A New Decomposition Linear Programming Model to Solve Zero Sum Two Person Matrix Game in Fully Fuzzy Trapezoidal Environment

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Abstract
This article targets to unriddle the problem of a non-cooperative fully fuzzified 'Zero Sum Two Person Matrix Game' (ZSTPMG) with payoff matrix equipped with Trapezoidal fuzzy numbers (TrFNs). To achieve the target a unique and novel decomposition technique has been introduced. First, we develop two auxiliaries fully fuzzified linear programming problem (FFLPP) models for both the players and then we decompose these two FFLPP models into four linear programming (LP) models each, for both the players. These eight LP models are then solved by using the software TORA-2.0. The solutions of these eight LP models ascertain the optimal strategies and the optimal value of the fully fuzzified ZSTPMG for both the players. Our technique has an advantage over the existing ones as it can solve fully fuzzified ZSTPMG with all kind of TrFNs such as symmetric, asymmetric, positive or negative TrFNs. To establish this fact, the proposed methodology has been illustrated by taking three numericals equipped with various kinds of TrFNs.

Keywords- Fully fuzzy matrix games, Trapezoidal fuzzy numbers, Fully fuzzy linear programming problem.

1. Introduction

A game is an analytical tool of mathematics that is used to explain strategic mutual mediation or interaction among various players also known as DMs (Decision Makers). The trailblazing work of two American economists, John and Oskar (1944) titled "Theory of Games and Economic Behaviour" set the advent of game theory in 1944. Since the advent of game theory various types of games such as ZSTPMG, bimatrix games, constrained games, differential games, have been introduced and their solution techniques proposed. Broadly games can be classified into two categories, cooperative games and non-cooperative games. Zero sum two-person matrix game (ZSTPMG) is a significant class of non-cooperative games where the output payoffs are assumed to be well defined and crisp for both the players, although in real life games, this may not be the case always. In fact, in a real-life game, DMs are not always able to gauge the outcome payoff exactly because of the inadequacy present in the available information or imprecision present in the expert's opinions as it is estimated by using the approximate data available to the experts.

To counter and overcome such type of imprecision and vagueness in the available information, Zadeh (1965) introduced the revolutionary concept of fuzziness to the already existing set theory in 1965. Then in 1978, for the first time Butnariu (1978) used this concept of fuzziness in a non-cooperative game where he modelled each player's action against the other as fuzzy sets. Although the credit of being the first to study non-cooperative game in a fuzzy environment goes to (Campos, 1989). He used Yager's resolution method (Yager, 1981) and fuzzy ranking index to transform FMG (fuzzy matrix game) problem into a pair of auxiliary FLPP (fuzzy linear programming problem). Nishizaki and Sakawa (1995, 2000) and Sakawa and Nishizaki (1994) investigated matrix games with fuzzy goals and fuzzy payoffs and gave max-min solution concept for multi objective fuzzy matrix games. Maeda (2003) described fuzzy max order-based equilibrium strategy to unriddle the problem of a particular type of games in fuzzy environment. Ganesan and Veeramani (2006) dealt with FLPPs involving TrFNs and introduced a unique method to solve FLPPs without transforming them into crisp LP problems. Li (2008, 2012) and Dengfeng (1999) introduced a two-tier LP Model to solve FMG with TFN (triangular fuzzy numbers) payoff matrix. Bector and Chandra (2005) famously called it 'The Li Model'. Dutta and Gupta (2014) elaborated the work of Maeda (2003) and Cunlin and Qiang (2011) and investigated the Nash equilibrium strategy of ZSTPMG with TrFN payoff matrix. Seikh et al. (2013, 2015) studied Intuitionistic fuzzy matrix games (IFMGs) and introduced a novel IFO (Intuitionistic fuzzy optimization) technique to unriddle IFMGs. Further in 2021, Seikh et al. (2021) used triangular fuzzy dense sets in payoff matrix to make the game problem more realistic and used a defuzzification function to solve it. Bhaumik et al. (2017) and Jana and Roy (2018) introduced a solution method to unriddle FMGs with TrFN payoffs by converting it to crisp MG by means of an appropriately chosen linear ranking function. Hosseinzadeh and Edalatpanah (2016) explored Lexicographic technique in colligation with crisp LP to develop a novel model to solve FFLP problem with L-R fuzzy numbers. Brikaa et al. (2020) investigated the constrained matrix games under completely rough fuzzy environment and developed an efficacious multi-objective fuzzy model algorithm to solve constrained MGs. Again Brikaa et al. (2022) introduced a new technique called 'Mehar Approach' to solve a matrix game with payoff of dual triangular hesitant fuzzy sets. Interval number and symmetric TFNs were considered as fuzzy payoffs by Nayak and Pal (2009) for their study of FMGs. Bandyopadhyay and Nayak (2013) considered symmetrical TrFNs only for their study of FMGs. Kumar et al. (2016) took only the positive TrFNs for their study of FMGs. However, in real life game problems we may have asymmetric TrFNs and TrFNs with negative entries as well. Das and Chakraborty (2021), Das et al. (2017, 2019) and Akram et al. (2022) have provided new mathematical models for solving FLP problems in their recent works.

Developing new techniques to solve decision taking problems under uncertain and vague environment has been a motivating field for researchers all over the globe. Fuzzy game theory is a very potent tool to solve decision making models in uncertain and ambiguous environment. TrFNs have played a pivotal role in handling uncertainty and vagueness in decision science. Various articles have already been published involving TrFNs to unfold the problem of decision science under uncertain conditions. In their articles researchers have considered only a particular type of TrFNs for their study. A question arises- Can we develop a methodology by which we can handle models involving all kind of TrFNs? We try to build this research article from this aspect.

Here in this paper, we propose a novel, effective and easy decomposition fuzzy linear programming (FLP) model. With the help of our proposed solution methodology, we can handle fully fuzzified matrix games (FFMG) having payoff matrix equipped with all kind of TrFNs, such as symmetric, asymmetric, and with positive, negative entries as well. Also, we have not used any ranking or defuzzification function to crispify TrFNs, which almost all the researchers have used. Therefore, ours is a novel concept to solve such problems and has an advantage over the existing ones. (Refer subsection 3.3)

This paper has been developed as follows. Section 2 briefly introduces some of the fundamental concepts and definitions of fuzzy theory. Section 3 throws light on the concepts of matrix games (MG), both crisp and fuzzy, shortcomings of existing methods are discussed as well. Section 4 communicates the proposed solution methodology to solve the fully fuzzified matrix games (FFMG), followed by a flow chart (Figure 2) depicting the solution procedure. Section 5 illustrates three fuzzy numerical examples to showcase the applicability and prove validity of the proposed method. Comparison of our work with some others esteem researchers has been presented by means of Tables 1, 2 and 3 in section 6 and lastly this paper culminates after deriving the conclusion in section 7.

2. Preliminaries

Definition 1. (Bector and Chandra, 2005) Let X be a universe of discourse. Then a fuzzy set $\tilde{A}$ of $X$is completely determined by its degree function $\tilde{A}(z): X \to [0,1]$, that specifies the degree of belongingness of each element of X in $\tilde{A}$, i.e., for each $z \in X$, $\tilde{A}(z) \in [0,1]$ specifies degree of belongingness of z in $\tilde{A}$. A fuzzy set $\tilde{A}$ is also characterized as the set of ordered pair of elements z and its degree of belongingness $\tilde{A}(z)$ and is usually written $\tilde{A} = \{(z, \tilde{A}(z)): z \in X\}$. If $\tilde{A}(z) = 0$ then it means $z \notin \tilde{A}$.

Definition 2. (Bector and Chandra, 2005) A fuzzy set $\tilde{A}$ is called **Normal** if $\exists$ some $x_0 \in X$ s.t. $\tilde{A}(x_0) = 1$.

Definition 3. (Bector and Chandra, 2005) A fuzzy set $\tilde{A}$ is called **Convex** if for any

$$x_1,\ x_2 \in X,\ \tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq min\left(\tilde{A}(x_1),\ \tilde{A}(x_2)\right).$$

Definition 4. (Bector and Chandra, 2005) Any fuzzy set $\tilde{A}$ in $\mathbb{R}$ (Real set) is called a **fuzzy number** if

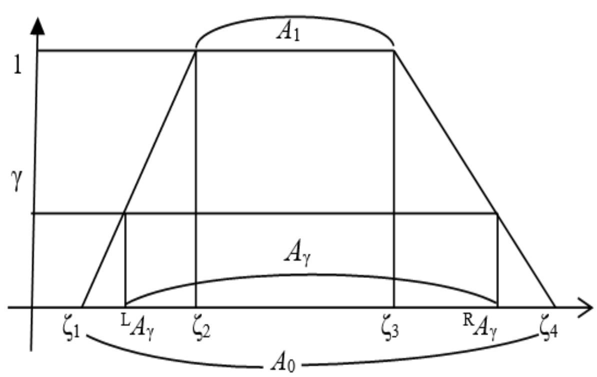
(i) $\tilde{A}$ is Normal.

(ii) $\tilde{A}$ is Convex.

(iii) $\tilde{A}(x)$ is piecewise continuous function.

Definition 5. (Bector and Chandra, 2005) For $0 \leq \gamma \leq 1$, the $\boldsymbol{\gamma}$ **−level** set or $\boldsymbol{\gamma}$ **−cut** of fuzzy number $\tilde{A}$ is defined and denoted as $\tilde{A}_\gamma(x) = \{x: \tilde{A}(x) \geq \gamma\}$;$\gamma$ −cut of fuzzy number $\tilde{A}$ is a crisp set.

Definition 6. (Bandyopadhyay and Nayak, 2013) A quadruple $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = \tilde{A}\ (say), \zeta_1 \leq \zeta_2 \leq \zeta_3 \leq \zeta_4$is known as **Trapezoidal fuzzy number** (TrFN) if its membership function $\tilde{A}(x): \mathbb{R} \to [0,1]$ is given by



$$\tilde{A}(x) = \begin{cases} \dfrac{x-\zeta_1}{\zeta_2-\zeta_1} & if \quad \zeta_1 \leq x \leq \zeta_2 \\ 1 & if \quad \zeta_2 \leq x \leq \zeta_3 \\ \dfrac{\zeta_4-x}{\zeta_4-\zeta_3} & if \quad \zeta_3 \leq x \leq \zeta_4 \\ 0 & if\ x < \zeta_1;\ x > \zeta_4 \end{cases}$$

Figure 1. Membership Function of TrFN.

