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Approximation of Two-Dimensional Time-Fractional Navier-Stokes Equations involving Atangana-Baleanu Derivative

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Abstract
This article addresses the two analytical methods, i.e., the new iterative transform method (NITM) and the homotopy perturbation transform method (HPTM), along with an Aboodh transform (AT), to approximate the nonlinear system of two-dimensional (2D) time-fractional Navier-Stokes (TFNS) equations. We take the time-fractional derivative in the form of Atangana-Baleanu (AB). The article's suggested examples examine the accuracy and efficacy of the proposed methods, while the graphs demonstrate their potential and effectiveness. The article also provides demonstrations of uniqueness and convergence. The aforementioned techniques are straightforward and support a high rate of convergence, which helps in understanding the dynamics of fractional nonlinear systems.

Keywords- 2D-TFNS equations, NITM, HPTM, Integral transform, AB fractional derivative.

1. Introduction

The study of fractional calculus (FC), which is considered an arbitrary order's derivatives and integrals, originated in the 17th century. It has attracted researchers more in the last three decades due to its high accuracy, applicability and numerous applications in various fields of science and mathematics (Miller and Ross, 1993; Podlubny, 1999; Caponetto et al., 2010). A great tool for characterizing the memory and inherited qualities of different materials and processes is the fractional derivative. Due to the greater degree of freedom in the fractional-order model, irrespective of the of the classical one, and its non-local property, i.e., the system's subsequent state is found not immediately by its present state but also by all of its preceding states, this is the drawback of the integer-order differential operator. This is more sensible, which is one fact that FC is becoming more common. The theory of fractional derivatives was found to be more appropriate for modeling real-world problems than ordinary ones (Rossikhin and Shitikova, 2012;

Łabedzki and Pawlikowski, 2023; Amilo et al., 2024).

In literature, several definitions of fractional order operators exist, like Riemann-Liouville, Grunwald-Letnikov, Caputo, etc. Researchers generally use Riemann-Liouville fractional derivatives; however, they are unsuitable for real-world problems since they require the formulation of initial conditions of fractional order. To overcome this problem, Caputo proposed a new concept with the advantage of describing the initial conditions of integer order (Podlubny, 1998). In both Riemann-Liouville as well as Caputo fractional derivatives, the singular kernels exist, but the constant of Riemann-Liouville is not zero. Caputo and Fabrizio (2015) resolved it by constructing a new differential operator by substituting the singular kernel with a non-singular kernel. Researchers highlighted non-locality and non-singular kernels as significant difficulties in the CF derivative, despite their critical role in understanding the behavior and nature of issues. Furthermore, Atangana and Baleanu (2016) expanded Caputo-Fabrizio's formulation by introducing a Mittag-Leffler (ML) function in place of an exponential function as a kernel and handling all the stated issues. Two types of fractional operators were established, i.e., Atangana-Baleanu Riemann as well as Caputo derivatives. These derivatives attract a lot of attention from researchers due to their intriguing properties (see Algahtani, 2016; Abdeljawad, 2017; Saad et al., 2018).

The fractional differential equations (FDEs) are regarded as the most widely used and significant tool in numerous disciplines of engineering science, such as polymer physics, chemical physics, bio-engineering, propagation of seismic waves, relaxation processes in complex systems, control theory, fluid mechanics, analytical chemistry, quantum physics, image processing, etc. (see Harris, 2020; Sharma and Arora, 2020; Kaur and Kuldeep, 2022; Kovalnogov et al., 2021; Singh, 2023; Shah and Li, 2018). Because most physical systems are nonlinear, nonlinear partial differential equations (NLPDEs) have long been a prominent topic, addressing problems in a wide range of fields. Furthermore, for certain challenging issues, fractional PDEs are more perfect than integer-order PDEs and have a considerable impact on the qualitative behavior of the mathematical models. It has extensive applications in solid-state physics, plasma physics, thermal conductivity, economics, mathematical biology, diffusion processes, turbulent flow, and material science (Patra and Ray, 2014; Kumar et al., 2020; Yasmin et al., 2020; Nazir et al., 2023).

Claude-Louis Navier and Sir George Gabriel Stokes first proposed the Navier-Stokes (N-S) equations in the nineteenth century. This equation is a set of PDEs that describe the motion of viscous fluid substances, considering factors such as pressure, velocity, and viscosity. Despite its complexity and challenges, understanding fluid flow using the N-S equations has been fundamental to advancements in various fields, including aerospace engineering, meteorology, climate estimation, blood flow, and biomedical applications (Adomian, 1995; Herreros and Ligrzana, 2020; Krasnoschok et al., 2020). The literature discusses several methods for solving the TFNS equations. El-Shahed and Salem (2004) used Laplace, Fourier sine, and Hankel transforms to solve TFNS. The authors of (Birajdar, 2014; Kumar et al., 2015; Maitama, 2018; Hajira et al., 2020) used homotopy perturbation and Adomian decomposition methods, respectively, to solve TFNS equations analytically. Recently, researchers and mathematicians have been focusing on analytical solutions to the multi-dimensional fractional-order N-S equation (Singh and Kumar, 2018; Chu et al., 2021; Elsayed et al., 2022; Mukhtar et al., 2022; Singh et al., 2024). In the same vein, the present work is the analysis of a two-dimensional fractional (N-S) equation with the initial condition using HPTM (He, 2009; Maurya et al., 2019; Kashyap et al., 2023) and NITM by Daftardar-Gejji and Jafari (2006) techniques, along with the Aboodh transform. We apply the suggested analytical scheme directly, without any limiting assumptions, linearization, or transformation. The obtained results are entirely consistent with those of the established techniques.

The paper's originality lies in the examination and combination of two different analytical approaches, i.e., the new iterative transform method and the homotopy perturbation transform method, to determine the solution of the nonlinear system of two-dimensional time-fractional Navier-Stokes equations. Furthermore, the use of the Atangana-Baleanu fractional derivative undoubtedly makes the analysis much more intricate and pertinent in this regard. Through several illustrative examples and plotted graphs, this paper demonstrates the accuracy, efficiency, and potential of the methods utilized to solve TFNS equations. In addition, several significant features, such as the novelty and convergence nature of the presented techniques, are also outlined in this paper due to their simplicity and fast rate of convergence. Because of this, it is easy to say that it is an important contribution to understanding how fractional nonlinear systems move and where future research should go in the fields of fractional calculus and fluid dynamics.

The article is structured as follows: In section 2, some basic definitions and properties are discussed. In section 3, the interpretation of the NITM and HPTM are explained for the solution of the fractional PDEs. In section 4, uniqueness and convergence are presented for the method described in section 3. In section 5, the outcome of the suggested method is illustrated by examples. In section 6, the results are verified by the graphical interpretation of the suggested examples.

