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Novel Solution for Time-fractional Klein-Gordon Equation with Different Applications

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Abstract
In this paper, for the first time, the Laplace Homotopy Perturbation Method (LHPM), which is the coupling of the Laplace transform and the Homotopy Perturbation Method, is employed to solve non-linear time-fractional Klein-Gordon (TFKG) equations. In other words, for the first time in literature, LHPM is used to solve non-linear TFKG equations. First of all, the procedure of LHPM on TFKG with Caputo fractional derivative is developed. Further, the developed approach of LHPM on TFKG is used for two different examples. This in turn proves the versatile nature of the proposed method. In addition, the validity of the approach is proved by comparing the numerical solutions of both examples with their exact solution. Finally, the comparison of relative errors calculated in each example proves the efficiency and effectiveness of the proposed method on TFKG equations.

Keywords- Caputo fractional derivative, Laplace transform, Perturbation, Homotopy, Klein-Gordon equation.

1. Introduction

The application of Partial differential equations (PDE) is widespread in a variety of disciplines, including biology, chemistry, economics, engineering, and physics. Research on the application of PDE has recently been discovered in the fields of nanotechnology, and electronic communication, including blogs and Facebook, relativistic physics, condensed type matter physics, fluid mechanics, non-linear optics, chemical kinetics, wave phenomena, etc. (Ahmed et al., 2020; Inc et al., 2020; Shehata et al., 2019; Zayed et al., 2020). The fact that ODE or PDE is used to express real-world complex system problems. Since it may be challenging to achieve accurate results for many PDEs; therefore, approximate solutions to their fractional partial differential equation (FPDE) of different orders could easily be established using various numerical techniques. An overview of different fractional derivatives, fractional differential equations, and methods for solving them is discussed (Miller and Ross, 1993; Podlubny, 1999; Kilbas et al., 2006). Since 2000 onwards, fractional calculus has been indeed playing a vital role in research. Many researchers have used different fractional derivative operators such as Riemann-Liouville fractional differential operators, Caputo, Caputo-Fabrizio, Grunwald-Letnikov, etc. in fractional partial differential equations of the any system. Authors have been focusing on the numerical solutions of FPDEs with these novel fractional derivative operators over the past few years while considering a variety of fractional orders. According to the article (Liu et al., 2020), authors have developed a new four-dimensional fractional-order chaotic system using fractional calculus for the simplest memristive circuit. However, some authors, in the reference (Wang et al., 2022), have analyzed the numerical solution of travelling waves in chemical kinetics. Interestingly, Naeem et al. (2022) have discussed the solution of the Fuzzy fractional order of the KdV equation in the sense of the Caputo-Fabrizio derivative. Although a physical phenomenon typically depends on both space and time, FPDE is also specified for time, space, and time-space FPDE with a variety of fractional derivative operators (Podlubny, 1999). For the last few years, researchers have been showing interest in analytical and approximation solutions of FPDE in boundary value and initial value problems (Javeed et al., 2018; Rezazadeh et al., 2019; Zafar et al., 2022). Many researchers and scientists have demonstrated the importance of one such non-linear and complex PDE called the Klein-Gordon, which has widespread applications in non-linear optics, condensed-type matter physics, quantum mechanics, etc. (Kanth and Aruna, 2009; Kumar et al., 2014; Mahdy et al., 2015; Inc et al., 2020).

In many works of literature, it is found that there are various methods to solve the non-linear and linear Klein–Gordon type equations. Further, these numerical techniques have also been investigated and used to evaluate the approximate numerically fractional Klein-Gordon equations. In the past decades, the Transforms method (Abuteen et al., 2016), Integral method (Eslami and Rezazadeh, 2016), Wavelet method (Hariharan, 2013), Adomian decomposition method (ADM) (Ghadle and Khan, 2017), Variation iteration method (VIM) (Odibat and Momani, 2009), Homotopy Perturbation method (HPM) (Golmankhaneh et al., 2011), Homotopy Adomian method (HAM) (Jafari and Seifi, 2009; Kurulay, 2012; Jafari, 2016) etc. were applied on fractional Klein–Gordon equations for its analytical and numerical solutions. Also, Tamsir and Srivastava (2016) used the fractional reduced differential transform method (FRDTM) to the TFKG equation with fractional orders from 0 to 1. Further, in some other literatures(Johnston et al., 2016; Satsanit and Arnuphap, 2019), authors applied the Laplace Homotopy Perturbation Method (LHPM) on linear or non-linear PDEs, fractional partial differential equations, or ordinary differential equations to obtain more accurate results compared to other methods like HAM, ADM, VIM, etc.