Definition 7. (Bandyopadhyay and Nayak, 2013) Theγ −cut $\tilde{A}_\gamma$ of TrFN $\tilde{A} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, using definition-6, is defined as the crisp interval $\tilde{A}_\gamma = [\gamma\zeta_2 + (1-\gamma)\zeta_1, \gamma\zeta_3 + (1-\gamma)\zeta_4] = [{}^L A_\gamma, {}^R A_\gamma]$ for $0 \le \gamma \le 1$. Clearly $\tilde{A}_1 = [\zeta_2,\ \zeta_3]$ and $\tilde{A}_0 = [\zeta_1,\ \zeta_4]$ (refer Figure 1).

Definition 8. (Kaufmann and Gupta, 1991; Kaur and Kumar, 2012), A TrFN $\tilde{E} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is defined as:-

(i) **Non-negative TrFN** if $\zeta_1 \ge 0$ and is denoted as $\tilde{E} \succcurlyeq \tilde{0}$.
(ii) **Non-positive TrFN** if $\zeta_4 \le 0$ and is denoted as $\tilde{E} \preccurlyeq \tilde{0}$.
(iii) **Unrestricted TrFN** if $\zeta_1 \le 0$ and $\zeta_4 \ge 0$.
(iv) **Scalar TrFN** if $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = k$ and is denoted as$(\tilde{k})$.
  i. **Zero TrFN** if $k = 0$ and is denoted as$(\tilde{0})$.
  ii. **Unit TrFN** if $k = 1$ and is denoted as$(\tilde{1})$.
(v) **Triangular Fuzzy Number** (TFN) if $\zeta_2 = \zeta_3$.

Definition 9. (Kaufmann and Gupta, 1991) Two TrFNs $\tilde{E}_1 = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ and $\tilde{E}_2 = (\xi_1, \xi_2, \xi_3, \xi_4)$ are defined to be **equal** if $\zeta_1 = \xi_1; \zeta_2 = \xi_2; \zeta_3 = \xi_3; \zeta_4 = \xi_4$.

Definition 10. (Liou and Wang, 1992) A **Ranking or Defuzzification function** $\Re$ is a real function defined on the set of all fuzzy numbers, which maps each fuzzy numbers into the real line, where the order exists naturally, i.e., $\Re: F(\mathbb{R}) \to \mathbb{R}$, where, $F(\mathbb{R})$ is a set of all fuzzy numbers. If $\tilde{E} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in F(\mathbb{R})$ is a TrFN then we define ranking of $\tilde{E}$ as

$$\Re(\tilde{E}) = \frac{\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4}{4}.$$

Definition 11. (Liou and Wang, 1992) If $\tilde{E}_1 = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ and $\tilde{E}_2 = (\xi_1, \xi_2, \xi_3, \xi_4)$ are two TrFNs then
(i) $\tilde{E}_1 \preccurlyeq \tilde{E}_2 \Leftrightarrow \Re(\tilde{E}_1) \le \Re(\tilde{E}_2)$.
(ii) $\tilde{E}_1 \succcurlyeq \tilde{E}_2 \Leftrightarrow \Re(\tilde{E}_1) \ge \Re(\tilde{E}_2)$.
(iii) $\tilde{E}_1 \approx \tilde{E}_2 \Leftrightarrow \Re(\tilde{E}_1) = \Re(\tilde{E}_2)$.
where, $\preccurlyeq$, $\succcurlyeq$, and $\approx$ are fuzzy versions of $\le, \ge$ and $=$ respectively.

Definition 12. (Kaufman and Gupta, 1991) **Arithmetic operations** on two TrFNs $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{E} = (\xi_1, \xi_2, \xi_3, \xi_4)$ are defined as follows:

(i) **Addition**: $\tilde{A} \oplus \tilde{E} = (a_1 + \xi_1, a_2 + \xi_2, a_3 + \xi_3, a_4 + \xi_4)$.
(ii) **Negation**: $(-\tilde{E}) = (-\xi_4, -\xi_3, -\xi_2, -\xi_1)$.
(iii) **Substraction**: $\tilde{A} \ominus \tilde{E} = (a_1 - \xi_4, a_2 - \xi_3, a_3 - \xi_2, a_4 - \xi_1)$.
(iv) **Multiplication**: $\tilde{A} \otimes \tilde{E} = \begin{pmatrix} min(a_1\xi_1, a_1\xi_4, a_4\xi_1, a_4\xi_4), min(a_2\xi_2, a_2\xi_3, a_3\xi_2, a_3\xi_3), \\ max(a_2\xi_2, a_2\xi_3, a_3\xi_2, a_3\xi_3), max(a_1\xi_1, a_1\xi_4, a_4\xi_1, a_4\xi_4) \end{pmatrix}$.

Particular Cases:
If$\tilde{E} = (\xi_1, \xi_2, \xi_3, \xi_4)$ is a non-negative TrFN and $\tilde{A} = (a_1, a_2, a_3, a_4)$ is any TrFN then:
(a) $\tilde{A} \otimes \tilde{E} = (a_1\xi_4, a_2\xi_2, a_3\xi_3, a_4\xi_4)\ if a_1 < 0, and\ a_2, a_3, a_4 \ge 0$.
(b) $\tilde{A} \otimes \tilde{E} = (a_1\xi_4, a_2\xi_3, a_3\xi_3, a_4\xi_4)\ if a_1 < 0, a_2 < 0\ and\ a_3, a_4 \ge 0$.
(c) $\tilde{A} \otimes \tilde{E} = (a_1\xi_4, a_2\xi_3, a_3\xi_2, a_4\xi_4)\ if a_1 < 0, a_2 < 0, a_3 < 0\ and\ a_4 \ge 0$.
(d) $\tilde{A} \otimes \tilde{E} = (a_1\xi_4, a_2\xi_3, a_3\xi_2, a_4\xi_1)\ if a_1 < 0, a_2 < 0, a_3 < 0\ and\ a_4 \le 0$.

(v) **Division**:
$$\tilde{A}\oslash\tilde{E}=\left(min\left(\frac{a_1}{\xi_1},\frac{a_1}{\xi_4},\frac{a_4}{\xi_1},\frac{a_4}{\xi_4}\right),min\left(\frac{a_2}{\xi_2},\frac{a_2}{\xi_3},\frac{a_3}{\xi_2},\frac{a_3}{\xi_3}\right),max\left(\frac{a_2}{\xi_2},\frac{a_2}{\xi_3},\frac{a_3}{\xi_2},\frac{a_3}{\xi_3}\right),max\left(\frac{a_1}{\xi_1},\frac{a_1}{\xi_4},\frac{a_4}{\xi_1},\frac{a_4}{\xi_4}\right)\right).$$

(vi) **Reciprocal**: $\left(\tilde{E}\right)^{-1}=\left(\frac{1}{\xi_4},\frac{1}{\xi_3},\frac{1}{\xi_2},\frac{1}{\xi_1}\right) if \tilde{E}\succ 0 or \tilde{E}\prec 0.$

(vi) **Scalar Multiplication**: $\left(k\tilde{E}\right)=\begin{cases}(k\xi_1,k\xi_2,k\xi_3,k\xi_4) if k\geq 0\\(k\xi_4,k\xi_3,k\xi_2,k\xi_1) if k<0\end{cases}.$

Remark 1: The ranking of a TrFN given in definition-10 is a linear function, i.e.
$$\Re(\alpha\tilde{E}_1+\beta\tilde{E}_2)=\alpha\Re\left(\tilde{E}_1\right)+\beta\Re\left(\tilde{E}_2\right);\text{ for any scalars } \alpha \text{ and } \beta.$$

Remark 2: If $\tilde{E}_1=(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ and $\tilde{E}_2=(\xi_1,\xi_2,\xi_3,\xi_4)$ are two TrFNs such that $\zeta_1\leq\xi_1$, $\zeta_2\leq\xi_2,\zeta_3\leq\xi_3,\zeta_4\leq\xi_4$; then clearly, we have $\Re\left(\tilde{E}_1\right)\preccurlyeq\Re\left(\tilde{E}_2\right)$.

3. Matrix Games (MGs)

3.1 Crisp Matrix Game (CMG)

We shall initiate this section by first explaining a crisp matrix game (CMG) then we will explain FMG in next subsection. Let $E=\left(a_{jk}\right)_{m\times n}\in\mathbb{R}^{m\times n}$ be real matrix. By a crisp zero sum two-person matrix game (CMG) we shall mean a triplet (S^I,S^{II},E), where, S^I,S^{II} denotes set of all possible mixed strategies for player-I and player-II respectively i.e.
$$S^I=\left\{\xi=(\xi_1,\xi_2,\ldots,\xi_m):\textstyle\sum_{j=1}^m\xi_j=1,\xi_j\in\mathbb{R}\right\},\text{ and}$$
$$S^{II}=\{\eta=(\eta_1,\eta_2,\ldots,\eta_n):\textstyle\sum_{k=1}^n\eta_k=1,\ \eta_k\in\mathbb{R}\}.$$

The matrix $E$is called payoff matrix for player-I and the real quantity $(\xi^T E\eta)$ for $\xi\in S^I$and$\eta\in S^{II}$ is called expected payoff value to player-I. The triplet $(\xi^*,\eta^*,\ u^*)\in S^I\times S^{II}\times\mathbb{R}$ is solution of CMG if
$$\xi^{*T}E\eta\geq u^*\quad\forall\ \eta\in S^{II}\text{ and }\xi^T E\eta^*\leq u^*\quad\forall\ \xi\in S^I.$$

Since CMG is zero sum the payoff value for player-II is$-(\xi^T E\eta)$. For a given CMG it is well known to write following pair of LPPs for player I and II respectively.