2. Basic Concepts

Definition 2.1 The Aboodh transform of an exponentially ordered function $\mathbb{Q}(t)$ on the set of functions (Aboodh, 2013),

$$\mathcal{F} = \left\{\mathbb{Q}: |\mathbb{Q}(t)| < \mathcal{C}e^{\alpha_i|t|}, if\ t \in (-1)^i \times [0,\infty)\right\}, i = 1,2; (\mathcal{C}, \alpha_1, \alpha_2 > 0),$$

is written as

$$\mathcal{A}[\mathbb{Q}(t)] = \mathbb{M}(s),$$

by the following integral,

$$\mathcal{A}[\mathbb{Q}(t)] = \mathbb{Q}(s) = \frac{1}{s}\int_0^\infty \mathbb{Q}\,(t)e^{-st}dt,\ \alpha_1 \leq s \leq \alpha_2.$$

Definition 2.2 The inversion of Aboodh transform for the function $\mathbb{Q}(t)$ is defined as,

$$\mathbb{Q}(t) = \mathcal{A}^{-1}[\mathbb{Q}(s)].$$

Definition 2.3 The Mittag-Leffler function $E_\alpha(t)$ is a special function and a generalization of the exponential series. For $\alpha = 1, E_\alpha(t) = \exp(t)$, and is defined (Mittag-Leffler, 1903).

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+1)},\ \ \alpha \in \mathbb{C},\ \Re(\alpha) > 0.$$

Definition 2.4 Let $\mathbb{Q} \in \mathcal{H}^1(0,1)$ and $0 < \alpha < 1$, then the AB fractional derivative in Caputo sense is defined as (Atangana and Baleanu, 2016),

$${}_0^{ABC}D_t^\alpha\big(\mathbb{Q}(t)\big) = \frac{\mathfrak{T}(\alpha)}{1-\alpha}\int_0^t \mathbb{Q}'\,(x)E_\alpha\left(\frac{-\alpha(1-t)^\alpha}{1-\alpha}\right)dx.$$

where, $\mathfrak{T}(\alpha)$ is a normalization function $\mathfrak{T}(\alpha) > 0$ satisfies the condition $\mathfrak{T}(0) = \mathfrak{T}(1) = 1$.

Theorem 2.1 The Aboodh transform of AB fractional derivative in the Caputo sense is defined as (Awuya and Subasi, 2021),

$$\mathcal{A}\left({}^{\mathrm{ABC}}_{\ \ 0}\mathrm{D}^{\alpha}_{\mathrm{t}}(\mathbb{Q}(\mathrm{t}))\right)=\frac{\mathfrak{T}(\alpha)\left(\mathbb{Q}(\mathrm{s})-\mathrm{s}^{-2}\mathbb{Q}(0)\right)}{1-\alpha+\alpha\mathrm{s}^{-\alpha}}.$$

3. Methodology

In this section, we use the AB fractional derivative operator to examine the two-dimensional TFNS equation by using NITM and HPTM with the Aboodh transform.

Consider the fractional partial differential equation of the form:

$$ {}^{AB}_{\ \ 0}\mathcal{D}^{\alpha}_{\ell}\mathbb{Q}(\wp,\varepsilon,\ell)+\mathcal{R}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right)+\mathcal{N}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right)-\mathcal{P}(\wp,\varepsilon,\ell)=0 \quad (1)$$

subject to initial condition

$$\mathbb{Q}(\wp,\varepsilon,0)=h(\wp,\varepsilon) \quad (2)$$

${}^{ABC}_{\ \ 0}\mathcal{D}^{\alpha}_{\ell}$ is the Antangana-Baleanu fractional differential operator of order $\alpha,\mathcal{R}$ and $\mathcal{N}$ are linear and non-linear terms, and $\mathcal{P}$ is the source term, respectively.

Case 1 (Procedure of NITM): By employing the Aboodh transform on both side of Equation (1), we get

$$\mathcal{A}[\mathbb{Q}(\wp,\varepsilon,\ell)]=s^{-2}\mathbb{Q}(\wp,\varepsilon,0)+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\{\mathcal{P}(\wp,\varepsilon,\ell)-[\mathcal{R}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right)+\mathcal{N}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right)]\} \quad (3)$$

By using the inverse Aboodh transform, Equation (3), reduced to the form,

$$\mathbb{Q}(\wp,\varepsilon,\ell)=\mathcal{A}^{-1}\left\{s^{-2}\mathbb{Q}(\wp,\varepsilon,0)+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\{\mathcal{P}(\wp,\varepsilon,\ell)-[\mathcal{R}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right)+\mathcal{N}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right)]\}\right\} \quad (4)$$

The nonlinear operator $\mathcal{N}$, as in Daftardar-Gejji and Jafari (2006), can be decomposed as,

$$\begin{aligned}\mathcal{N}\left(\mathbb{Q}(\wp,\varepsilon,\ell)\right) &=\mathcal{N}\left(\textstyle\sum_{r=0}^{\infty}\mathbb{Q}_r(\wp,\varepsilon,\ell)\right)\\ &=\mathcal{N}\left(\mathbb{Q}_0(\wp,\varepsilon,\ell)\right)+\textstyle\sum_{r=1}^{\infty}\left\{\mathcal{N}\left(\sum_{l=0}^{r}\mathbb{Q}(\wp,\varepsilon,\ell)\right)-\mathcal{N}\left(\sum_{l=0}^{r-1}\mathbb{Q}(\wp,\varepsilon,\ell)\right)\right\}\end{aligned} \quad (5)$$

Now, define a m[th] order approximate series as,

$$\begin{aligned}\mathcal{D}^{(m)}(\wp,\varepsilon,\ell) &=\textstyle\sum_{r=0}^{m}\mathbb{Q}_r(\wp,\varepsilon,\ell)\\ &=\mathbb{Q}_0(\wp,\varepsilon,\ell)+\mathbb{Q}_1(\wp,\varepsilon,\ell)+\mathbb{Q}_2(\wp,\varepsilon,\ell)+---+\mathbb{Q}_m(\wp,\varepsilon,\ell),\quad m\in N\end{aligned} \quad (6)$$

Consider that the solution of Equation (1) is in a series form is

$$\mathbb{Q}(\wp,\varepsilon,\ell)=\lim_{m\to\infty}\mathcal{D}^{(m)}(\wp,\varepsilon,\ell)=\textstyle\sum_{r=0}^{\infty}\mathbb{Q}_r(\wp,\varepsilon,\ell) \quad (7)$$

