimal selling price, S_p^*. The discount parameter, α, and time the time from which deterioration starts, t_3, have the greatest impact on the optimal selling price S_p^*. On the other hand, as the shape parameter of demand b, and the time from which price discount is offered t_4 increase, the optimal selling price determined by the model decreases. The shape parameter, b, has the greatest effect on the value of the optimal selling price.

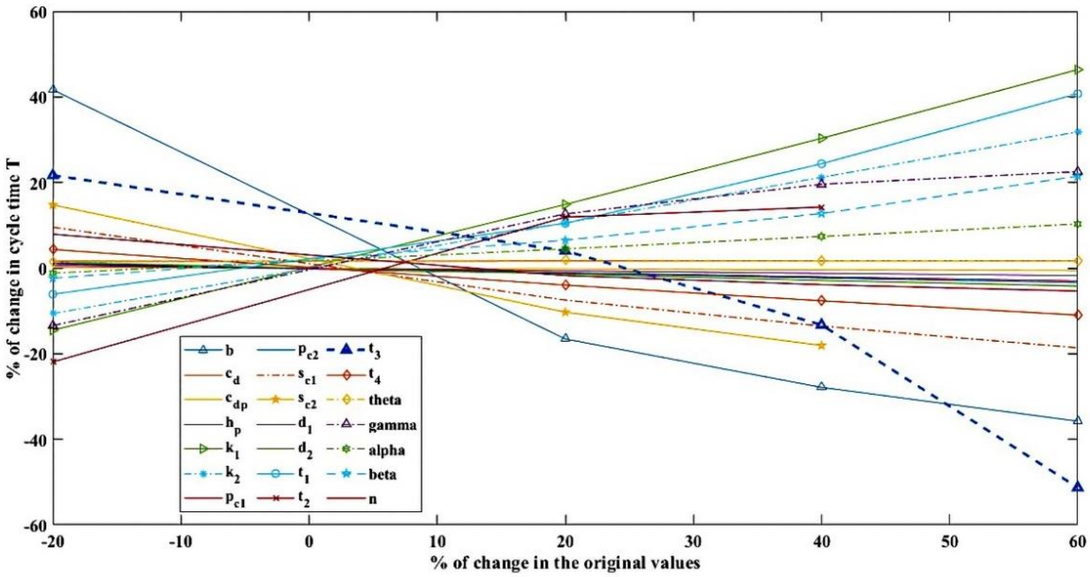

Figure 4. Effect of changing parameters on T^*.

Table 4. Influence of proposed model parameters on S_p^*.

Change in parameter	Change in S_p^* (%)																				
	b	C_d	C_{dp}	h_p	k_1	k_2	p_{c1}	p_{c2}	S_{C1}	S_{C2}	d_1	d_2	t_1	t_2	t_3	t_4	θ	γ	α	β	n
-20	17.8	0	0.3	0.2	0.1%	0.0	0.1	0.1	-1.5	-2.4	0.0	0.0	0.2	-0.6	-1.7	1.7	0	-0.7	-3.6	0.1	-0.1
+20	-12.8	0	0.4	0.5	0.6	0.7	0.6	0.6	2.1	2.8	0.0	0.0	0.6	1.6	1.2	-1.1	0	0.3	3.9	-0.1	0.1
+40	-22.3	0	0.4	0.6	1	1.1	0.8	0.8	3.7	5.1	0.1	0.0	1.1	2.8	5.5	-2.5	0	0.4	8.1	-0.2	0.1
+60	-29.6	0	0.4	0.7	1.4	1.5	1	1.1	5.3	*	0.1	0.0	1.6	*	13.5	-4	0.	0.5	12.6	-0.3	0.2