LPP for player-I

Maximize (u)

Subject to
$$\begin{aligned}&\textstyle\sum_{j=1}^m a_{jk}\xi_j\geq u;(k=1,2,\ldots,n)\\&e^T\xi=1\\&\xi\geq 0\end{aligned}\tag{1}$$

LPP for player - II

Minimize (v)

Subject to
$$\begin{aligned}&\textstyle\sum_{k=1}^n a_{jk}\eta_k\leq v;(j=1,2,\ldots,m)\\&e^T\eta=1\\&\xi\geq 0\end{aligned}\tag{2}$$

where, $e^T=(1,\ 1,\ldots,\ 1)$ is row matrix of ones having the context specific order.

3.2 Fully Fuzzified Matrix Games (FFMG)

Let $\widetilde{\boldsymbol{E}} = \left(\widetilde{\boldsymbol{a}}_{\boldsymbol{jk}}\right)_{\boldsymbol{m\times n}} \in \left(\boldsymbol{TrFN}(\mathbb{R})\right)^{\boldsymbol{m\times n}}$ be fuzzy matrix with TrFN entries where $\tilde{a}_{jk} = \left(a_{jk}^1, a_{jk}^2, a_{jk}^3, a_{jk}^4\right) \in TrFN(\mathbb{R})$, where $TrFN(\mathbb{R})$ being the set of all TrFNs.

The matrix $\tilde{E}$ is called the fuzzy payoff matrix for player - I and the fuzzy entity $\left(\tilde{\xi}^T \otimes \tilde{E} \otimes \tilde{\eta}\right)$ for $\tilde{\xi} \in \tilde{S}^I$ and $\tilde{\eta} \in \tilde{S}^{II}$ is called fuzzy expected TrFN payoff value to player - I. Then by a fully fuzzified matrix game we mean a triplet $\left(\tilde{S}^I, \tilde{S}^{II}, \tilde{E}\right)$= FFMG (say) where $\tilde{S}^I$ and $\tilde{S}^{II}$ are sets of all fuzzy mixed TrFN strategies for player-I and player-II respectively, i.e.

$$\tilde{S}^I = \left\{\tilde{\xi} = \left(\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_m\right): \sum_{j=1}^m \tilde{\xi}_j \simeq \tilde{1}, \tilde{\xi}_j \in TrFN(\mathbb{R})\right\}.$$
$$\tilde{S}^{II} = \left\{\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n): \sum_{k=1}^n \tilde{\eta}_k \simeq \tilde{1}, \tilde{\eta}_k \in TrFN(\mathbb{R})\right\}.$$

Then the couplet $(\tilde{u}, \ \tilde{v})$ where $\tilde{u}, \tilde{v} \in TrFN(\mathbb{R})$, is known as a **Reasonable Solution** of FFMG if there exists $\tilde{\xi}^* \in \tilde{S}^I$ and $\tilde{\eta}^* \in \tilde{S}^{\ II}$ satisfying

$$\left((\tilde{\xi}^*)^T \otimes \tilde{E} \otimes \tilde{\eta}\right) \succcurlyeq \tilde{u} \ \ \forall \ \ \tilde{\eta} \in \tilde{S}^{II}.$$
$$\left(\tilde{\xi}^T \otimes \tilde{E} \otimes \tilde{\eta}^*\right) \preccurlyeq \tilde{v} \quad \forall \ \ \tilde{\xi} \in \tilde{S}^I.$$

and $\tilde{u}$, $\tilde{v}$ are known as **Reasonable Values** of FFMG for player-I and player-II respectively.

Let's take $\tilde{\Lambda}_1, \ \tilde{\Lambda}_2$ as reasonable value sets for player-I and player-II respectively.
Then if $\exists \ \tilde{u}^* \in \tilde{\Lambda}_1$ and $\tilde{v}^* \in \tilde{\Lambda}_2$ such that

$$\tilde{u}^* \succcurlyeq \tilde{u} \ \ \forall \ \ \tilde{u} \in \tilde{\Lambda}_1 \text{ and } \tilde{v}^* \preccurlyeq \tilde{v} \qquad \forall \ \tilde{v} \in \tilde{\Lambda}_2.$$

Then the quadruple $\left(\tilde{\xi}^*, \ \tilde{\eta}^*, \ \tilde{u}^*, \ \tilde{v}^*\right)$ is called the solution of FFMG = $\left(\tilde{S}^I, \tilde{S}^{II}, \tilde{E}\right)$ and $\tilde{u}^*$, $\tilde{v}^*$ are called the fuzzy optimum TrFN values of FFMG for player-I and player-II respectively and $\tilde{\xi}^*$, $\tilde{\eta}^*$ are called the fuzzy optimal TrFN strategies of FFMG for player-I and player-II respectively. Equations similar to equations (1) and (2) can be written in fuzzy environment to obtain the optimum fuzzy solution of a given FFMG (Refer to equation (5) and equation (6)).

3.3 Existing Methods and their Limitations

In this subsection we discuss the shortfalls of the existing methods given by various researchers like (Bandopadhyay and Nayak, 2013; Kumar et al., 2016; Krishnavani and Ganesan, 2018) to compute fuzzy optimum solution of a given FFMG with payoff of TrFNs.

(i) All the above mentioned researchers have used a particular type of TrFN payoffs matrix for their study e.g. (Bandopadhyay and Nayak, 2013) considered only symmetric TrFNs for their study, (Kumar et al., 2016) illustrated only positive TrFNs. In the example taken by Krishnavani and Ganesan (2018), in his study, has used the old domination law by ranking TrFNs with the help of a ranking function. Their results can be influenced by changing the ranking function, also the strategies of the two players are calculated as TrFNs with negative entries and their sum is also not equal to one, hence the solution can't be the best relied on. (Refer Table 2).

(ii) Almost all the researchers mentioned above have not computed the value of the FMG for player-II, they assume it as negative of the value obtained for player-I, as the FMG is zero sum. But this is not true as the sum of any TrFN and its negative is never zero unless it is a scalar TrFN (Refer Tables 1, 2, 3).

(iii) Almost all the researchers have taken the strategies to be crisp, so their solutions are a particular case, not the generalized ones. In our work strategies are also taken as fuzzy.

4. Proposed Solution Methodology

In this section we now present a new model to overcome the shortcomings of the existing methods. The steps of the method are as follows:

Step 1: We write FFLPPs for the two players respectively

For Player-I: $(\text{FFLPP})^{I}$

$$\begin{aligned} &\text{Maximize} && (\tilde{u}) \quad \text{where } \tilde{u} = (u_1, u_2, u_3, u_4) \text{ is a TrFN.} \\ &\text{Subject to} && \tilde{\xi} \otimes \tilde{A} \otimes \tilde{\eta} \succcurlyeq \tilde{u} \quad \forall \ \tilde{\eta} \in \tilde{S}^{II} \\ & && \tilde{\xi} \in \tilde{S}^{I} \end{aligned} \tag{3}$$

where, $\tilde{\xi}$, $\tilde{\eta}$ are TrFNs giving the fuzzy strategies and $\tilde{S}^{I}$, $\tilde{S}^{II}$ are fuzzy TrFN strategy spaces of player-I and II respectively, $\tilde{A} = (\tilde{a}_{jk})_{m\times n}$ is payoff matrix with TrFN entries $\tilde{a}_{jk}$.

For Player-II: $(\text{FFLPP})^{II}$

$$\begin{aligned} &\text{Minimize} && (\tilde{v}) \quad \text{where } \tilde{v} = (v_1, v_2, v_3, v_4) \text{ is a TrFN.} \\ &\text{Subject to} && \tilde{\xi} \otimes \tilde{A} \otimes \tilde{\eta} \preccurlyeq \tilde{v} \quad \forall \ \tilde{\xi} \in \tilde{S}^{I} \\ & && \tilde{\eta} \in \tilde{S}^{II} \end{aligned} \tag{4}$$

Taking the stationary points (pure strategies) of convex polytope S^{I} and S^{II} in the constraints, we get the following FFLPPs

$(\text{FFLPP})^{I}$

$$\begin{aligned} &\text{Maximize} && (\tilde{u}) \\ &\text{Subject to} && \tilde{\xi} \otimes \tilde{A}_k \succcurlyeq \tilde{u} (k = 1,2,\dots,n). \\ & && \tilde{e}^{T} \otimes \tilde{\xi} \approx \tilde{1} \\ & && \tilde{\xi} \succcurlyeq \tilde{0} \end{aligned} \tag{5}$$

where, $\tilde{A}_k (k = 1,2,\dots,n)$ is k^{th} column of pay off matrix $\tilde{A}$ and $\tilde{e}^{T} = (\tilde{1}, \ \tilde{1}, \dots, \ \tilde{1})_{1\times m}$

$(\text{FFLPP})^{II}$

$$\begin{aligned} &\text{Minimize} && (\tilde{v}) \\ &\text{Subject to} && \tilde{A}_j \otimes \tilde{\eta} \preccurlyeq \tilde{v} (j = 1,2,\dots,m). \\ & && \tilde{e}^{T} \otimes \tilde{\eta} \approx \tilde{1} \\ & && \tilde{\eta} \succcurlyeq \tilde{0} \end{aligned} \tag{6}$$

where, $\tilde{A}_j (j = 1,2,\dots,m)$ is j^{th} row of pay off matrix $\tilde{A}$ and $\tilde{e}^{T} = (\tilde{1}, \ \tilde{1}, \dots, \ \tilde{1})_{1\times n}$.