By substituting Equations (5) and (6) in Equation (4), we get,

$$\begin{aligned}&\textstyle\sum_{r=0}^{\infty}\mathbb{Q}_r(\wp,\varepsilon,\ell)=\mathcal{A}^{-1}\left\{s^{-2}\mathbb{Q}(\wp,\varepsilon,0)+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.\\ &\left.\times\mathcal{A}\left[\mathcal{P}(\wp,\varepsilon,\ell)-\left[\mathcal{R}\left(\mathbb{Q}_0(\wp,\varepsilon,\ell)\right)+\mathcal{N}\left(\mathbb{Q}_0(\wp,\varepsilon,\ell)\right)\right]\right]\right\}-\mathcal{A}^{-1}\left\{\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.\\ &\left.\times\mathcal{A}\left[\textstyle\sum_{r=1}^{\infty}\left\{\mathcal{R}\left(\mathbb{Q}_r(\wp,\varepsilon,\ell)\right)+\left[\mathcal{N}\left(\sum_{i=0}^{r}\mathbb{Q}_i(\wp,\varepsilon,\ell)\right)-\mathcal{N}\left(\sum_{i=0}^{r-1}\mathbb{Q}_i(\wp,\varepsilon,\ell)\right)\right]\right\}\right]\right\}\end{aligned} \quad (8)$$

From Equation (8), following iterations are obtained.

$$\mathbb{Q}_0(\wp,\varepsilon,\ell)=\mathcal{A}^{-1}\left[s^{-2}\mathbb{Q}(\wp,\varepsilon,0)+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}[\mathcal{P}(\wp,\varepsilon,\ell)]\right] \tag{9}$$

$$\mathbb{Q}_1(\wp,\varepsilon,\ell)=-\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\big[\mathcal{R}\big(\mathbb{Q}_0(\wp,\varepsilon,\ell)\big)+\mathcal{N}\big(\mathbb{Q}_0(\wp,\varepsilon,\ell)\big)\big]\right] \tag{10}$$

$$\mathbb{Q}_{r+1}(\wp,\varepsilon,\ell)=-\mathcal{A}^{-1}\left\{\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\times\mathcal{A}\left[\sum_{r=1}^{\infty}\left\{\mathcal{R}\big(\mathbb{Q}_r(\wp,\varepsilon,\ell)\big)+\left[\mathcal{N}\Big(\sum_{i=0}^{r}\mathbb{Q}_i(\wp,\varepsilon,\ell)\Big)-\mathcal{N}\left(\sum_{i=0}^{r-1}\mathbb{Q}_i(\wp,\varepsilon,\ell)\right)\right]\right\}\right]\right\} \tag{11}$$

Case 2 (Procedure of HPTM): By employing the Aboodh transform on both side of Equation (1), we get

$$\mathcal{A}[\mathbb{Q}(\wp,\varepsilon,\ell)]=\varpi(\mathbb{Q}(\wp,\varepsilon,s)-\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\big\{\big[\mathcal{R}\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)+\mathcal{N}\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)\big]\big\} \tag{12}$$

where,

$$\varpi\big(\mathbb{Q}(\wp,\varepsilon,s)\big)=s^{-2}\mathbb{Q}(\wp,\varepsilon,0)+\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\tilde{P}(\wp,\varepsilon,s).$$

By applying the inverse Aboodh transform, Equation (12), reduced to the form

$$\mathbb{Q}(\wp,\varepsilon,\ell)=\varpi\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)-\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\big\{\big[\mathcal{R}\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)+\mathcal{N}\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)\big]\big\}\right] \tag{13}$$

where, $\varpi\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)$ represents the term arising from the source term. Now apply the HPTM to find the solution to Equation (13).

$$\mathbb{Q}(\wp,\varepsilon,\ell)=\sum_{r=0}^{\infty}z^r\,\mathbb{Q}_r(\wp,\varepsilon,\ell) \tag{14}$$

and the non-linear tern can be decomposed as

$$\mathcal{N}\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)=\sum_{r=0}^{\infty}z^r\,\mathcal{H}_r(\wp,\varepsilon,\ell) \tag{15}$$

For some He's polynomials (Ghorbani, 2009) given as

$$\mathcal{H}_r(\mathbb{Q}_0,\mathbb{Q}_1,\ldots,\mathbb{Q}_r)=\frac{1}{r!}\frac{\partial^r}{\partial z^r}\Big[\mathcal{N}\Big(\sum_{j=0}^{\infty}z^j\,\mathbb{Q}_j\Big)\Big],\ \ r=0,1,2,\ldots \tag{16}$$

By substituting Equations (14) and (15) in the Equation (13), we get,

$$\sum_{r=0}^{\infty}\mathbb{Q}_r(\wp,\varepsilon,\ell)z^r=\varpi\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big)$$

$$-z\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\{\mathcal{R}\sum_{r=0}^{\infty}z^r\,\mathbb{Q}_r(\wp,\varepsilon,\ell)+\mathcal{N}\sum_{r=0}^{\infty}z^r\,\mathcal{H}_r(\wp,\varepsilon,\ell)\}\right] \tag{17}$$

Comparing the coefficients of like power of z, the following approximations are obtained

$$z^0:\mathbb{Q}_0(\wp,\varepsilon,\ell)=\varpi\big(\mathbb{Q}(\wp,\varepsilon,\ell)\big) \tag{18}$$

$$z^1:\mathbb{Q}_1(\wp,\varepsilon,\ell)=-\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\{\mathcal{R}[\mathbb{Q}_0(\wp,\varepsilon,\ell)]+\mathcal{H}_0(\mathbb{Q})\}\right] \tag{19}$$

$$z^{r+1}:\mathbb{Q}_{r+1}(\wp,\varepsilon,\ell)=-\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\{\mathcal{R}[\mathbb{Q}_r(\wp,\varepsilon,\ell)]+\mathcal{H}_r(\mathbb{Q})\}\right] \tag{20}$$

4. Uniqueness and Convergence

Theorem 4.1 The solution derived with the aid of $\mathrm{NITM}_{\mathrm{AB}}$ of Equation (1) is unique, whenever $0<$

$$(\Theta_1+\Theta_2)\left[1-\alpha+\alpha\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right]<1.$$

Proof: Let $X=(C[J],\|.\|)$ be the Banach space for all continuous functions over the interval $J=[0,T]$, with the norm $\|\phi(\ell)=\max_{\ell\in J}|\phi(\ell)|$.

Define the mapping $\mathcal{F}:X\to X$, where,

$$u_{r+1}^{C}=u_0^{C}-\mathbb{A}^{-1}\left[\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\mathcal{R}(u(\wp,\varepsilon,\ell))+\mathcal{N}(u(\wp,\varepsilon,\ell))-\mathcal{P}(\wp,\varepsilon,\ell)\}\right],\quad r\geq 0.$$

Now assume that, $\mathcal{R}(u)$ and $\mathcal{N}(u)$ satisfy the Lipschitz conditions with Lipschitz constant Θ_1,Θ_2 and $|\mathcal{R}(u)-\mathcal{R}(\bar{u})|<\Theta_1|u-\bar{u}|$, $|\mathcal{N}(u)-\mathcal{N}(\bar{u})|<\Theta_2|u-\bar{u}|$, where $u=u(\wp,\varepsilon,\ell)$ and $\bar{u}=u(\wp,\varepsilon,\ell)$ are the values of two distinct functions.