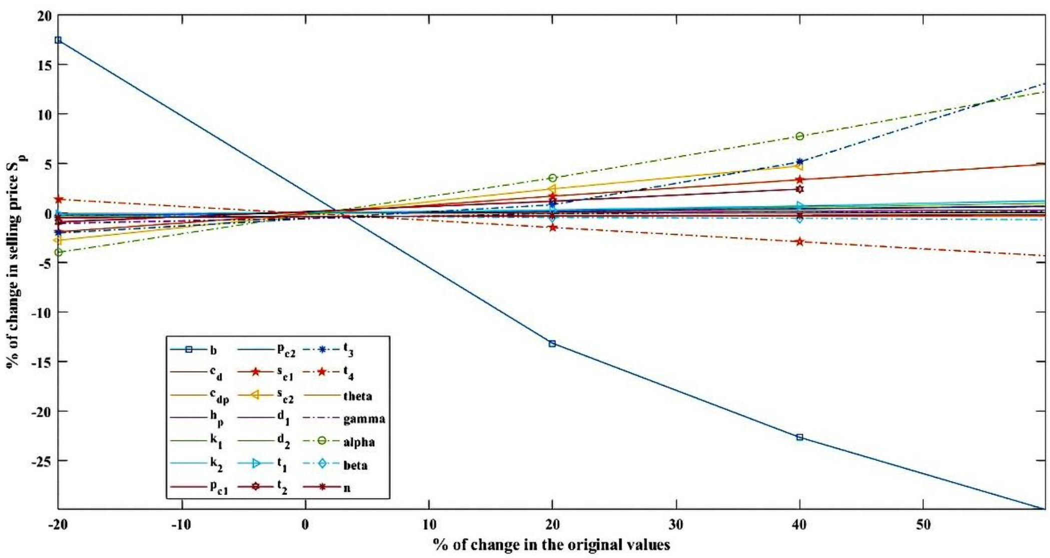

Figure 5. Effect of changing parameters on S_p^*.

The total profit per unit time, NP^*, is highly sensitive with respect to changes in setup cost parameters, S_{c1}, S_{c2}, the shape parameter, β, and the shelf-life period, n. The total profit per unit time, NP^*, is moderately sensitive to changes in parameter γ, disposal cost C_{dp}, inventory holding cost h_p; unit production costs $p_{c1,}\, p_{c2}$, time parameters t_1, t_2, t_3 and t_4. The total profit per unit time NP^* exhibits a strong positive sensitivity to changes in setup cost parameters S_{c1}, S_{c2}, disposal cost C_{dp}, inventory holding cost h_p; unit production costs $p_{c1,}\, p_{c2}$, time parameters $t_2, t_{4,}$ and the shelf-life period n. Among these parameters, the set-up cost parameters, the shelf live period n and time parameter t_4 have the greatest impact on the optimal profit. NP^* is highly sensitive in a negative to changes in shape parameters b, γ and β, production rates k_1, k_2, time t_1 and t_3. Among these parameters, shape parameters γ and β, and time parameter t_1 have the greatest impact on the optimal profit per cycle. The change in C_d, α, and θ has a minimal effect on the net profit NP^*, indicating that these parameters have relatively insignificant influence on revenue when compared to other factors that affect profitability.

Table 5. Influence of proposed model parameters on NP^*.

	Change in NP^* (%)																				
Change in parameter	**b**	C_d	C_{dp}	h_p	k_1	k_2	p_{c1}	p_{c2}	S_{C1}	S_{C2}	d_1	d_2	t_1	t_2	t_3	t_4	θ	γ	α	β	n
-20	3.7	0	0.9	0.7	2.7	2	0.4	0.3	-4	-6.6	-0.3	-0.2	2	1.9	5.4	-0.4	0	3.4	-0.2	3.9	-5.6
+20	-6.6	0	1.1	1.3	-0.9	0	1.6	1.7	5.6	7.5	0.2	0.2	-0.8	1.9	-2.9	2.2	0	-2.7	0.2	-4.4	3.23
+40	-12.6	0	1.2	1.7	-2.9	-1	2.2	2.3	9.8	13.4	0.5	0.5	-3.2	4.5	-3	3.4	0	-4.6	0.3	-9.5	5.4
+60	-17.8	0	1.3	2	-5	-2.1	2.7	3	13.8	*	0.7	0.7	-5.9	*	-4.5	4.5	0	-5.6	0.5	-15.7	7