Equation (5) and (6) respectively gives

$(\text{FFLPP})^{I}$

$$\begin{aligned} &\text{Maximize} && (u_1, u_2, u_3, u_4) \\ &\text{Subject to} && \textstyle\sum_{j=1}^{m} \tilde{a}_{jk} \otimes \tilde{\xi}_j \succcurlyeq (u_1, u_2, u_3, u_4) (k = 1,2,\dots,n) \\ & && \textstyle\sum_{j=1}^{m} \tilde{\xi}_j \approx (1,1,1,1) \\ & && \tilde{\xi}_j \succcurlyeq \tilde{0} (j = 1,2,\dots,m) \end{aligned} \tag{7}$$

(FFLPP)^{II}

Minimize (v_1, v_2, v_3, v_4)

Subject to $\sum_{k=1}^{n} \tilde{a}_{jk} \otimes \tilde{\eta}_k \preccurlyeq (v_1, v_2, v_3, v_4)(j = 1,2, \dots, m)$

$\sum_{k=1}^{n} \tilde{\eta}_k \approx (1,1,1,1)$

$\tilde{\eta}_k \succcurlyeq \tilde{0}(k = 1,2, \dots, n)$ (8)

Now if we take $\tilde{a}_{jk} = (a_{jk}^1, a_{jk}^2, a_{jk}^3, a_{jk}^4), \tilde{\xi}_j = (\xi_j^1, \xi_j^2, \xi_j^3, \xi_j^4), \tilde{\eta}_k = (\eta_k^1, \eta_k^2, \eta_k^3, \eta_k^4)$, equation (7) and (8) becomes

(FFLPP)^{I}

Maximize (u_1, u_2, u_3, u_4)

Subject to $\sum_{j=1}^{m} (a_{jk}^1, a_{jk}^2, a_{jk}^3, a_{jk}^4) \otimes (\xi_j^1, \xi_j^2, \xi_j^3, \xi_j^4) \succcurlyeq (u_1, u_2, u_3, u_4)(k = 1,2, \dots, n)$

$\sum_{j=1}^{m} (\xi_j^1, \xi_j^2, \xi_j^3, \xi_j^4) \approx (1,1,1,1)$

$(\xi_j^1, \xi_j^2, \xi_j^3, \xi_j^4) \succcurlyeq \tilde{0}(j = 1,2, \dots, m)$ (9)

(FFLPP)^{II}

Minimize (v_1, v_2, v_3, v_4)

subject to $\sum_{k=1}^{n} (a_{jk}^1, a_{jk}^2, a_{jk}^3, a_{jk}^4) \otimes (\eta_k^1, \eta_k^2, \eta_k^3, \eta_k^4) \preccurlyeq (v_1, v_2, v_3, v_4)(j = 1,2, \dots, m)$

$\sum_{j=1}^{m} (\eta_k^1, \eta_k^2, \eta_k^3, \eta_k^4) \approx (1,1,1,1)$

$(\eta_k^1, \eta_k^2, \eta_k^3, \eta_k^4) \succcurlyeq \tilde{0}(k = 1,2, \dots, n)$ (10)

Step 2: Decomposing FFLPPs into four crisp LPPs each for player-I and II respectively.

Now we decompose the above FFLPPs (9) and (10) into four crisp LPP's for both players, by solving them using the algebra of TrFNs (refer to definition 12) as follows.

For Player-I

(LPP-1)^{I}

Maximize (u_1)

subject to $\sum_{j=1}^{m} a_{jk}^1 \xi_j^1 \geq u_1; (k = 1,2, \dots, n)$

$\sum_{j=1}^{m} \xi_j^1 = 1$

$\xi_j^1 \geq 0(j = 1,2, \dots, m)$ (11)

If optimal solutions to (LPP-1)^{I} is $\alpha_j^1 : (j = 1,2, \dots, m)$ then go on to solve LPP-2 for player-I.

(LPP-2)^{I}

Maximize (u_2)

subject to $\sum_{j=1}^{m} a_{jk}^2 \xi_j^2 \geq u_2; (k = 1,2, \dots, n)$

$\sum_{j=1}^{m} \xi_j^2 = 1$

$u_2 \geq u_1, \xi_j^2 \geq \alpha_j^1 (j = 1,2, \dots, m)$ (12)

If optimal solution to (LPP-2)^{I} is $\alpha_j^2 : (j = 1,2, \dots, m)$ then go on to solve LPP-3 for player-I.

(LPP-3)^{I}

Maximize (u_3)

subject to
$$\sum_{j=1}^{m} a_{jk}^{3}\xi_j^3 \geq u_3; (k = 1,2,\dots,n)$$
$$\sum_{j=1}^{m} \xi_j^3 = 1$$
$$u_3 \geq u_2, \xi_j^3 \geq \alpha_j^2 (j = 1,2,\dots,m) \quad (13)$$

If optimal solution to (LPP-3)^{I} is $\alpha_j^3 : (j = 1,2,\dots,m)$ then go on to solve LPP-4 for player-I.

(LPP-4)^{I}

Maximize (u_4)

Subject to
$$\sum_{j=1}^{m} a_{jk}^{4}\xi_j^4 \geq u_4; (k = 1,2,\dots,n)$$
$$\sum_{j=1}^{m} \xi_j^4 = 1$$
$$u_4 \geq u_3, \xi_j^4 \geq \alpha_j^3 (j = 1,2,\dots,m) \quad (14)$$

Now for Player-II

(LPP-I)^{II}

Minimize (v_1)

Subject to
$$\sum_{k=1}^{n} a_{jk}^{1}\eta_k^1 \leq v_1; (j = 1,2,\dots,m)$$
$$\sum_{k=1}^{n} \eta_k^1 = 1$$
$$\eta_k^1 \geq 0 (k = 1,2,\dots,n) \quad (15)$$

If optimal solution to this (LPP-I)^{II} is $\beta_k^1 : (k = 1,2,\dots,n)$ then go on to write next LPP-2 for player-II.

(LPP-2)^{II}

Minimize (v_2)

Subject to
$$\sum_{k=1}^{n} a_{jk}^{2}\eta_k^2 \leq v_2; (j = 1,2,\dots,m)$$
$$\sum_{k=1}^{n} \eta_k^2 = 1$$
$$v_2 \geq v_1, \eta_k^2 \geq \beta_k^1 (k = 1,2,\dots,n) \quad (16)$$

If optimal solution to this (LPP-2)^{II} is $\beta_k^2 : (k = 1,2,\dots,n)$ then go on to next LLP for player-II.

(LPP-3)^{II}

Minimize (v_3)

Subject to
$$\sum_{k=1}^{n} a_{jk}^{3}\eta_k^3 \leq v_3; (j = 1,2,\dots,m)$$
$$\sum_{k=1}^{n} \eta_k^3 = 1$$
$$v_3 \geq v_2, \eta_k^3 \geq \beta_k^2 (k = 1,2,\dots,n) \quad (17)$$

If optimal solution to this (LPP-3)^{II} is $\beta_k^3 : (k = 1,2,\dots,n)$ then go on to next LLP for player-II.

(LPP-4)^{II}

Minimize (v_4)

Subject to
$$\sum_{k=1}^{n} a_{jk}^{4}\eta_k^4 \leq v_4; (j = 1,2,\dots,m)$$
$$\sum_{k=1}^{n} \eta_k^4 = 1$$
$$v_4 \geq v_3, \eta_k^4 \geq \beta_k^3 (k = 1,2,\dots,n) \quad (18)$$

Step 3: Solving the crisp LPPs for optimum values.
Now we solve the LPP equations (11), (12), (13) and (14) using TORA-2.0 software for optimum solution for player-I. Let $\alpha_j^1, \alpha_j^2, \alpha_j^3, \alpha_j^4, (j = 1,2, \ldots, m)$ be the optimal solution and u_1, u_2, u_3, u_4 be the optimum objective function value of these equations respectively.

Similarly for player II, let $\beta_k^1, \beta_k^2, \beta_k^3, \beta_k^4, (k = 1,2, \ldots, n)$ be the optimum solution and v_1, v_2, v_3, v_4be the optimum objective function value of LPP equations (15), (16), (17) and (18).

Step 4: Combining the results of step-3 to achieve the complete optimal solution.
According to the data obtained in step-3, we get the best/optimal strategies for player-I as $(\alpha_j^1, \alpha_j^2, \alpha_j^3, \alpha_j^4)$: $(j = 1,2, \ldots, m)$ and the best/optimal value of the game as $(u_1, u_2, u_3, u_4)$for player-I. Similarly, we get the best/optimal strategies for player -II as$(\beta_k^1, \beta_k^2, \beta_k^3, \beta_k^4), (k = 1,2, \ldots, n)$, and the best/optimal value of the game is (v_1, v_2, v_3, v_4) for player-II.

4.1 Flow Chart
For an easy understanding, a visual representation of the proposed solution methodology has been depicted by the flowchart in Figure 2.

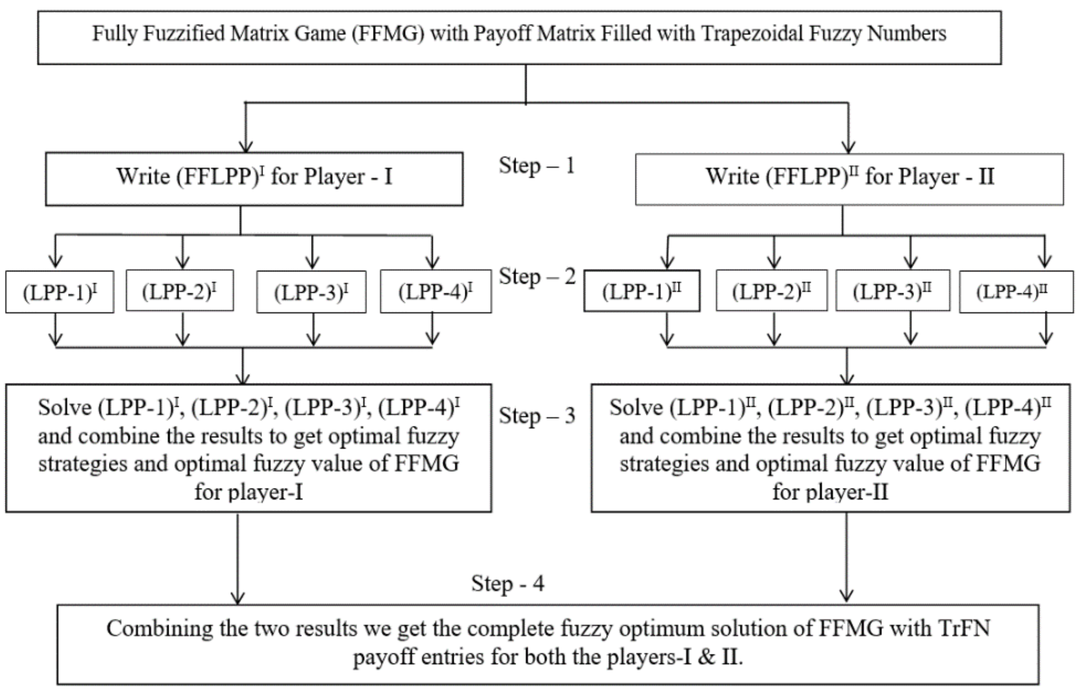


Figure 2. Flow chart of the proposed solution model.