$$\begin{aligned}\|F(u)-F(\bar{u})\|\leq \max_{\ell\in J}\Big|\ &\mathbb{A}^{-1}\left[\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\mathcal{R}(u(\wp,\varepsilon,\ell))-\mathcal{R}(\bar{u}(\wp,\varepsilon,\ell))\}\right]\\ &+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\mathcal{N}(u(\wp,\varepsilon,\ell))-\mathcal{N}(\bar{u}(\wp,\varepsilon,\ell))\}\ \Big|\\ \leq \max_{\ell\in J}&\left[\Theta_1\mathbb{A}^{-1}\left\{\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}[u(\wp,\varepsilon,\ell)-\bar{u}(\wp,\varepsilon,\ell)]\right\}\right]\\ +&\left[\Theta_2\mathbb{A}^{-1}\left\{\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}[u(\wp,\varepsilon,\ell)-\bar{u}(\wp,\varepsilon,\ell)]\right\}\right]\\ \leq &\max_{\ell\in J}(\Theta_1+\Theta_2)\left[\mathbb{A}^{-1}\left\{\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}[u(\wp,\varepsilon,\ell)-\bar{u}(\wp,\varepsilon,\ell)]\right\}\right]\\ \leq &(\Theta_1+\Theta_2)\left[\mathbb{A}^{-1}\left\{\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}[u(\wp,\varepsilon,\ell)-\bar{u}(\wp,\varepsilon,\ell)]\right\}\right]\\ \leq &(\Theta_1+\Theta_2)\left[1-\alpha+\alpha\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right]\|u-\bar{u}\|.\end{aligned}$$

$\mathcal{F}$ is contraction as $0<(\Theta_1+\Theta_2)\left[1-\alpha+\alpha\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right]<1$. Thus, the result of (1) is unique with the aid of Banach fixed point theorem.

Theorem 4.2 The solution derived of Equation (1), using $NITM_{AB}$ converges if $0<\omega<1$ and $\|u_i\|<\infty$, where $\omega=(\Theta_1+\Theta_2)\left[1-\alpha+\alpha\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right]$.

Proof: Let $u_n=\sum_{r=0}^{n}u_r(\wp,\varepsilon,\ell)$ is a partial sum of series. To prove that $\{u_n\}$ is a Cauchy sequence in the Banach space X, we consider

$$\begin{aligned}\|(u_m-u_n\| &= max_{\ell\in J}|\textstyle\sum_{r=n+1}^{m}u_r(\wp,\varepsilon,\ell)|,\quad n=1,2,3,\dots\\ &\leq max_{\ell\in J}\left|\mathbb{A}^{-1}\left[\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\textstyle\sum_{r=n+1}^{m}[\mathcal{R}(u_{r-1}(\wp,\varepsilon,\ell))+\mathcal{N}(u_{r-1}(\wp,\varepsilon,\ell))]\}\right]\right|\\ &\leq max_{\ell\in J}\left|\mathbb{A}^{-1}\left[\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\mathcal{R}(u_{m-1})-\mathcal{R}(u_{n-1})+\mathcal{N}(u_{m-1})-\mathcal{N}(u_{n-1})\}\right]\right|\\ &\leq \Theta_1\ max_{\ell\in J}\left|\mathbb{A}^{-1}\left[\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\mathcal{R}(u_{m-1})-\mathcal{R}(u_{n-1})\}\right]\right|\\ &\quad+\Theta_2\ max_{\ell\in J}\left|\mathbb{A}^{-1}\left[\frac{(1-\alpha+\alpha s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathbb{A}\{\mathcal{N}(u_{m-1})-\mathcal{N}(u_{n-1})\}\right]\right|\\ &= (\Theta_1+\Theta_2)\left[1-\alpha+\alpha\frac{\ell^{\alpha}}{\Gamma(\alpha+1)}\right]\|u_{m-1}-u_{n-1}\|.\end{aligned}$$

If $m = n + 1$, then

$$\| u_{n+1} - u_{n-1} \| \leq \Theta \| u_n - u_{n-1} \| \leq \Theta^2 \| u_{n-1} - u_{n-2} \| \leq \ldots \leq \Theta^n \| u_1 - u_0 \|,$$

where, $\Theta = (\Theta_1 + \Theta_2)[1 - \alpha + \alpha\ell]$.

In the similar way,

$$\begin{aligned} \| u_m - u_n \| \quad & \leq \| u_{n+1} - u_n \| \leq \| u_{n+2} - u_{n+1} \| \leq \ldots \| u_m - u_{m-1} \|, \\ & \leq (\Theta^n + \Theta^{n+1} + \ldots + \Theta^{m-1}) \| u_1 - u_0 \|, \\ & \leq \Theta^n \left(\frac{1-\Theta^{m-n}}{1-\Theta} \right) \| u_1 \|, \end{aligned}$$

We see that, $1 - \Theta^{m-n} < 1$, as $0 < \Theta < 1$. Thus,

$$\| u_m - u_n \| \leq \left(\frac{\Theta^n}{1-\Theta} \right) max_{\ell \in J} \| u_1 \|.$$

Since, $\| u_1 \| < \infty$. Therefore, $\| u_m - u_n \| \to 0$ as $n \to \infty$. Hence u_m is a Cauchy sequence in X. So, the series u_m is convergent.

Theorem 4.3 The solution derived with the aid of $HPTM_{AB}$ of Equation (1) is unique, whenever $0 < (\Theta_1 + \Theta_2) \left[1 - \alpha + \alpha \frac{\ell^\alpha}{\Gamma(\alpha+1)} \right] < 1$.

Proof: The proof is similar to Theorem (4.1). Thus, it is skipped.

Theorem 4.4 The solution derived of equation (1), using $\mathrm{HPTM_{AB}}$ converges if $0 < \omega < 1$ and $\| u_i \| < \infty$, where, $\omega = (\Theta_1 + \Theta_2) \left[1 - \alpha + \alpha \frac{\ell^\alpha}{\Gamma(\alpha+1)} \right]$.

Proof: The proof is similar to Theorem (4.2). Thus, it is skipped.

5. Results and Discussion

In this section, we demonstrate the effectiveness of NITM and HPTM with Aboodh transform to solve 2D-TFNS-equations.

Example 1. Consider the 2D fractional Navier-Stokes equations (Singh and Kumar, 2018).

$$\begin{aligned} {}^{AB}_{0}\mathcal{D}^{\alpha}_{\ell}(\mu) + \mu \frac{\partial \mu}{\partial \wp} + \nu \frac{\partial \mu}{\partial \varepsilon} = \rho \left[\frac{\partial^2 \mu}{\partial \wp^2} + \frac{\partial^2 \mu}{\partial \varepsilon^2} \right] + q, \\ {}^{AB}_{0}\mathcal{D}^{\alpha}_{\ell}(\nu) + \mu \frac{\partial \nu}{\partial \wp} + \nu \frac{\partial \nu}{\partial \varepsilon} = \rho \left[\frac{\partial^2 \nu}{\partial \wp^2} + \frac{\partial^2 \nu}{\partial \varepsilon^2} \right] - q, \end{aligned} \tag{21}$$

where, the parameter α describe the order of the time fractional derivative. $\mu = \mu(\wp, \varepsilon, \ell)$, $\nu = \nu(\wp, \varepsilon, \ell)$, and $q = -\frac{1}{\rho}\frac{\partial p}{\partial z}$, while ℓ, ρ, p are time, density and pressure, respectively. $\wp$, ε are the spatial components.