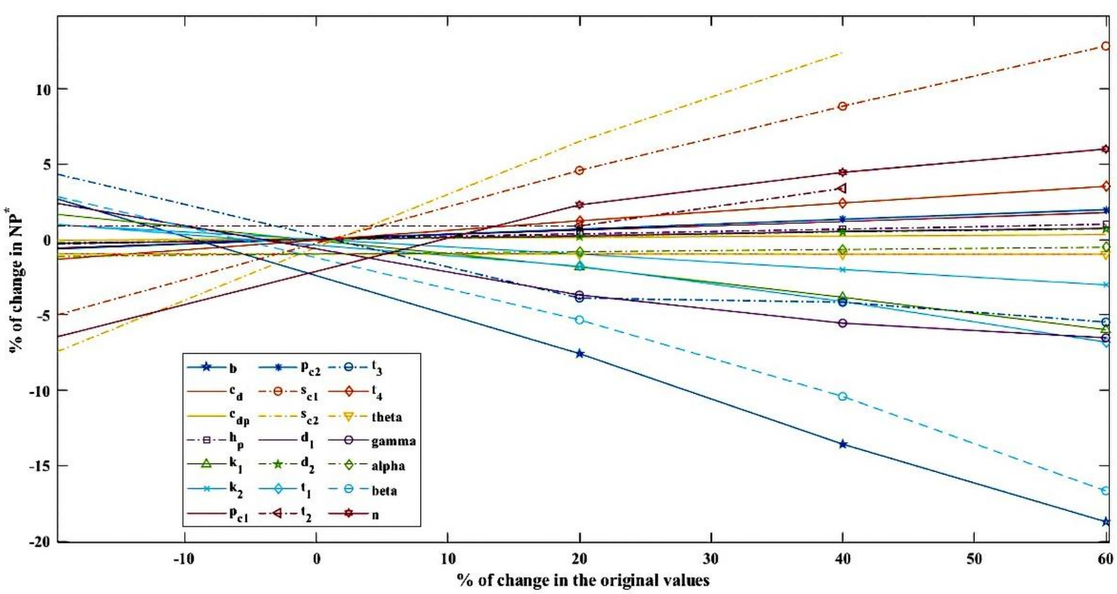

Figure 6. Effect of changing parameters on NP^*.

6. Managerial Implications

The findings from the sensitivity analysis provide valuable suggestions to managers and decision-makers for enhancing the total profit. These suggestions aim to optimize various factors and improve the overall profitability.

1) It is important to note that the demand elasticity factor, b, is an important parameter in determining the net profit, the selling price, and the cycle time. This may provide the managers with the necessary alternative optima, especially in situations where there may be operating constraints on possible practical values. For instance, market forces may impose limitations on the achievable cycle time, necessitating managers to operate within specific range of values. It can be observed from Figure 4 from the changes in the gradients that the changes in cycle time is more sensitive to decrease in demand elasticity than its increase, hence, the manager may appear to have some leeway in allowing increase in this value. However, seeing the implications on the net profit function in Figure 6 discourages this assumption.

2) The setup cost parameters, S_{c1} and S_{c2}, are important parameters in determining the cycle time and the net profit. Analysis of the provided tables reveals a notable decrease in cycle time, which proves to be beneficial in optimizing the revenue. This reduction in cycle time helps mitigate holding costs, particularly in production systems where storage costs, deterioration, or obsolescence of inventory are significant concerns. With this increase in setup cost, it can be seen that the model recommended small increase in selling price, which may account for increase in profit. This must, however, be carefully done as the effect of demand sensitivity to price needs to be factored in, else the profit may start dropping more quickly.