In the next section we present three examples with different type TrFN payoffs that illustrate the computational process of our proposed model.

5. Illustration and Computational Results
Let us illustrate the proposed method by numerical examples:

Example 1: (Kumar et al., 2016) Take a specific FMG of Payoff Matrix$\tilde{A}$for player-I equipped with positive TrFN entries:

$$\tilde{A} = \begin{bmatrix} (175,180,185,190) & (150,155,157,158) \\ (80,87,92,100) & (175,180,185,190) \end{bmatrix}.$$

Assuming that $\tilde{\xi}_1, \tilde{\xi}_2$ are fuzzy TrFN strategies for player-I and $\tilde{\eta}_1, \tilde{\eta}_2$ are fuzzy TrFN strategies for player-II. The TrFNs $\tilde{u}$ and $\tilde{v}$ are the best (optimal) fuzzy TrFN values of game for player-I and player-II respectively, where,
$\tilde{\xi}_1 = (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4), \tilde{\xi}_2 = (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4), \tilde{\eta}_1 = (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4), \tilde{\eta}_2 = (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4), \tilde{u} = (u_1, u_2, u_3, u_4).$

and $\tilde{v} = (v_1, v_2, v_3, v_4)$. Now using the theory of proposed solution methodology (Refer section-4), we get the following FFLPPs for Player-I and Player-II respectively.

For player-I (FFLPP)^{I}
Maximize (u_1, u_2, u_3, u_4)
subject to

$$\begin{aligned} &(175,180,185,190) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (80,87,92,100) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq (u_1, u_2, u_3, u_4) \\ &(150,155,157,158) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (175,180,185,190) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq (u_1, u_2, u_3, u_4) \\ &(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \approx (1,1,1,1) \\ &(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \succcurlyeq \tilde{0} \\ &(\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq \tilde{0} \end{aligned} \tag{19}$$

For player-II (FFLPP)^{II}
Minimize (v_1, v_2, v_3, v_4)
subject to

$$\begin{aligned} &(175,180,185,190) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (150,155,157,158) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \preccurlyeq (v_1, v_2, v_3, v_4) \\ &(80,87,92,100) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (175,180,185,190) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \preccurlyeq (v_1, v_2, v_3, v_4) \\ &(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \approx (1,1,1,1) \\ &(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \succcurlyeq \tilde{0} \\ &(\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \succcurlyeq \tilde{0} \end{aligned} \tag{20}$$

Now for the player-I, we decompose equation (19) into four crisp LPP's as follows

(LPP-I)^{I}
Maximize (u_1)
subject to

$$\begin{aligned} &175\xi_1^1 + 80\xi_2^1 \geq u_1 \\ &150\xi_1^1 + 175\xi_2^1 \geq u_1 \\ &\xi_1^1 + \xi_2^1 = 1 \\ &\xi_1^1 \geq 0, \xi_2^1 \geq 0 \end{aligned} \tag{21}$$

Solving equation (21) we get $\xi_1^1 = 0.7917, \xi_2^1 = 0.2083, u_1 = 155.21$. Now go on to solve LPP-2,

(LPP-2)^{I}
Maximize (u_2)
subject to

$$\begin{aligned} &180\xi_1^2 + 87\xi_2^2 \geq u_2 \\ &155\xi_1^2 + 180\xi_2^2 \geq u_2 \end{aligned} \tag{22}$$

$$\xi_1^2 + \xi_2^2 = 1$$
$$\xi_1^2 \geq 0.7917, \xi_2^2 \geq 0.2083, u_2 \geq 155.21$$

Solving equation (22) we get $\xi_1^2 = 0.7917, \xi_2^2 = 0.2083, u_2 = 160.21$, now we go on to write the next LPP

(LPP-3)^{I}

Maximize (u_3)

subject to
$$185\xi_1^3 + 92\xi_2^3 \geq u_3$$
$$157\xi_1^3 + 185\xi_2^3 \geq u_3$$
$$\xi_1^3 + \xi_2^3 = 1 \quad (23)$$
$$\xi_1^3 \geq 0.7917, \xi_2^3 \geq 0.2083, u_3 \geq 160.21$$

Solving equation (23) we get $\xi_1^3 = 0.7917, \xi_2^3 = 0.2083, u_3 = 162.83$, now we solve the following LPP

(LPP-4)^{I}

Maximize (u_4)

subject to
$$190\xi_1^4 + 100\xi_2^4 \geq u_4$$
$$158\xi_1^4 + 190\xi_2^4 \geq u_4$$
$$\xi_1^4 + \xi_2^4 = 1 \quad (24)$$
$$\xi_1^4 \geq 0.7917, \xi_2^4 \geq 0.2083, u_4 \geq 162.83$$

Solving this equation (24) we get $\xi_1^4 = 0.7917, \xi_2^4 = 0.2083, u_4 = 164.67$, From the solutions of equations (21), (22), (23) and (24) we get the best (optimal) fuzzy TrFN strategy for player-I is $(0.7917, 0.7917, 0.7917, 0.7917)$ and $(0.2083, 0.2083, 0.2083, 0.2083)$ and the best (optimal) value of the game for player-I is $(155.21, 160.21, 162.83, 164.67)$ i.e. player-I will win at least $(155.21, 160.21, 162.83, 164.67)$ if he/she opts his/her $I^{st}$and II^{nd}strategies with probabilities 0.7917and 0.2083 respectively.

Now for player-II, we decompose equation (20) into four crisp LPP's as follows

(LPP-1)^{II}

Minimize (v_1)

subject to
$$175\eta_1^1 + 150\eta_2^1 \leq v_1$$
$$80\eta_1^1 + 175\eta_2^1 \leq v_1$$
$$\eta_1^1 + \eta_2^1 = 1$$
$$\eta_1^1 \geq 0, \eta_2^1 \geq 0 \quad (25)$$

Solving this equation (25) we get $\eta_1^1 = 0.2083, \eta_2^1 = 0.7917, v_1 = 155.21$, now go to write next LPP

(LPP-2)^{II}

Minimize (v_2)

subject to
$$180\eta_1^2 + 155\eta_2^2 \leq v_2$$
$$87\eta_1^2 + 180\eta_2^2 \leq v_2$$
$$\eta_1^2 + \eta_2^2 = 1 \quad (26)$$
$$\eta_1^2 \geq 0.2083, \eta_2^2 \geq 0.7917, v_2 \geq 155.21.$$

Solving this equation (26) we get $\eta_1^2 = 0.2083, \eta_2^2 = 0.7917, v_2 = 160.63$, now go on to write next LPP

(LPP-3)^{II}

Minimize (v_3)

subject to

$$\begin{aligned}&185\eta_1^3 + 157\eta_2^3 \leq v_3\\&92\eta_1^3 + 185\eta_2^3 \leq v_3\\&\eta_1^3 + \eta_2^3 = 1\\&\eta_1^3 \geq 0.2083, \eta_2^3 \geq 0.7917, v_3 \geq 160.63.\end{aligned} \tag{27}$$

Solving this equation (27) we get $\eta_1^3 = 0.2083, \eta_2^3 = 0.7917, v_3 = 165.63$, now we finally go on to solve the following LPP.

(LPP-4)^{II}

Minimize (v_4)

subject to

$$\begin{aligned}&190\eta_1^4 + 158\eta_2^4 \leq v_4\\&100\eta_1^4 + 190\eta_2^4 \leq v_4\\&\eta_1^4 + \eta_2^4 = 1\\&\eta_1^4 \geq 0.2083, \eta_2^4 \geq 0.7917, v_4 \geq 165.63.\end{aligned} \tag{28}$$

Solving this we get $\eta_1^4 = 0.2083, \eta_2^4 = 0.7917, v_4 = 171.253$. From the solution of equations (25), (26), (27) and (28) we get the best (optimal) strategies for the player-II are (0.2083, 0.2083, 0.2083, 0.2083) and $(0.7917,\ 0.7917,\ 0.7917,\ 0.7917)$ and the best (optimal) value of the game for player-II is $(155.21{,}160.63{,}165.63{,}171.25)$ i.e. player-II will lose at the most $(155.21, 160.63, 165.63, 171.25)$ if he/she opts for his/her $I^{st}$and II^{nd}strategies with probabilities 0.2083and 0.7917respectively.

Example 2: (Krishnaveni and Ganesan, 2018) Now we take an example where payoff matrix contains TrFNs with negative entries as well:

$$\tilde{A} = \begin{bmatrix} (1,4,5,6) & (1,2,4,5) & (3,4,7,8) & (4,5,7,8) \\ (5,10,12,17) & (8,10,11,19) & (5,7,10,14) & (7,10,11,12) \\ (-1,0,2,3) & (-1,2,3,4) & (12,14,18,20) & (8,17,21,30) \end{bmatrix}.$$

Assuming that $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$ are fuzzy TrFN strategies for player-I and $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4$ are fuzzy TrFN strategies for player-II. The TrFNs $\tilde{u}$ and $\tilde{v}$are the best/optimal TrFN fuzzy values of game for player-I and player-II respectively. We get the following FFLPPs for both the players.