Initial conditions are $\begin{cases} \mu(\wp, \varepsilon, 0) = -\sin(\wp + \varepsilon) \\ \nu(\wp, \varepsilon, 0) = \sin(\wp + \varepsilon) \end{cases}$ (22)

Applying the Aboodh transform and inversion in Equation (21), we get,

$$\mu(\wp,\varepsilon,\ell)=\mu(\wp,\varepsilon,0)+\mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right]+\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.$$
$$\left.\times\,\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right)-\left(\mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}\right)\right\}\right],$$
$$\nu(\wp,\varepsilon,\ell)=\nu(\wp,\varepsilon,0)-\mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right]+\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.$$
$$\left.\times\,\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right)-\left(\mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}\right)\right\}\right], \tag{23}$$

(By NITM): From Equations (21), and (22), we set the following.

$$\begin{cases} \mathcal{P}_1(\wp,\varepsilon,\ell)=q,\ \mathcal{R}(\mu(\wp,\varepsilon,\ell))=-\rho\left(\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right),\quad \mathcal{N}(\mu(\wp,\varepsilon,\ell))=\left(\mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}\right)\\ \mathcal{P}_2(\wp,\varepsilon,\ell)=-q,\ \mathcal{R}(\nu(\wp,\varepsilon,\ell))=-\rho\left(\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right),\quad \mathcal{N}(\nu(\wp,\varepsilon,\ell))=\left(\mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}\right)\\ \mu_0(\wp,\varepsilon,0)=-\sin(\wp+\varepsilon),\quad \nu_0(\wp,\varepsilon,0)=-\sin(\wp+\varepsilon)\end{cases}$$

Using the iteration process outlined in Section (3) of NITM, we obtain the following

$$\begin{aligned}\mu_0(\wp,\varepsilon,\ell)&=\mathcal{A}^{-1}\left[s^{-2}\mu(\wp,\varepsilon,0)+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}[\mathcal{P}_1(\wp,\varepsilon,\ell)]\right],\ \ 0<\alpha\le 1\\ &=-\sin(\wp+\varepsilon)+\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],\\ \nu_0(\wp,\varepsilon,\ell)&=\mathcal{A}^{-1}\left[s^{-2}\nu(\wp,\varepsilon,0)+\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}[\mathcal{P}_2(\wp,\varepsilon,\ell)]\right]\\ &=\sin(\wp+\varepsilon)-\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],\end{aligned} \tag{24}$$

$$\begin{aligned}\mu_1(\wp,\varepsilon,\ell)&=-\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left(\mathcal{R}(\mu_0(\wp,\varepsilon,\ell))+\mathcal{N}(\mu_0(\wp,\varepsilon,\ell))\right)\right]\\ &=\sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],\\ \nu_1(\wp,\varepsilon,\ell)&=-\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left(\mathcal{R}(\nu_0(\wp,\varepsilon,\ell))+\mathcal{N}(\nu_0(\wp,\varepsilon,\ell))\right)\right]\\ &=-\sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],\end{aligned} \tag{25}$$

$$\begin{aligned}\mu_2(\wp,\varepsilon,\ell)=&\ -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left[(\mathcal{R}(\mu_1(\wp,\varepsilon,\ell))\right.\right.\\ &\left.\left.+\{\mathcal{N}(\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell))-\mathcal{N}(\mu_0(\wp,\varepsilon,\ell))\}\right]\right]\\ &=-\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right],\\ \nu_2(\wp,\varepsilon,\ell)=&\ -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left[(\mathcal{R}(\nu_1(\wp,\varepsilon,\ell))\right.\right.\\ &\left.\left.+\{\mathcal{N}(\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell))-\mathcal{N}(\nu_0(\wp,\varepsilon,\ell))\}\right]\right]\\ &=\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right],\end{aligned} \tag{26}$$

$$\mu_3(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left[(\mathcal{R}(\mu_2(\wp,\varepsilon,\ell)) + \{\mathcal{N}(\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)+\mu_2(\wp,\varepsilon,\ell)) - \mathcal{N}(\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell))\}\right]\right]$$
$$= \sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3 + \frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)} + \frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right],$$

$$\nu_3(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left[(\mathcal{R}(\nu_2(\wp,\varepsilon,\ell)) + \{\mathcal{N}(\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)+\nu_2(\wp,\varepsilon,\ell)) - \mathcal{N}(\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell))\}\right]\right]$$
$$= -\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3 + \frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)} + \frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right] \quad (27)$$

In general way,

$$\mu(\wp,\varepsilon,\ell) = \sum_{r=0}^{\infty}\mu_r(\wp,\varepsilon,\ell) = \mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)+\mu_2(\wp,\varepsilon,\ell)+\ldots$$

$$\nu(\wp,\varepsilon,\ell) = \sum_{r=0}^{\infty}\nu_r(\wp,\varepsilon,\ell) = \nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)+\nu_2(\wp,\varepsilon,\ell)+\ldots$$

In the addition of all μ and ν, we have

$$\mu(\wp,\varepsilon,\ell) = -\sin(\wp+\varepsilon) + \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right] + \sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right]$$
$$-\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right] + \sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3$$
$$\times\left[(1-\alpha)^3 + \frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)} + \frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]-\ldots$$

$$\nu(\wp,\varepsilon,\ell) = \sin(\wp+\varepsilon) - \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right] - \sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right]$$
$$+\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right] - \sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3$$
$$\times\left[(1-\alpha)^3 + \frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)} + \frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]+\ldots$$

The exact solution of Equation (21) at $\alpha = 1$ and $q = 0$.

$$\mu(\wp,\varepsilon,\ell) = -e^{-2\rho\ell}\sin(\wp+\varepsilon)$$
$$\nu(\wp,\varepsilon,\ell) = e^{-2\rho\ell}\sin(\wp+\varepsilon) \quad (28)$$

(By HPTM): By implementing HPTM in Equation (23), we get,

$$\sum_{r=0}^{\infty} z^r\,\mu(\wp,\varepsilon,\ell) = -\sin(\wp+\varepsilon) + \mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right] + z.\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\times\mathcal{A}\left\{\rho\sum_{r=0}^{\infty}z^r\left(\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right) - \sum_{r=0}^{\infty}z^r\,\mathcal{H}_r(\wp,\varepsilon)\right\}\right],$$
$$\sum_{r=0}^{\infty} z^r\,\nu(\wp,\varepsilon,\ell) = \sin(\wp+\varepsilon) - \mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right] + z.\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\times\mathcal{A}\left\{\rho\sum_{r=0}^{\infty}z^r\left(\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right) - \sum_{r=0}^{\infty}z^r\,\mathcal{I}_r(\wp,\varepsilon)\right\}\right], \quad (29)$$

where, $\mathcal{H}_r(\wp,\varepsilon)=\mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}$ and $\mathcal{J}_r(\wp,\varepsilon)=\mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}$, represent the nonlinear term.