Now, in this paper, we propose this method i.e., LHPM, a numerical and computational approach that has never been used on time-fractional Klein-Gordon type equations. However, Rajaraman et al. (2012) applied this approach to the Klein-Gordon equation in his work. Compared to other numerical approaches, this method provides a quick converging solution to exact solutions. However, this paper presents a novel solution for fractional Klein-Gordon type equations with different fractional order from 1 to 2 in the Caputo sense, among the various analytical numerical solutions with better approximate and convergence properties, with specific applications. Earlier, Golmankhaneh et al. (2011) introduced the homotopy perturbation method to time-fractional Klein-Gordon while taking fractional order from 0 to 1. In 2011, Madani et al. (2011) proposed the coupling method of the transform method and HPM for approximated solutions of the partial differential equations. The concept of perturbation was first introduced by He (2005). Later, proposing new analytical approximated solutions of fractional partial differential equations (FPDEs) considering Liouville-Caputo and Caputo-Fabrizio derivative operators, Morales-Delgado et al. (2016) also discussed another method based on a combination of the Laplace transform and Homotopy Adomian Method (LHAM). Eventually, many authors have proven that the Laplace homotopy perturbation method (LHPM), among the numerical methods, outperforms in terms of convergence. Once again, it is indeed important to emphasize that, to the best of our knowledge; no open literature proposes a solution for TFKG equations using LHPM. In addition, using the proposed LHPM, we are aiming to compute approximate solutions to various numerical problems based on time-fractional Klein-Gordon equations.

Finally, this paper is categorized into various sections. The analytical approximate solution for TFKG using LHPM is suggested in section 2. As in section 3, examples are used to demonstrate the effectiveness of the proposed method. Section 4 presents the numerical quantification of other given analytical expressions while in the last section, we conclude the paper.

2. Proposed Analytical Expression for TFKG Equation using LHPM

In this section, the proposed LHPM method is implemented to present an analytical approximate solution for the time-fractional Klein-Gordon (TFKG) equation using Caputo differential operator discussed in (Miller and Ross, 1993; Podlubny, 1999; Kilbas et al., 2006). We have considered a time-fractional Klein-Gordon equation type, given in reference (Jafari, 2016), as:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + bN\big(u(x,t)\big) = g(x,t), 1 < \alpha \leq 2 \quad (1)$$

$$D_t{}^\alpha u(x,t) - D_{xx}u(x,t) + au(x,t) + bN\big(u(x,t)\big) = g(x,t), 1 < \alpha \leq 2 \quad (2)$$

with the initial conditions

$$u(x,0) = f_1(x), \ \frac{\partial u(x,0)}{\partial t} = f_2(x) \quad (3)$$

where, $D_t{}^\alpha$ is fractional Caputo derivative of order α and $a, b \in R$, $g(x,t)$ is a known analytic function and $N\big(u(x,t)\big)$ is a non–linear function. Consider the time-fractional Klein-Gordon equation (TFKG) with initial conditions. Let us apply the Laplace transform to Eq. (2), we have

$$L\ \Big(D_t{}^\alpha u(x,t) - D_{xx}u(x,t) + au(x,t) + bN\big(u(x,t)\big)\Big) = L\, g(x,t), 1 < \alpha \leq 2 \quad (4)$$

with the use of the properties of fractional Caputo derivative on Eq. (4), we get

$$s^\alpha \overline{U(x,s)} - s^{\alpha-1}u(x,0) - s^{\alpha-2}u'(x,0) - D_{xx}\overline{U(x,s)} + a\overline{U(x,s)} + bL\Big(N\big(u(x,t)\big)\Big) = \overline{G(x,s)} \quad (5)$$

where, L $(u(x,t))$ = $\overline{U(x,s)}$ and L($g(x$,t)) =$\overline{G(x,s)}$.