For player - I, (FFLPP)^{I}

Maximize $(\tilde{u})$

subject to

$$\begin{aligned}&\textstyle\sum_{j=1}^{3} \tilde{a}_{jk} \otimes \tilde{\xi}_j \succcurlyeq \tilde{u}\ \ (k = 1,2,3,4)\\&\textstyle\sum_{j=1}^{3} \tilde{\xi}_j \approx \tilde{1}\\&\tilde{\xi}_j \succcurlyeq \tilde{0}(j = 1,2,3)\end{aligned} \tag{29}$$

For player- II, (FFLPP)^{II}

Minimize $(\tilde{v})$

subject to

$$\begin{aligned}&\textstyle\sum_{k=1}^{4} \tilde{a}_{jk} \otimes \tilde{\eta}_k \preccurlyeq \tilde{v}(j = 1,2,3)\\&\textstyle\sum_{k=1}^{4} \tilde{\eta}_k \approx \tilde{1}\\&\tilde{\eta}_k \succcurlyeq \tilde{0}(k = 1,2,3,4)\end{aligned} \tag{30}$$

Equation (29) gives us the following FFLPP for Player-I

$(FFLPP)^I$
Maximize $(\tilde{u})$
subject to

$$(1,4,5,6) \otimes \tilde{\xi}_1 \oplus (5,10,12,17) \otimes \tilde{\xi}_2 \oplus (-1,0,2,3) \otimes \tilde{\xi}_3 \succcurlyeq \tilde{u}$$
$$(1,2,4,5) \otimes \tilde{\xi}_1 \oplus (8,10,11,19) \otimes \tilde{\xi}_2 \oplus (-1,2,3,4) \otimes \tilde{\xi}_3 \succcurlyeq \tilde{u}$$
$$(3,4,7,8) \otimes \tilde{\xi}_1 \oplus (5,7,10,14) \otimes \tilde{\xi}_2 \oplus (12,14,18,20) \otimes \tilde{\xi}_3 \succcurlyeq \tilde{u}$$
$$(4,5,7,8) \otimes \tilde{\xi}_1 \oplus (7,10,11,12) \otimes \tilde{\xi}_2 \oplus (8,17,21,30) \otimes \tilde{\xi}_3 \succcurlyeq \tilde{u}$$
$$\tilde{\xi}_1 \oplus \tilde{\xi}_2 \oplus \tilde{\xi}_3 \approx \tilde{1}$$
$$\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3 \succcurlyeq \tilde{0} \tag{31}$$

let us take $\tilde{\xi}_1 = (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4), \tilde{\xi}_2 = (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4), \tilde{\xi}_3 = (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4)$ and $\tilde{u} = (u_1, u_2, u_3, u_4)$in equation (31), we get,

$(FFLPP)^I$
Maximize (u_1, u_2, u_3, u_4)
subject to

$$(1,4,5,6) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (5,10,12,17) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \oplus (-1,0,2,3) \otimes (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(1,2,4,5) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (8,10,11,19) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \oplus (-1,2,3,4) \otimes (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(3,4,7,8) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (5,7,10,14) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \oplus (12,14,18,20) \otimes (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(4,5,7,8) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (7,10,11,12) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \oplus (8,17,21,30) \otimes (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \oplus (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \approx \tilde{1}$$
$$(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \succcurlyeq \tilde{0}$$
$$(\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq \tilde{0}$$
$$(\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \succcurlyeq \tilde{0}. \tag{32}$$

$(FFLPP)^I$ further yields
Maximize (u_1, u_2, u_3, u_4)
subject to

$$(\xi_1^1, 4\xi_1^2, 5\xi_1^3, 6\xi_1^4) \oplus (5\xi_2^1, 10\xi_2^2, 12\xi_2^3, 17\xi_2^4) \oplus (-\xi_3^4, 0\xi_3^2, 2\xi_3^3, 3\xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(\xi_1^1, 2\xi_1^2, 4\xi_1^3, 5\xi_1^4) \oplus (8\xi_2^1, 10\xi_2^2, 11\xi_2^3, 19\xi_2^4) \oplus (-\xi_3^4, 2\xi_3^2, 3\xi_3^3, 4\xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(3\xi_1^1, 4\xi_1^2, 7\xi_1^3, 8\xi_1^4) \oplus (5\xi_2^1, 7\xi_2^2, 10\xi_2^3, 14\xi_2^4) \oplus (12\xi_3^1, 14\xi_3^2, 18\xi_3^3, 20\xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(4\xi_1^1, 5\xi_1^2, 7\xi_1^3, 8\xi_1^4) \oplus (7\xi_2^1, 10\xi_2^2, 11\xi_2^3, 12\xi_2^4) \oplus (8\xi_3^1, 17\xi_3^2, 21\xi_3^3, 30\xi_3^4) \succcurlyeq (u_1, u_2, u_3, u_4)$$
$$(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \oplus (\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \approx \tilde{1}$$
$$(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \succcurlyeq \tilde{0}$$
$$(\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq \tilde{0}$$
$$(\xi_3^1, \xi_3^2, \xi_3^3, \xi_3^4) \succcurlyeq \tilde{0}. \tag{33}$$

Now we decompose equation (33) into four crisp LPPs as follows for Player-1.
$(LPP\text{-}1)^I$
Maximize (u_1)

subject to

$$\xi_1^1 + 5\xi_2^1 + (-1)\xi_3^4 \geq u_1$$
$$\xi_1^1 + 8\xi_2^1 + (-1)\xi_3^4 \geq u_1$$
$$3\xi_1^1 + 5\xi_2^1 + 12\xi_3^1 \geq u_1$$
$$4\xi_1^1 + 7\xi_2^1 + 8\xi_3^1 \geq u_1$$
$$\xi_1^1 + \xi_2^1 + \xi_3^1 = 1$$
$$\xi_3^4 - \xi_3^1 \geq 0$$
$$\xi_1^1, \xi_2^1, \xi_3^1, \xi_3^4 \geq 0. \quad (34)$$

On solving equation (34) we get $\xi_1^1 = 0, \xi_2^1 = 1, \xi_3^1 = 0, \xi_3^4 = 0$ and $u_1 = 5$, now let us go on to solve the following LPP.

(LPP-2)^I

Maximize (u_2)

subject to

$$4\xi_1^2 + 10\xi_2^2 + 0\xi_3^2 \geq u_2$$
$$2\xi_1^2 + 10\xi_2^2 + 2\xi_3^2 \geq u_2$$
$$4\xi_1^2 + 7\xi_2^2 + 14\xi_3^2 \geq u_2$$
$$5\xi_1^2 + 10\xi_2^2 + 17\xi_3^2 \geq u_2$$
$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$
$$\xi_1^2 \geq 0, \xi_2^2 \geq 1, \xi_3^2 \geq 0, u_2 \geq 5. \quad (35)$$

On solving equation (35) we get $\xi_1^2 = 0, \xi_2^2 = 1, \xi_3^2 = 0$ and $u_2 = 7$. Now let us go on to solve LPP-3

(LPP-3)^I

Maximize (u_3)

subject to

$$5\xi_1^3 + 12\xi_2^3 + 2\xi_3^3 \geq u_3$$
$$4\xi_1^3 + 11\xi_2^3 + 3\xi_3^3 \geq u_3$$
$$7\xi_1^3 + 10\xi_2^3 + 18\xi_3^3 \geq u_3$$
$$7\xi_1^3 + 11\xi_2^3 + 21\xi_3^3 \geq u_3$$
$$\xi_1^3 + \xi_2^3 + \xi_3^3 = 1$$
$$\xi_1^3 \geq 0, \xi_2^3 \geq 1, \xi_3^3 \geq 0, u_3 \geq 7 \quad (36)$$

On solving equation (36) we get $\xi_1^3 = 0, \xi_2^3 = 1, \xi_3^3 = 0$ and $u_3 = 10$. now finally let us go on to solve the next LPP for player I

(LPP-4)^I

Maximize (u_4)

subject to

$$6\xi_1^4 + 17\xi_2^4 + 3\xi_3^4 \geq u_4$$
$$5\xi_1^4 + 19\xi_2^4 + 4\xi_3^4 \geq u_4$$
$$8\xi_1^4 + 14\xi_2^4 + 20\xi_3^4 \geq u_4$$
$$8\xi_1^4 + 12\xi_2^4 + 30\xi_3^4 \geq u_4$$
$$\xi_1^4 + \xi_2^4 + \xi_3^4 = 1$$
$$\xi_3^4 - \xi_3^1 \geq 0$$
$$\xi_1^4 \geq 0, \xi_2^4 \geq 1, \xi_3^4 \geq 0, u_4 \geq 10 \quad (37)$$

On solving equation (37) we get $\xi_1^4 = 0, \xi_2^4 = 1, \xi_3^4 = 0$ and $u_4 = 12$. So for player-1 we get the best fuzzy TrFN strategies as $\tilde{\xi}_1 = (0,0,0,0), \tilde{\xi}_2 = (1,1,1,1), \tilde{\xi}_3 = (0,0,0,0)$ and the best fuzzy TrFN value of the game $\tilde{u} = (5,7,10,12)$ i.e., Player- 1 will never play his/her first and third strategy. He/she will win at least (5,7,10,12) by playing the second strategy only.

Now for player-II

$(FFLPP)^{II}$

Minimize $(\tilde{v})$

subject to

$$\begin{aligned}
&(1,4,5,6) \otimes \tilde{\eta}_1 \oplus (1,2,4,5) \otimes \tilde{\eta}_2 \oplus (3,4,7,8) \otimes \tilde{\eta}_3 \oplus (4,5,7,8) \otimes \tilde{\eta}_4 \preccurlyeq (\tilde{v})\\
&(5,10,12,17) \otimes \tilde{\eta}_1 \oplus (8,10,11,19) \otimes \tilde{\eta}_2 \oplus (5,7,10,14) \otimes \tilde{\eta}_3 \oplus (7,10,11,12) \otimes \tilde{\eta}_4 \preccurlyeq (\tilde{v})\\
&(-1,0,2,3) \otimes \tilde{\eta}_1 \oplus (-1,2,3,4) \otimes \tilde{\eta}_2 \oplus (12,14,18,20) \otimes \tilde{\eta}_3 \oplus (8,17,21,30) \otimes \tilde{\eta}_4 \preccurlyeq (\tilde{v})\\
&\tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3 \oplus \tilde{\eta}_4 \approx \tilde{1}\\
&\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \tilde{\eta}_4 \succcurlyeq \tilde{0} \qquad (38)
\end{aligned}$$

Let us take $\tilde{\eta}_1 = (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4), \tilde{\eta}_2 = (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4), \tilde{\eta}_3 = (\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4), \tilde{\eta}_4 = (\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4)$ and $\tilde{v} = (v_1, v_2, v_3, v_4)$ in equation (38). This gives.