From Equation (29), comparing the power of z, we get,

$$z^0:\mu_0(\wp,\varepsilon,\ell)=-\sin(\wp+\varepsilon)+\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],$$
$$z^0:\nu_0(\wp,\varepsilon,\ell)=\sin(\wp+\varepsilon)-\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right], \tag{30}$$

$$\begin{aligned}z^1:\mu_1(\wp,\varepsilon,\ell)&=\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu_0}{\partial\wp^2}+\frac{\partial^2\mu_0}{\partial\varepsilon^2}\right)-\mathcal{H}_0(\wp,\varepsilon)\right\}\right]\\&=\sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],\\z^1:\nu_1(\wp,\varepsilon,\ell)&=\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu_0}{\partial\wp^2}+\frac{\partial^2\nu_0}{\partial\varepsilon^2}\right)-\mathcal{J}_0(\wp,\varepsilon)\right\}\right]\\&=-\sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],\end{aligned}\tag{31}$$

where, $\mathcal{H}_0(\wp,\varepsilon)=\mu_0\frac{\partial\mu_0}{\partial\wp}+\nu_0\frac{\partial\mu_0}{\partial\varepsilon}$, and $\mathcal{J}_0(\wp,\varepsilon)=\mu_0\frac{\partial\nu_0}{\partial\wp}+\nu_0\frac{\partial\nu_0}{\partial\varepsilon}$.

$$\begin{aligned}z^2:\mu_2(\wp,\varepsilon,\ell)&=\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu_1}{\partial\wp^2}+\frac{\partial^2\mu_1}{\partial\varepsilon^2}\right)-\mathcal{H}_1(\wp,\varepsilon)\right\}\right]\\&=-\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right],\\z^2:\nu_2(\wp,\varepsilon,\ell)&=\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu_1}{\partial\wp^2}+\frac{\partial^2\nu_1}{\partial\varepsilon^2}\right)-\mathcal{J}_1(\wp,\varepsilon)\right\}\right]\\&=\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right],\end{aligned}\tag{32}$$

where, $\mathcal{H}_1(\wp,\varepsilon)=\left(\mu_0\frac{\partial\mu_1}{\partial\wp}+\mu_1\frac{\partial\mu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\mu_1}{\partial\varepsilon}+\nu_1\frac{\partial\mu_0}{\partial\varepsilon}\right)$,
and $\mathcal{J}_1(\wp,\varepsilon)=\left(\mu_0\frac{\partial\nu_1}{\partial\wp}+\mu_1\frac{\partial\nu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\nu_1}{\partial\varepsilon}+\nu_1\frac{\partial\nu_0}{\partial\varepsilon}\right)$.
$\vdots$

$$\begin{aligned}z^3:\quad&\mu_3(\wp,\varepsilon,\ell)=\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu_2}{\partial\wp^2}+\frac{\partial^2\mu_2}{\partial\varepsilon^2}\right)-\mathcal{H}_2(\wp,\varepsilon)\right\}\right]\\&=\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right],\\z^3:\quad&\nu_3(\wp,\varepsilon,\ell)=\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu_2}{\partial\wp^2}+\frac{\partial^2\nu_2}{\partial\varepsilon^2}\right)-\mathcal{J}_2(\wp,\varepsilon)\right\}\right]\\&=-\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right],\end{aligned}\tag{33}$$

where, $\mathcal{H}_2(\wp,\varepsilon)=\left(\mu_0\frac{\partial\mu_2}{\partial\wp}+\mu_1\frac{\partial\mu_1}{\partial\wp}+\mu_2\frac{\partial\mu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\mu_2}{\partial\varepsilon}+\nu_1\frac{\partial\mu_1}{\partial\varepsilon}+\nu_2\frac{\partial\mu_0}{\partial\varepsilon}\right)$,
and $\mathcal{J}_2(\wp,\varepsilon)=\left(\mu_0\frac{\partial\nu_2}{\partial\wp}+\mu_1\frac{\partial\nu_1}{\partial\wp}+\mu_2\frac{\partial\nu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\nu_2}{\partial\varepsilon}+\nu_1\frac{\partial\nu_1}{\partial\varepsilon}+\nu_2\frac{\partial\nu_0}{\partial\varepsilon}\right)$.
$\vdots$

In general way,

$$\mu(\wp,\varepsilon,\ell)=\sum_{r=0}^{\infty}\mu_r(\wp,\varepsilon,\ell)=\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)+\mu_2(\wp,\varepsilon,\ell)+\ldots$$

$$\nu(\wp,\varepsilon,\ell)=\sum_{r=0}^{\infty}\nu_r(\wp,\varepsilon,\ell)=\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)+\nu_2(\wp,\varepsilon,\ell)+\ldots$$

In addition of all μ and ν, we have,

$$\mu(\wp,\varepsilon,\ell)=-\sin(\wp+\varepsilon)+\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right]+\sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$-\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right]+\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3$$
$$\times\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]-\ldots$$
$$\nu(\wp,\varepsilon,\ell)=\sin(\wp+\varepsilon)-\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right]-\sin(\wp+\varepsilon)\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$+\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right]-\sin(\wp+\varepsilon)\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3$$
$$\times\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]+\ldots$$

The exact solution of Equation (21) at $\alpha=1$ and $q=0$.

$$\begin{aligned}\mu(\wp,\varepsilon,\ell)&=-e^{-2\rho\ell}\sin(\wp+\varepsilon)\\ \nu(\wp,\varepsilon,\ell)&=e^{-2\rho\ell}\sin(\wp+\varepsilon)\end{aligned}\tag{34}$$

Example 2. Consider the 2D fractional Navier-Stokes equations (Singh and Kumar, 2018).

$$\begin{aligned}{}^{AB}_{\;\,0}\mathcal{D}^{\alpha}_{\ell}(\mu)+\mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}&=\rho\left[\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right]+q,\\ {}^{AB}_{\;\,0}\mathcal{D}^{\alpha}_{\ell}(\nu)+\mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}&=\rho\left[\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right]-q,\end{aligned}\tag{35}$$

with the initial conditions,

$$\begin{cases}\mu(\wp,\varepsilon,0)=-e^{(\wp+\varepsilon)}\\ \nu(\wp,\varepsilon,0)=e^{(\wp+\varepsilon)}\end{cases}\tag{36}$$

Applying the Aboodh transform and inversion in Equation (35), we get,

$$\begin{aligned}\mu(\wp,\varepsilon,\ell)&=\mu(\wp,\varepsilon,0)+\mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right]+\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.\\ &\qquad\left.\times\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right)-\left(\mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}\right)\right\}\right],\\ \nu(\wp,\varepsilon,\ell)&=\nu(\wp,\varepsilon,0)-\mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right]+\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.\\ &\qquad\left.\times\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right)-\left(\mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}\right)\right\}\right],\end{aligned}\tag{37}$$