After substituting initial conditions, from Eq. (3), on the above equation, the following result is obtained.

$$\overline{U(x,s)} = \frac{f_1(x)}{s} + \frac{f_2(x)}{s^2} + \frac{D_{xx}\overline{U(x,s)}}{s^\alpha} - \frac{a\overline{U(x,s)}}{s^\alpha} - \frac{bL\big(N(u(x,t))\big)}{s^\alpha} + \frac{\overline{G(x,s)}}{s^\alpha} \quad (6)$$

When we apply the inversion of Laplace transform to the preceding equation, we get

$$u(x,t) = f_1(x) + t.f_2(x) + L^{-1}\left\{\frac{D_{xx}\overline{U(x,s)}}{s^\alpha}\right\} - aL^{-1}\left\{\frac{\overline{U(x,s)}}{s^\alpha}\right\} - bL^{-1}\left\{\frac{L\big(N(u(x,t))\big)}{s^\alpha}\right\} + L^{-1}\left\{\frac{\overline{G(x,s)}}{s^\alpha}\right\} \quad (7)$$

Rewriting Eq. (7) into linear and non-linear parts along with linear and non-linear operators ***L*** and ***N*** for the Homotopy method. we have

$$\boldsymbol{L}(u) = f_1(x) + t.f_2(x), \quad \boldsymbol{N}(u) = L^{-1}\left\{\frac{D_{xx}\overline{U(x,s)}}{s^\alpha}\right\} - aL^{-1}\left\{\frac{\overline{U(x,s)}}{s^\alpha}\right\} - bL^{-1}\left\{\frac{L\big(N(u(x,t))\big)}{s^\alpha}\right\}$$

$$\text{and } f(x,t) = L^{-1}\left\{\frac{\overline{G(x,s)}}{s^\alpha}\right\}.$$

We construct homotopy of Eq. (7) as

$$H(v,p) = v(x,t) - f_1(x) - tf_2(x) - L^{-1}\left\{\frac{\overline{G(x,s)}}{s^\alpha}\right\} - p\begin{Bmatrix} L^{-1}\left\{\frac{D_{xx}\overline{U(x,s)}}{s^\alpha}\right\} \\ -aL^{-1}\left\{\frac{\overline{U(x,s)}}{s^\alpha}\right\} - bL^{-1}\left\{\frac{L\left(N(u(x,t))\right)}{s^\alpha}\right\}\end{Bmatrix} = 0 \quad (8)$$

In this LHPM algorithm, we apply the perturbation technique on Eq. (8) as

$$v(x,t) = \sum_{n=0}^{\infty} p^n\, v_n\,(x,t) \quad (9)$$

Thereafter, equating the coefficient of p^0, p^1, p^2, and others in Eq. (8) to get the values of v's and thus the solution is obtained as

$$v(x,t) = v_0 + p\, v_1 + p^2\, v_2 + p^3\, v_3 + \cdots \quad (10)$$

But when $p = 1$, Eq. (10) results in the solution of time -fractional Klein-Gordon Eq. (1), i.e.

$$u(x,t) = v_0 + v_1 + v_2 + v_3 + \cdots + \cdots \quad (11)$$

Later in this paper, we will refer this approximate solution as LHPM solution of TFKG type equations.

3. Different Applications of Proposed Analytical Expression

In this section, we derive the expression for the approximate solution or LHPM solution of the TFKG equation with various functions $g\,(x, t)$.

Example 1. Let us consider a problem based on non-linear, non-homogeneous TFKG (Golmankhaneh et al., 2011; Jafari, 2016), $a = 0$, $b = 1$ in Eq. (1), as

$$D_t{}^\alpha u(x,t) - D_{xx}u(x,t) + u^2 = x^2t^2,\ 1 < \alpha \le 2 \quad (12)$$

with initial conditions

$$u\,(x\,,0) = 0,\ u_t\,(x,0) = x \quad (13)$$

The exact solution of Eq. (12), as in the reference (Kanth & Aruna, 2009), is.

$$u\,(x,\mathrm{t}) = x\,\mathrm{t},\ \alpha = 2. \quad (14)$$

Applying Laplace transform on Eq. (12), we get

$$s^\alpha u(x,s) - s^{\alpha-1}u(x,0) - s^{\alpha-2}u'(x,0) - L(D_{xx}u - u^2) = \frac{2x^2}{s^3} \quad (15)$$

Putting initial conditions and then after rearranging the terms, we obtain

$$u(x,s) = \frac{x}{s^2} + \frac{L\left(D_{xx}u-u^2\right)}{s^\alpha} + \frac{2x^2}{s^{\alpha+3}} \quad (16)$$