$(FFLPP)^{II}$

Minimize (v_1, v_2, v_3, v_4)

subject to

$$\begin{aligned}
&(1,4,5,6) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (1,2,4,5) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \oplus (3,4,7,8) \otimes (\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4)\\
&\oplus (4,5,7,8) \otimes (\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \preccurlyeq (v_1, v_2, v_3, v_4)\\
&(5,10,12,17) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (8,10,11,19) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \oplus (5,7,10,14) \otimes (\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4)\\
&\oplus (7,10,11,12) \otimes (\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \preccurlyeq (v_1, v_2, v_3, v_4)\\
&(-1,0,2,3) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (-1,2,3,4) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \oplus (12,14,18,20) \otimes (\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4)\\
&\oplus (8,17,21,30) \otimes (\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \preccurlyeq (v_1, v_2, v_3, v_4)\\
&(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \oplus (\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4) \oplus (\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \approx (1,1,1,1)\\
&(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \succcurlyeq (0,0,0,0)\\
&(\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \succcurlyeq (0,0,0,0)\\
&(\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4) \succcurlyeq (0,0,0,0)\\
&(\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \succcurlyeq (0,0,0,0) \qquad (39)
\end{aligned}$$

Equation (39) further gives

$(FFLPP)^{II}$

Minimize (v_1, v_2, v_3, v_4)

subject to

$$\begin{aligned}
&(\eta_1^1, 4\eta_1^2, 5\eta_1^3, 6\eta_1^4) \oplus (\eta_2^1, 2\eta_2^2, 4\eta_2^3, 5\eta_2^4) \oplus (3\eta_3^1, 4\eta_3^2, 7\eta_3^3, 8\eta_3^4) \oplus (4\eta_4^1, 5\eta_4^2, 7\eta_4^3, 8\eta_4^4)\\
&\qquad \preccurlyeq (v_1, v_2, v_3, v_4)\\
&(5\eta_1^1, 10\eta_1^2, 12\eta_1^3, 17\eta_1^4) \oplus (8\eta_2^1, 10\eta_2^2, 11\eta_2^3, 19\eta_2^4) \oplus (5\eta_3^1, 7\eta_3^2, 10\eta_3^3, 14\eta_3^4)\\
&\qquad \oplus (7\eta_4^1, 10\eta_4^2, 11\eta_4^3, 12\eta_4^4) \preccurlyeq (v_1, v_2, v_3, v_4)\\
&(-\eta_1^4, 0\eta_1^2, 2\eta_1^3, 3\eta_1^4) \oplus (-\eta_2^4, 2\eta_2^2, 3\eta_2^3, 4\eta_2^4) \oplus (12\eta_3^1, 14\eta_3^2, 18\eta_3^3, 20\eta_3^4) \oplus (8\eta_4^1, 17\eta_4^2, 21\eta_4^3, 30\eta_4^4)\\
&\qquad \preccurlyeq (v_1, v_2, v_3, v_4)\\
&(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \oplus (\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4) \oplus (\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \approx (1,1,1,1)\\
&(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \succcurlyeq (0,0,0,0)\\
&(\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \succcurlyeq (0,0,0,0)\\
&(\eta_3^1, \eta_3^2, \eta_3^3, \eta_3^4) \succcurlyeq (0,0,0,0)\\
&(\eta_4^1, \eta_4^2, \eta_4^3, \eta_4^4) \succcurlyeq (0,0,0,0) \qquad (40)
\end{aligned}$$

Now for player-II we decompose equation (40) into four crisp LPPs as follows:

(LPP-1)^{II}

Minimize (v_1)

subject to

$$\eta_1^1 + \eta_2^1 + 3\eta_3^1 + 4\eta_4^1 \le v_1$$
$$5\eta_1^1 + 8\eta_2^1 + 5\eta_3^1 + 7\eta_4^1 \le v_1$$
$$-\eta_1^4 - \eta_2^4 + 12\eta_3^1 + 8\eta_4^1 \le v_1$$
$$\eta_1^1 + \eta_2^1 + \eta_3^1 + \eta_4^1 = 1$$
$$\eta_1^4 - \eta_1^1 \ge 0$$
$$\eta_2^4 - \eta_2^1 \ge 0$$
$$\eta_1^1 \ge 0, \eta_2^1 \ge 0, \eta_3^1 \ge 0, \eta_4^1 \ge 0 \tag{41}$$

The solution of equation (41) gives $\eta_1^1 = 0.54$, $\eta_2^1 = 0$, $\eta_3^1 = 0.46$, $\eta_4^1 = 0, \eta_1^4 = 0.54$, $\eta_2^4 = 0$ and $v_1 = 5$.

Now we go on to solve the next LPP-2 for player-II

(LPP-2)^{II}

Minimize (v_2)

subject to

$$4\eta_1^2 + 2\eta_2^2 + 4\eta_3^2 + 5\eta_4^2 \le v_2$$
$$10\eta_1^2 + 10\eta_2^2 + 7\eta_3^2 + 10\eta_4^2 \le v_2$$
$$0\eta_1^2 + 2\eta_2^2 + 14\eta_3^2 + 17\eta_4^2 \le v_2$$
$$\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1$$
$$\eta_1^2 \ge 0.54, \eta_2^2 \ge 0, \eta_3^2 \ge 0.46, \eta_4^2 \ge 0, v_2 \ge 5 \tag{42}$$

The solution of equation (42) gives $\eta_1^2 = 0.54, \eta_2^2 = 0, \eta_3^2 = 0.46, \eta_4^2 = 0$and $v_2 = 8.62$. Now we go on to solve the next LPP-3 for player-II

(LPP-3)^{II}

Minimize (v_3)

subject to

$$5\eta_1^3 + 4\eta_2^3 + 7\eta_3^3 + 7\eta_4^3 \le v_3$$
$$12\eta_1^3 + 11\eta_2^3 + 10\eta_3^3 + 11\eta_4^3 \le v_3$$
$$2\eta_1^3 + 3\eta_2^3 + 18\eta_3^3 + 21\eta_4^3 \le v_3$$
$$\eta_1^3 + \eta_2^3 + \eta_3^3 + \eta_4^3 = 1$$
$$\eta_1^3 \ge 0.54, \eta_2^3 \ge 0, \eta_3^3 \ge 0.46, \eta_4^3 \ge 0, v_3 \ge 8.62 \tag{43}$$

The solution of equation (43) gives $\eta_1^3 = 0.54$, $\eta_2^3 = 0$, $\eta_3^3 = 0.46$, $\eta_4^3 = 0$ and v_3 = 11.08. Now finally we go on to solve the LPP-4 for player-II

(LPP-4)^{II}

Minimize (v_4)

subject to

$$6\eta_1^4 + 5\eta_2^4 + 8\eta_3^4 + 8\eta_4^4 \le v_4$$
$$17\eta_1^4 + 19\eta_2^4 + 14\eta_3^4 + 12\eta_4^4 \le v_4$$
$$3\eta_1^4 + 4\eta_2^4 + 20\eta_3^4 + 30\eta_4^4 \le v_4$$
$$\eta_1^4 + \eta_2^4 + \eta_3^4 + \eta_4^4 = 1$$
$$\eta_1^4 \ge .54, \eta_2^4 \ge 0, \eta_3^4 \ge .46, \eta_4^4 \ge 0, v_4 \ge 11.08 \tag{44}$$

The solution of equation (44) gives $\eta_1^4 = 0.54, \eta_2^4 = 0, \eta_3^4 \ge 0.46, \eta_4^4 = 0, v_4 = 15.62$. Therefore for player- II, the best value of the game is (5, 8.62, 11.08, 15.62) and the best strategies are given as $\tilde{\eta}_1$ = (0.54, 0.54, 0.54, 0.54), $\tilde{\eta}_2$= (0, 0, 0, 0), $\tilde{\eta}_3$ = (0.46, 0.46, 0.46, 0.46) and $\tilde{\eta}_4$ = (0, 0, 0, 0) i.e player- II will never use his /her IInd and IVth strategies and will lose at the most (5, 8.62, 11.08, 15.62) if he/she plays the first and third strategies with probabilities 0.54 and 0.46 respectively.

Example 3: Bandyopadhyay and Nayak (2013): Now we take a specific FMG with positive symmetric TrFN payoff matrix:

$$\tilde{A} = \begin{bmatrix} (1,2,3,4) & (2.5,5,7.5,10) \\ (3.5,7,10.5,14) & (1.5,3,4.5,6) \end{bmatrix}.$$

Assuming that $\tilde{\xi}_1, \tilde{\xi}_2$ are fuzzy TrFN strategies for player-I and $\tilde{\eta}_1, \tilde{\eta}_2$ are fuzzy TrFN strategies for player-II. The TrFNs $\tilde{u}$ and $\tilde{v}$ are the best fuzzy TrFN values of game for player-I and player-II respectively, where
$\tilde{\xi}_1 = (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4), \tilde{\xi}_2 = (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4), \tilde{\eta}_1 = (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4), \tilde{\eta}_2 = (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4), \tilde{u} = (u_1, u_2, u_3, u_4)$ and $\tilde{v} = (v_1, v_2, v_3, v_4)$. Now using the theory of proposed solution methodology (Refer section-4), we get the following FFLPPs for Player-I and Player-II respectively.

For player-I (FFLPP)^{I}

Maximize (u_1, u_2, u_3, u_4)

subject to

$$\begin{aligned} &(1,2,3,4) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (3.5,7,10.5,14) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq (u_1, u_2, u_3, u_4) \\ &(2.5,5,7.5,10) \otimes (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (1.5,3,4.5,6) \otimes (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq (u_1, u_2, u_3, u_4) \\ &(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \oplus (\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \approx (1,1,1,1) \\ &(\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4) \succcurlyeq \tilde{0} \\ &(\xi_2^1, \xi_2^2, \xi_2^3, \xi_2^4) \succcurlyeq \tilde{0} \end{aligned} \tag{45}$$

For player-II (FFLPP)^{II}

Minimize (v_1, v_2, v_3, v_4)

subject to

$$\begin{aligned} &(1,2,3,4) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (2.5,5,7.5,10) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \preccurlyeq (v_1, v_2, v_3, v_4) \\ &(3.5,7,10.5,14) \otimes (\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (1.5,3,4.5,6) \otimes (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \preccurlyeq (v_1, v_2, v_3, v_4) \\ &(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \oplus (\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \approx (1,1,1,1) \\ &(\eta_1^1, \eta_1^2, \eta_1^3, \eta_1^4) \succcurlyeq \tilde{0} \\ &(\eta_2^1, \eta_2^2, \eta_2^3, \eta_2^4) \succcurlyeq \tilde{0} \end{aligned} \tag{46}$$

Now for the player-I, we decompose equation (45) into four crisp LPP's as follows

(LPP-I)^{I}

Maximize (u_1)

subject to

$$\begin{aligned} &1.0\xi_1^1 + 3.5\xi_2^1 \geq u_1 \\ &2.5\xi_1^1 + 1.5\xi_2^1 \geq u_1 \\ &\xi_1^1 + \xi_2^1 = 1 \\ &\xi_1^1 \geq 0, \xi_2^1 \geq 0 \end{aligned} \tag{47}$$

Solving equation (47) we get $\xi_1^1 = 0.57, \xi_2^1 = 0.43, u_1 = 2.07$. Now we go on towrite the next LPP.