(By NITM): From Equations (35), and (36), we set the following,

$$\begin{cases}\mathcal{P}_1(\wp,\varepsilon,\ell)=q,\ \mathcal{R}\big(\mu(\wp,\varepsilon,\ell)\big)=-\rho\left(\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right),\quad\mathcal{N}\big(\mu(\wp,\varepsilon,\ell)\big)=\left(\mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}\right)\\ \mathcal{P}_2(\wp,\varepsilon,\ell)=-q,\ \mathcal{R}\big(\nu(\wp,\varepsilon,\ell)\big)=-\rho\left(\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right),\quad\mathcal{N}\big(\nu(\wp,\varepsilon,\ell)\big)=\left(\mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}\right)\\ \mu_0(\wp,\varepsilon,0)=-e^{(\wp+\varepsilon)},\quad\nu_0(\wp,\varepsilon,0)=e^{(\wp+\varepsilon)}\end{cases}$$

Using the iteration process outlined in Section (3) of NITM, we obtain the following,

$$\mu_0(\wp,\varepsilon,\ell) = \mathcal{A}^{-1}\left[s^{-2}\mu(\wp,\varepsilon,0) + \frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}[\mathcal{P}_1(\wp,\varepsilon,\ell)]\right],\ \ 0<\alpha\le 1$$
$$= -e^{\wp+\varepsilon} + \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],$$

$$\nu_0(\wp,\varepsilon,\ell) = \mathcal{A}^{-1}\left[s^{-2}\nu(\wp,\varepsilon,0) + \frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}[\mathcal{P}_2(\wp,\varepsilon,\ell)]\right]$$
$$= e^{\wp+\varepsilon} - \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right] \tag{38}$$

$$\mu_1(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left(\mathcal{R}\big(\mu_0(\wp,\varepsilon,\ell)\big)+\mathcal{N}\big(\mu_0(\wp,\varepsilon,\ell)\big)\right)\right]$$
$$= -e^{\wp+\varepsilon}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],$$
$$\nu_1(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left(\mathcal{R}\big(\nu_0(\wp,\varepsilon,\ell)\big)+\mathcal{N}\big(\nu_0(\wp,\varepsilon,\ell)\big)\right)\right] \tag{39}$$
$$= e^{\wp+\varepsilon}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],$$

$$\mu_2(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\Big[\big(\mathcal{R}\big(\mu_1(\wp,\varepsilon,\ell)\big)\right.$$
$$\left.+\left\{\mathcal{N}\big(\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)\big)-\mathcal{N}\big(\mu_0(\wp,\varepsilon,\ell)\big)\right\}\Big]\right]$$
$$= -e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$
$$\nu_2(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\Big[\big(\mathcal{R}\big(\nu_1(\wp,\varepsilon,\ell)\big)\right. \tag{40}$$
$$\left.+\left\{\mathcal{N}\big(\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)\big)-\mathcal{N}\big(\nu_0(\wp,\varepsilon,\ell)\big)\right\}\Big]\right]$$
$$= e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right],$$

$$\mu_3(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\Big[\big(\mathcal{R}\big(\mu_2(\wp,\varepsilon,\ell)\big)\right.$$
$$\left.+\left\{\mathcal{N}\big(\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)+\mu_2(\wp,\varepsilon,\ell)\big)-\mathcal{N}\big(\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)\big)\right\}\Big]\right]$$
$$= -e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right],$$

$$\nu_3(\wp,\varepsilon,\ell) = -\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\Big[\big(\mathcal{R}\big(\nu_2(\wp,\varepsilon,\ell)\big)\right. \tag{41}$$
$$\left.+\left\{\mathcal{N}\big(\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)+\nu_2(\wp,\varepsilon,\ell)\big)-\mathcal{N}\big(\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)\big)\right\}\Big]\right]$$
$$= e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]$$
$$\vdots$$

In general way,

$$\mu(\wp,\varepsilon,\ell) = \sum_{r=0}^{\infty}\mu_r(\wp,\varepsilon,\ell) = \mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)+\mu_2(\wp,\varepsilon,\ell)+\ldots$$
$$\nu(\wp,\varepsilon,\ell) = \sum_{r=0}^{\infty}\nu_r(\wp,\varepsilon,\ell) = \nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)+\nu_2(\wp,\varepsilon,\ell)+\ldots$$

In the addition of all μ and ν, we have,

$$\mu(\wp,\varepsilon,\ell) = -e^{\wp+\varepsilon} + \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right] - e^{\wp+\varepsilon}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$-e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right] - e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3$$
$$\times\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]-\cdots$$

$$\nu(\wp,\varepsilon,\ell) = e^{\wp+\varepsilon} - \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right] + e^{\wp+\varepsilon}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right]$$
$$+e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right] + e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3$$
$$\times\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^{\alpha}}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]+\ldots$$

The exact solution of Equation (35) at $\alpha = 1$ and $q = 0$.

$$\mu(\wp,\varepsilon,\ell) = -e^{\wp+\varepsilon+2\rho\ell}$$
$$\nu(\wp,\varepsilon,\ell) = e^{\wp+\varepsilon+2\rho\ell} \tag{42}$$

(By HPTM): By implementing HPTM in Equation (37), we get,

$$\sum_{r=0}^{\infty} z^r\,\mu(\wp,\varepsilon,\ell) = -e^{(\wp+\varepsilon)} + \mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right] + z.\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.$$
$$\left.\times\mathcal{A}\left\{\rho\sum_{r=0}^{\infty} z^r\left(\frac{\partial^2\mu}{\partial\wp^2}+\frac{\partial^2\mu}{\partial\varepsilon^2}\right)-\sum_{r=0}^{\infty} z^r\,\mathcal{H}_r(\wp,\varepsilon)\right\}\right],$$

$$\sum_{r=0}^{\infty} z^r\,\nu(\wp,\varepsilon,\ell) = e^{(\wp+\varepsilon)} - \mathcal{A}^{-1}\left[\left(\frac{1-\alpha+\alpha.s^{-\alpha}}{\mathfrak{T}(\alpha)}\right)\mathcal{A}[q]\right] + z.\mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\right.$$
$$\left.\times\mathcal{A}\left\{\rho\sum_{r=0}^{\infty} z^r\left(\frac{\partial^2\nu}{\partial\wp^2}+\frac{\partial^2\nu}{\partial\varepsilon^2}\right)-\sum_{r=0}^{\infty} z^r\,\mathcal{J}_r(\wp,\varepsilon)\right\}\right], \tag{43}$$

where, $\mathcal{H}_r(\wp,\varepsilon) = \mu\frac{\partial\mu}{\partial\wp}+\nu\frac{\partial\mu}{\partial\varepsilon}$ and $\mathcal{J}_r(\wp,\varepsilon) = \mu\frac{\partial\nu}{\partial\wp}+\nu\frac{\partial\nu}{\partial\varepsilon}$, represent the nonlinear term.