Next, after applying the method of Inversion of Laplace transform on Eq. (16) result ends as

$$u(x,t) = x.t + \frac{2x^2t^{\alpha+2}}{\Gamma(\alpha+3)} + L^{-1}\left(\frac{L\left(D_{xx}u-u^2\right)}{s^\alpha}\right) \quad (17)$$

Now we construct homotopy with a parameter p ∈ [0,1], an embedded parameter, for Eq. (17) as

$$H(v,p) = \left(v(x,t) - x.t - \frac{2x^2t^{\alpha+2}}{\Gamma(\alpha+3)}\right) - pL^{-1}\left(\frac{L\left(D_{xx}v-v^2\right)}{s^\alpha}\right) = 0 \quad (18)$$

Further, applying the perturbation method on Eq. (18) as
$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) = v_0 + p\, v_1 + p^2 v_2 + p^3 v_3 + \cdots + \cdots \quad (19)$$

Substituting Eq. (19) in Eq. (18) and after that comparing the coefficient of p^n, we obtain
$$p^0: v_0(x,t) = x.t + \frac{2x^2 t^{\alpha+2}}{\Gamma(\alpha+3)}, \quad (20)$$

$$p^1: v_1(x,t) = \frac{4t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{2x^2 t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{4x^3(\alpha+3)t^{2\alpha+3}}{\Gamma(2\alpha+4)} - \frac{4x^4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(3\alpha+5)\Gamma(\alpha+3)} \quad (21)$$

$$p^2 : v_2(x,t) = \frac{-4.t^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{24x.(\alpha+3).t^{3\alpha+3}}{\Gamma(3\alpha+4)} - \frac{48.x^2\Gamma(2\alpha+5).t^{4\alpha+4}}{(\Gamma(\alpha+3))^2\Gamma(4\alpha+5)} + \frac{8x.\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(2\alpha+3).\Gamma(2\alpha+4)} + \frac{4.x^3(\alpha+3).t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \frac{8.x^4(\alpha+3).(2\alpha+4)t^{3\alpha+4}}{\Gamma(3\alpha+5)} + \frac{8.x^5(3\alpha+5).\Gamma(2\alpha+5).t^{4\alpha+5}}{\Gamma(4\alpha+6)(\Gamma(\alpha+3))^2} - \frac{16.x^2\Gamma(3\alpha+5).t^{4\alpha+4}}{\Gamma(\alpha+3)\Gamma(2\alpha+5)\Gamma(4\alpha+5)} + \frac{8.x^4.\Gamma(2\alpha+5).t^{3\alpha+4}}{\Gamma(3\alpha+5)(\Gamma(\alpha+3))^2} + \frac{16.x^5(\alpha+3).\Gamma(3\alpha+6).t^{4\alpha+5}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)\Gamma(4\alpha+6)} + \frac{16.x^6.\Gamma(2\alpha+5).\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(3\alpha+5)(\Gamma(\alpha+3))^2\Gamma(5\alpha+7)} \quad (22)$$

Similarly, we could have v_3, v_4, v_5 ......v_n. From Eqs. (19), (20), (21), and (22), we have solutions of Eq. (12), following Eq. (11) with p tends to 1 as
$$u(x,t) = xt - \frac{4.x^4(\Gamma(2\alpha+5)).t^{2\alpha+4}}{\Gamma(3\alpha+5)(\Gamma(\alpha+3))^2} - \frac{8x.\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(2\alpha+3)\Gamma(2\alpha+4)} + \frac{8.x^4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(3\alpha+5)(\Gamma(\alpha+3))^2} + \frac{8.x^4(\alpha+3).(2\alpha+4)t^{3\alpha+4}}{\Gamma(3\alpha+5)} - \frac{48.x^2.\Gamma(2\alpha+5).t^{4\alpha+4}}{\Gamma(4\alpha+5)(\Gamma(\alpha+3))^2} - \frac{16.x^2.\Gamma(3\alpha+5).t^{4\alpha+4}}{\Gamma(\alpha+3)\Gamma(2\alpha+5)\Gamma(4\alpha+5)} + \frac{8.x^5(3\alpha+5).\Gamma(2\alpha+5).t^{4\alpha+5}}{\Gamma(4\alpha+6)\Gamma(\alpha+3)} + \frac{16.x^5(\alpha+3).\Gamma(3\alpha+6).t^{4\alpha+5}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)\Gamma(4\alpha+6)} + \frac{16.x^6.\Gamma(2\alpha+5).\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(3\alpha+5)(\Gamma(\alpha+3))^2\Gamma(5\alpha+7)} \quad (23)$$