(LPP-2)^{I}

Maximize (u_2)

subject to
$$2.0\xi_1^2 + 7.0\xi_2^2 \geq u_2$$
$$5.0\xi_1^2 + 3.0\xi_2^2 \geq u_2$$
$$\xi_1^2 + \xi_2^2 = 1$$
$$\xi_1^2 \geq 0.57, \xi_2^2 \geq 0.43, u_2 \geq 2.07 \quad (48)$$

Solving equation (48) we get $\xi_1^2 = 0.57, \xi_2^2 = 0.43, u_2 = 4.14$, now we go on to write the next LPP
(
LPP-3)$^{\mathrm{I}}$

Maximize (u_3)

subject to
$$3.0\xi_1^3 + 10.5\xi_2^3 \geq u_3$$
$$7.5\xi_1^3 + 4.5\xi_2^3 \geq u_3$$
$$\xi_1^3 + \xi_2^3 = 1$$
$$\xi_1^3 \geq 0.57, \xi_2^3 \geq 0.43, u_3 \geq 4.14 \quad (49)$$

Solving equation (49) we get $\xi_1^3 = 0.57, \xi_2^3 = 0.43, u_3 = 6.21$, now we solve the following LPP

(LPP-4)$^{\mathrm{I}}$

Maximize (u_4)

subject to
$$4.0\xi_1^4 + 14\xi_2^4 \geq u_4$$
$$10\xi_1^4 + 6.0\xi_2^4 \geq u_4$$
$$\xi_1^4 + \xi_2^4 = 1$$
$$\xi_1^4 \geq 0.57, \xi_2^4 \geq 0.43, u_4 \geq 6.21 \quad (50)$$

Solving this LPP equation (50) we get $\xi_1^4 = 0.57, \xi_2^4 = 0.43, u_4 = 8.28$, From the solutions of equations (47), (48), (49) and (50) we get the best fuzzy TrFN strategy for player-I is$(0.57, 0.57, 0.57, 0.57)$ and $(0.43, 0.43, 0.43, 0.43)$ and the best value of the game for player-I is $(2.07, 4.14, 6.21, 8.28)$. i.e. player-I will win at least $(2.07, 4.14, 6.21, 8.28)$. if he/she opts his/her$I^{st}$and II^{nd}strategies with probabilities 0.57and 0.43 respectively.

Now for player-II, we decompose equation (46) into four crisp LPP's as follows

(LPP-1)$^{\mathrm{II}}$

Minimize (v_1)

subject to
$$1.0\eta_1^1 + 2.5\eta_2^1 \leq v_1$$
$$3.5\eta_1^1 + 1.5\eta_2^1 \leq v_1$$
$$\eta_1^1 + \eta_2^1 = 1$$
$$\eta_1^1 \geq 0, \eta_2^1 \geq 0 \quad (51)$$

Solving this equation (51) we get $\eta_1^1 = 0.29, \eta_2^1 = 0.71, v_1 = 2.07$, now go to write next LPP
(LPP-2)$^{\mathrm{II}}$

Minimize (v_2)

subject to
$$2.0\eta_1^2 + 5.0\eta_2^2 \leq v_2$$
$$7.0\eta_1^2 + 3.0\eta_2^2 \leq v_2$$
$$\eta_1^2 + \eta_2^2 = 1$$
$$\eta_1^2 \geq 0.29, \eta_2^2 \geq 0.71, v_2 \geq 2.07. \quad (52)$$

Solving this equation (52) we get $\eta_1^2 = 0.29, \eta_2^2 = 0.71, v_2 = 4.16,$ now go on to write next LPP-3

(LPP-3)^{II}

Minimize (v_3)

subject to

$$3.0\eta_1^3 + 7.5\eta_2^3 \leq v_3$$
$$10.5\eta_1^3 + 4.5\eta_2^3 \leq v_3$$
$$\eta_1^3 + \eta_2^3 = 1$$
$$\eta_1^3 \geq 0.29, \eta_2^3 \geq 0.71, v_3 \geq 4.16. \quad (53)$$

Solving this equation (53) we get $\eta_1^3 = 0.29, \eta_2^3 = 0.71, v_3 = 6.24$. Now we finally go on to solve the following LPP.

(LPP-4)^{II}

Minimize (v_4)

subject to

$$4.0\eta_1^4 + 10.0\eta_2^4 \leq v_4$$
$$14\eta_1^4 + 6.0\eta_2^4 \leq v_4$$
$$\eta_1^4 + \eta_2^4 = 1$$
$$\eta_1^4 \geq 0.29, \eta_2^4 \geq 0.71, v_4 \geq 6.24. \quad (54)$$

Solving this we get $\eta_1^4 = 0.29$, $\eta_2^4 = 0.71$, $v_4 = 8.32$. From the solution of equations (51), (52), (53) and (54) we get the best strategies (optimal) for the player-II as $(0.29, 0.29, 0.29, 0.29)$ and $(0.71, 0.71, 0.71, 0.71)$ and the best value of the game for player-II is $(2.07, 4.16, 6.24, 8.32)$ i.e. player-II will lose at the most $(2.07, 4.16, 6.24, 8.32)$ if he/she opts for his/her I^{st} and II^{nd} strategies with probabilities 0.29 and 0.71 respectively.

6. Comparison

In our paper, three numerical illustrations have been provided with different type of TrFN payoff matrix to show that our technique can unriddle all these problems, whereas existing methods given by researchers like (Bandopadhyay and Nayak, 2013; Kumar et al., 2016; Krishnaveni and Ganesan, 2018) has considered only a particular type of TrFNs for their study. Their methods may fail to solve FMGs with other type of TrFN payoffs. Our single technique is good enough to solve FFMGs with all these different types of TrFN payoffs and has given almost the same results. So, our method has an advantage over the others as we don't need three different methods, one for each type, as proposed by these researcher's. Also, they have considered crisp strategies, whereas ours is a fully fuzzified case, therefore it is more general and can handle FFMG with all type of TrFNs. Comparisons of our results with the results of other researchers are given in following tables.

Table 1. Comparison of our work with the results of other researchers for example 1.

Reference	Player – I's optimal mixed strategies	Player – II's optimal mixed strategies	Player – I's optimal TrFN value of the game	Player – II's optimal TrFN value of the game
Kumar et al. (2016)	(0.7377, 0.2623) (0.7917, 0.2083)	Not computed	(155.20, 160.29, 162.93, 166.39)	Assumed as –ve of player-I's value
Our work	(0.7917, 0.7917, 0.7917, 0.7917) (0.2083, 0.2083, 0.2083, 0.2083)	(0.2083, 0.2083, 0.2083, 0.2083) (0.7917, 0.7917, 0.7917, 0.7917)	(155.21, 160.21, 162.83, 164.67)	(155.21, 160.63, 165.63, 171.25)

Table 2. Comparsion of our work with the results of other researchers for example 2.

Reference	Player – I's optimal mixed strategies	Player – II's optimal mixed strategies	Player – I's optimal TrFN value of the game	Player – II's optimal TrFN value of the game
Krishnaveni and Ganesan (2018)	(0, 0, 0, 0) (-6.14, -1.14, 2.85, 7.85) (-6.66, -1.66, 2.33, 7.33)	(-6.57, -1.57, 2.42, 7.42) (0, 0, 0, 0) (-6.42, -1.42, 2.57, 7.57) (0, 0, 0, 0)	(2.5, 7.5, 11.5, 16.5)	Assumed as –ve of player-I's value
Our work	(0, 0, 0, 0) (1, 1, 1, 1) (0, 0, 0, 0)	(0.54, 0.54, 0.54, 0.54) (0, 0, 0, 0) (0.46, 0.46, 0.46, 0.46) (0, 0, 0, 0)	(5, 7, 10, 12)	(5, 8.62, 11.08, 15.62)

Table 3. Comparison of our work with the results of other researchers for example 3.

Reference	Player – I's optimal mixed strategies	Player – II's optimal mixed strategies	Player – I's optimal TrFN value of the game	Player – II's optimal TrFN value of the game
Bandopadhyay And Nayak (2013)	(0.5714, 0.4286)	(0.2857, 0.7143)	(2.0714, 4.1429, 6.2143, 8.2857)	Assumed as –ve of player-I's value
Our work	(0.57, 0.57, 0.57, 0.57) (0.43, 0.43, 0.43, 0.43)	(0.29, 0.29, 0.29, 0.29) (0.71, 0.71, 0 .71, 0 .71)	(2.07, 4.14, 6.21, 8.28)	(2.07, 4.14, 6.24, 8.32)

7. Conclusion

In our present article, we have proposed an easy to compute decomposition methodology to obtain the optimal solution of a fully fuzzified ZSTPMG with payoff matrix equipped with TrFNs. The methodology for solving matrix games, elaborated in our work, encompasses a wider class of ZSTPMGs as it does not restrict payoff matrix to a particular type of TrFNs. Numerical illustrations taken clearly show that our methodology is capable of solving ZSTPMGs with all type of TrFN payoff matrix. Also, we have not used any defuzzification function to crispify the TrFNs. So, our results cannot be influenced by changing any defuzzification function. With the help of TrFNs we can handle almost all kind of uncertainties and imprecision that occurs in day to day working of highly competitive real industrial world. Our work can help the competitive players of industrial world to make better decisions that can help them commercially and economically. In future we plan to further extend our work to a variety of games such as constrained fuzzy matrix games and bimatrix games.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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