3) The sensitivity analysis shows that an increase in the time from which price discount is offered, t_4, results in moderate decrease in the cycle time. The model then suggests a moderate increase in selling price, to result in a moderate increase in the optimal Net profit NP^*. At first glance, this behavior seems peculiar. This suggests that as the combined effects of both product deterioration and the freshness function kick in, the demand starts to drop, and spoilage starts to increase, and there is the need to promptly introduce demand stimulant, especially because of the impact of the freshness function. The extra sale will then compress the cycle time since the quantity produced is already fixed, and this can mitigate the overall effect of the holding cost and deteriorated quantity, amongst other, thereby rising profit.

4) Based on an observation derived from sensitivity analysis, it has been noted that any change in the freshness parameter (β) necessitates careful consideration to strike a balance between cycle time T^* and the total profit NP^*. As β increases, the product freshness drops, and the demand drops correspondingly. Consequently, since products are already made, the cycle time lengthens, which increases the quantity that deteriorated, makes the time the discounted product is sold longer, increases the holding cost, and subsequently depletes the profit. It can be seen from Figure 6 that this factor has one of the greatest impacts on profit, and as the cycle time increases further, comparing its slope to that of the demand elasticity factor, it can easily become the most important cause of profitability decline.

5) It is interesting to see that changes in the shelf life, n, does not have significant impact on the optimal cycle time, or on the optimal selling price. For instance, 60% increase in the product life, decreases the cycle time by just about 5 percent, and the selling price by less by less than 0.5 percent. However, it increases the profit moderately, by up to seven percent. This may be because longer shelf life dampens the effect of declining freshness (which reduces demand), decreases the cycle time (shortens the length of discount period on fresh products, since t_4 is fixed), hence, improving profit.

7. Conclusion

Companies are actively seeking effective inventory strategies for deteriorating products. While deteriorated products are unsellable, products that gradually lose freshness experience a decrease in demand as they age, although they can still be sold. Many previous models on inventory models for deteriorating products often assume that the demand remains unchanged regardless of the freshness level of the product. In reality, demand is affected by freshness. Another assumption is that the production rate is constant in a manufacturing system. This can be problematic as it doesn't consider the variability in and the complexity of manufacturing processes. In reality, production shifts from one rate to another due to a variety of factors such as variability of demand, machine breakdowns, material shortages, energy constraints, and quality issues. To effectively respond to changes in demand or unforeseen disruptions in the production process, many organizations implement flexible production systems for better efficiencies and profitability.

This study examined an EPQ model with alternating production rates and price-dependent demand while employing price discounting as a strategy to optimize profit. Additionally, we modified the inventory model to incorporate deteriorating products, where demand is not only influenced by the unit selling price but also by factors such as the level of stock displayed, and the freshness condition of the product. The primary objective is to determine both the selling price and the inventory cycle time that maximize the profit. The objective function of this model is highly non-linear, making it challenging to find an analytical solution, hence, solved numerically. We presented a numerical example and then conducted sensitivity analyses to gain managerial insights based on changes in parameter values. It is not always economically viable for the decision maker to sell products at the regular price, particularly when the production costs per unit or the set-up costs increase. In such cases, decision-makers can reduce their cycle time as much as possible and increase their unit selling price to protect their profit. Furthermore, when faced with an increase in discount, the analysis suggests that in such situations, the decision maker may respond by increasing their price and cycle time to maintain the profit. However, it is important for managers to consider market dynamics and customer behavior when contemplating cycle time and price adjustments. Adjustments in cycle time or price may be more effective in preserving profitability without compromising the overall profitability of the production system.
Future research can expand this model in several ways, such as considering non-linear shortages, inflation, incremental discount facilities, prepayment installment on pricing, and other factors. A potential extension of this work includes treating t_1, t_2, t_3 and t_4 as decision variables in the model, as opposed to fixed values, to determine how these variables may impact the model. Furthermore, one could explore models with demand dependent on advertisement and selling price, nonlinear stock dependent holding cost, non-instantaneous deterioration, and preservation technology, as well as introducing various credit policies (single level, two-level, partial, credit risk customers, etc.). The model can also be extended to stochastic, or fuzzy models.

Conflict of Interest
The authors confirm that there is no conflict of interest to declare for this publication.

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