We have investigated the TFKG's approximative solution up to the second term. Later, on Microsoft Excel 2007, the numerical values of the LHPM solution, at various fractional orders α = 1.75, 1.85, 1.95, and 2 at constant x= 0.1, will be calculated. Additionally, a detailed comprehension of the suggested method in terms of convergence will be achieved by the use of graphical depiction of these LHPM solutions created in MS Excel.

Example 2. Let us consider nonlinear, non-homogeneous TFKG equation (1) with g(x, t) = 0, a = 0, b = 0, (Golmankhaneh et al., 2011), as
$$D_t{}^{\alpha}u(x,t) - D_{xx}u(x,t) = -u^2, 0 \le x, 1 < \alpha \le 2 \quad (24)$$

and initial conditions are
$$u(x,0) = 1 + \sin x,\ u_t(x,0) = 0 \quad (25)$$

The steps of the Laplace Homotopy Pertubation Method (LHPM) described in section 2 will be applied on Eq. (24) along with initial condition i.e., Eq. (25). The following homotopy result is obtained as
$$v(x,t) = 1 + \sin x - \frac{pt^{\alpha}}{\Gamma(1+\alpha)}\{1 + 3\sin x + \sin^2 x\} + \frac{p^2 t^{\alpha}}{\Gamma(1+2\alpha)}\{11\sin x + 12\sin^2 x + 2\sin^3 x\} + .. \quad (26)$$

Finally, by considering $p \to 1$ in Eq. (26), $v(x,t)$ changes to $u(x,t)$.Thus, we get LHPM solution as
$$u(x,t) = 1 + \sin x - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\{1 + 3\sin x + \sin^2 x\} + \frac{t^{\alpha}}{\Gamma(1+2\alpha)}\{11\sin x + 12\sin^2 x + 2\sin^3 x\} + ... \quad (27)$$

The approximate analytical solution in Eq. (27) entirely agrees with the result given by Kanth and Aruna, (2009). Since the solution of this case is open so it is more important to compare all the analytical solutions, from Eq. (27), at α =1.75,1.85, 1.95, 2.

4. Results and Discussion

The behaviour of the approximation's solution evaluated to the problems in the previous section is now discussed here. The reliable Microsoft Excel program is employed to generate the numerical solutions of the TFKG problems, which are then used to analyze LHPM solutions graphically. Table 1 contains the exact numerical solutions of illustrated cases at certain x and for various time intervals.

Table 1. Exact solutions of the examples when x = 0.1.

Time	Exact solution		Time		Exact solution	
'*t*'	Example 1	Example 2		'*t*'	Example 1	Example 2
0	0	1.0998334		0.6	0.06	0.8707053
0.1	0.01	1.0932912		0.7	0.07	0.7911968
0.2	0.02	1.0737253		0.8	0.08	0.7015872
0.3	0.03	1.0413184		0.9	0.09	0.60279
0.4	0.04	0.996375		1	0.1	0.4958402
0.5	0.05	0.9393213				

Further, Table 2 displays the numerical values of the approximation's solution or LHPM solution of each example. For fractional orders α = 1.75, 1.85, 1.95, and 2 at a constant x = 0.1, we calculate the numerical solutions of both the cases for the time t, varying from 0 to 1, given in Table 2. According to Table 2, as in example 1 at t = 0.6, the approximate LHPM solutions at various fractional orders α = 1.75, 1.85, 1.95, and 2 are 0.0599998, 0.0599999, 0.06, and 0.06, respectively, and the exact solution for this case has a numerical value of 0.06. It is evident that these numerical LHPM solutions are very close to the exact solution.

Table 2. Approximate solutions of examples using LHPM.

Fractional order α	Time '*t*'	Approximate solutions			Fractional order α	Time '*t*'	Approximate solutions		
		Example 1		Example 2			Example1		Example 2
α = 1.75	0	0		1.0998334	*α* = 1.95	0	0		1.0998334
	0.1	0.01		1.0624535		0.1	0.01		1.0830832
	0.2	0.02		0.9741029		0.2	0.02		1.035115
	0.3	0.03		0.8442101		0.3	0.03		0.9571393
	0.4	0.04		0.676928		0.4	0.04		0.8497779
	0.5	0.05		0.4748971		0.5	0.05		0.7134566
	0.6	0.0599998		0.2400226		0.6	0.06		0.5484998
	0.7	0.0699994		-0.0262221		0.7	0.0699999		0.3551688
	0.8	0.0799981		-0.3226451		0.8	0.0799996		0.1336824
	0.9	0.0899948		-0.6482519		0.9	0.0899987		-0.1157714
	1	0.0999874		-1.0021933		1	0.0999967		-0.3930281
α = 1.85	0	0		1.0998334	*α* = 2.00	0	0		1.0998334
	0.1	0.01		1.0750374		0.1	0.01		1.0859874
	0.2	0.02		1.0104436		0.2	0.02		1.0444493
	0.3	0.03		0.9105743		0.3	0.03		0.9752193
	0.4	0.04		0.777583		0.4	0.04		0.8782971
	0.5	0.05		0.6128917		0.5	0.05		0.753683
	0.6	0.0599999		0.417554		0.6	0.06		0.6013768
	0.7	0.0699997		0.1924019		0.7	0.0699999		0.4213786
	0.8	0.0799991		-0.0618799		0.8	0.0799997		0.2136883
	0.9	0.0899974		-0.3447118		0.9	0.0899991		-0.021694
	1	0.0999935		-0.6555924		1	0.0999977		-0.2847683

Since example 2 does not produce the closed form of the solution (Kanth and Aruna, 2009). So, the numerical values of LHPM solutions are compared to that of the same approximate solution given in this reference. In both the tables, we observe that the approximate solution of the TFKG on various fractional orders comes close to the solution of the integer order "2" of the classical Klein-Gordon equation.

Figure 1 illustrates the graphical representation of the numerical LHPM solutions of example 1. Here, the exact solutions from Table 1, is compared visually with LHPM solution at various fractional orders presented in Table 2. We observe that the complexity of approximate multistep TFKG solutions and the exact solution are very similar for the example 1 graphically. Figure 2, defined for example 2, shows that all LHPM approximations at different periods are very similar. Although, all the approximation solutions in Figure 2 seem to be the same as fractional order rises in different time periods. For fractional order $\alpha = 2$, it has been observed that the presented results of TFKG, only up to two terms, have a perfect match with the exact solution of the Klein-Gordon equation of second order. Through these graphs, we again prove our reliability on LHPM. Finally, we calculate the relative errors of both the examples for further analysis. For this, we have calculated and presented the relative errors at different fractional orders in Table 3. The relative errors of example 2 at different fractional orders of 1.75, 1.85, 1.95, and 2 at $t = 0.8$ are, for instance, 1.4598788, 1.0881998, 0.8094571, and 0.6954216.

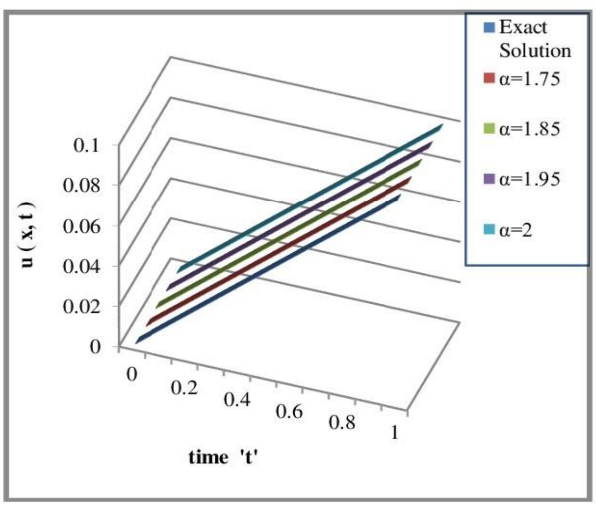

Figure 1. Comparison between the exact solution and LHPM solutions for example 1, when $x = 0.1$.

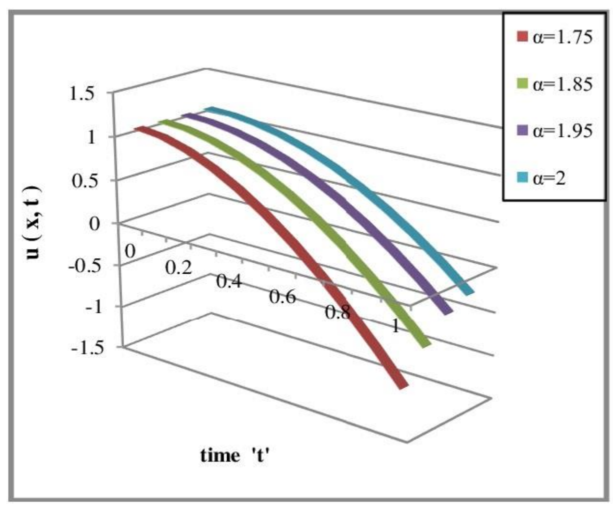

Figure 2. Comparison between LHPM solutions for example 2, when $x = 0.1$.

We can see that at a specific period, relative errors at various fractional orders of TFKG are significantly decreasing, much like in example 1.

The relative errors of the LHPM solution at different fractional orders of the TFKG are rapidly decreasing at a given time interval.

Table 3. Relative errors between exact and approximate solution of examples.

Fractional order 'α'	Time 't'	Relative Error		Fractional order 'α'	Time 't'	Relative Error	
		Example 1	Example 2			Example 1	Example 2
α = 1.75	0		1.804E-11	α = 1.95	0		1.80E-11
	0.1	6.507E-12	0.0282063		0.1	4.36E-13	0.0093369
	0.2	9.908E-10	0.092782		0.2	1.01E-10	0.0359592
	0.3	1.877E-08	0.1892872		0.3	2.43E-09	0.0808389
	0.4	1.516E-07	0.3206092		0.4	2.33E-08	0.1471305
	0.5	7.692E-07	0.4944253		0.5	1.35E-07	0.2404552
	0.6	2.91E-06	0.7243355		0.6	5.66E-07	0.3700511
	0.7	9.005E-06	1.0331423		0.7	1.91E-06	0.5510992
	0.8	2.407E-05	1.4598788		0.8	5.52E-06	0.8094571
	0.9	5.762E-05	2.0754191		0.9	1.41E-05	1.1920593
	1	0.0001265	3.0212021		1	3.28E-05	1.7926507
α = 1.85	0			α = 2.00	0		1.80E-11
	0.1	1.692E-12	1.804E-11		0.1	2.20E-13	0.0066805
	0.2	3.171E-10	0.0166962		0.2	5.64E-11	0.0272657
	0.3	6.78E-09	0.0589365		0.3	1.45E-09	0.0634764
	0.4	5.966E-08	0.1255563		0.4	1.45E-08	0.1185075
	0.5	3.232E-07	0.219588		0.5	8.65E-08	0.1976303
	0.6	1.289E-06	0.3475165		0.6	3.74E-07	0.3093222
	0.7	4.169E-06	0.5204417		0.7	1.29E-06	0.4674162
	0.8	1.157E-05	0.7568217		0.8	3.80E-06	0.6954216
	0.9	2.862E-05	1.0881998		0.9	9.87E-06	1.0359893
	1	6.466E-05	1.5718604		1	2.33E-05	1.5743146
			2.3221848				

5. Conclusion

We conclude that our proposed method on time-fractional Klein-Gordon type equations is an efficient, robust, and fast convergent method. This article presents the analytical approximate solution of time-fractional Klein-Gordon (TFKG) type equations in the Caputo sense using both the Laplace Transform and Homotopy perturbation method. The proposed method Laplace Homotopy perturbation method (LHPM), a simple method, can quickly evaluate the approximate solution of any fractional ordered PDE. Solutions of different examples have been computed with the help of the proposed method. In addition, the efficacy of LHPM has been presented in other figures and tables. Finally, the analysis of LHPM approximate solutions for different fractional order Klein-Gordon equations with exact solutions proves the efficiency and reliability of the proposed LHPM. The effectiveness, robustness, and fast convergence of the proposed method make it most viable in the different field of science and technology. However, most of the vibrant field of application include communication signaling and nanoscience specifically in the thermal assessment of nanoparticles.

Conflict of Interest

The authors declare that there is no conflict for this publication.

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