From Equation (43), comparing the power of z, we get,

$$z^0{:}\,\mu_0(\wp,\varepsilon,\ell) = -e^{(\wp+\varepsilon)} + \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],$$
$$z^0{:}\,\nu_0(\wp,\varepsilon,\ell) = e^{(\wp+\varepsilon)} - \frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right], \tag{44}$$

$$z^1{:}\,\mu_1(\wp,\varepsilon,\ell) = \mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu_0}{\partial\wp^2}+\frac{\partial^2\mu_0}{\partial\varepsilon^2}\right)-\mathcal{H}_0(\wp,\varepsilon)\right\}\right]$$
$$= -e^{(\wp+\varepsilon)}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right],$$
$$z^1{:}\,\nu_1(\wp,\varepsilon,\ell) = \mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu_0}{\partial\wp^2}+\frac{\partial^2\nu_0}{\partial\varepsilon^2}\right)-\mathcal{J}_0(\wp,\varepsilon)\right\}\right]$$
$$= e^{(\wp+\varepsilon)}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^{\alpha}}{\Gamma(\alpha+1)}\right], \tag{45}$$

where, $\mathcal{H}_0(\wp,\varepsilon) = \mu_0\frac{\partial\mu_0}{\partial\wp}+\nu_0\frac{\partial\mu_0}{\partial\varepsilon}$ and $\mathcal{J}_0(\wp,\varepsilon) = \mu_0\frac{\partial\nu_0}{\partial\wp}+\nu_0\frac{\partial\nu_0}{\partial\varepsilon}$.

$$\begin{aligned} z^2: \mu_2(\wp,\varepsilon,\ell) &= \mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu_1}{\partial\wp^2}+\frac{\partial^2\mu_1}{\partial\varepsilon^2}\right)-\mathcal{H}_1(\wp,\varepsilon)\right\}\right] \\ &= -e^{(\wp+\varepsilon)}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right], \\ z^2: \nu_2(\wp,\varepsilon,\ell) &= \mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu_1}{\partial\wp^2}+\frac{\partial^2\nu_1}{\partial\varepsilon^2}\right)-\mathcal{I}_1(\wp,\varepsilon)\right\}\right] \\ &= e^{(\wp+\varepsilon)}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right], \end{aligned} \tag{46}$$

where, $\mathcal{H}_1(\wp,\varepsilon)=\left(\mu_0\frac{\partial\mu_1}{\partial\wp}+\mu_1\frac{\partial\mu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\mu_1}{\partial\varepsilon}+\nu_1\frac{\partial\mu_0}{\partial\varepsilon}\right)$,

and $\mathcal{I}_1(\wp,\varepsilon)=\left(\mu_0\frac{\partial\nu_1}{\partial\wp}+\mu_1\frac{\partial\nu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\nu_1}{\partial\varepsilon}+\nu_1\frac{\partial\nu_0}{\partial\varepsilon}\right)$.

$$\begin{aligned} z^3: \mu_3(\wp,\varepsilon,\ell) &= \mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\mu_2}{\partial\wp^2}+\frac{\partial^2\mu_2}{\partial\varepsilon^2}\right)-\mathcal{H}_2(\wp,\varepsilon)\right\}\right] \\ &= -e^{(\wp+\varepsilon)}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right], \\ z^3: \nu_3(\wp,\varepsilon,\ell) &= \mathcal{A}^{-1}\left[\frac{(1-\alpha+\alpha.s^{-\alpha})}{\mathfrak{T}(\alpha)}\mathcal{A}\left\{\rho\left(\frac{\partial^2\nu_2}{\partial\wp^2}+\frac{\partial^2\nu_2}{\partial\varepsilon^2}\right)-\mathcal{I}_2(\wp,\varepsilon)\right\}\right] \\ &= e^{(\wp+\varepsilon)}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right], \end{aligned} \tag{47}$$

where, $\mathcal{H}_2(\wp,\varepsilon)=\left(\mu_0\frac{\partial\mu_2}{\partial\wp}+\mu_1\frac{\partial\mu_1}{\partial\wp}+\mu_2\frac{\partial\mu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\mu_2}{\partial\varepsilon}+\nu_1\frac{\partial\mu_1}{\partial\varepsilon}+\nu_2\frac{\partial\mu_0}{\partial\varepsilon}\right)$,
and $\mathcal{I}_2(\wp,\varepsilon)=\left(\mu_0\frac{\partial\nu_2}{\partial\wp}+\mu_1\frac{\partial\nu_1}{\partial\wp}+\mu_2\frac{\partial\nu_0}{\partial\wp}\right)+\left(\nu_0\frac{\partial\nu_2}{\partial\varepsilon}+\nu_1\frac{\partial\nu_1}{\partial\varepsilon}+\nu_2\frac{\partial\nu_0}{\partial\varepsilon}\right)$.

$$\vdots$$

In general way,

$$\mu(\wp,\varepsilon,\ell)=\sum_{r=0}^{\infty}\mu_r(\wp,\varepsilon,\ell)=\mu_0(\wp,\varepsilon,\ell)+\mu_1(\wp,\varepsilon,\ell)+\mu_2(\wp,\varepsilon,\ell)+\ldots$$
$$\nu(\wp,\varepsilon,\ell)=\sum_{r=0}^{\infty}\nu_r(\wp,\varepsilon,\ell)=\nu_0(\wp,\varepsilon,\ell)+\nu_1(\wp,\varepsilon,\ell)+\nu_2(\wp,\varepsilon,\ell)+\ldots$$

In the addition of all μ and ν, we have,

$$\begin{aligned} \mu(\wp,\varepsilon,\ell) &= -e^{\wp+\varepsilon}+\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right]-e^{\wp+\varepsilon}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right] \\ &\quad -e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right]-e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3 \\ &\times\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]-\cdots \end{aligned}$$

$$\begin{aligned} \nu(\wp,\varepsilon,\ell) &= e^{\wp+\varepsilon}-\frac{q}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right]+e^{\wp+\varepsilon}\frac{2\rho}{\mathfrak{T}(\alpha)}\left[(1-\alpha)+\frac{\alpha\ell^\alpha}{\Gamma(\alpha+1)}\right] \\ &\quad +e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^2\left[(1-\alpha)^2+\frac{2\alpha(1-\alpha)\ell^\alpha}{\Gamma(\alpha+1)}+\frac{\alpha^2\ell^{2\alpha}}{\Gamma(2\alpha+1)}\right]+e^{\wp+\varepsilon}\left(\frac{2\rho}{\mathfrak{T}(\alpha)}\right)^3 \\ &\times\left[(1-\alpha)^3+\frac{3\alpha(1-\alpha)^2\ell^\alpha}{\Gamma(\alpha+1)}+\frac{3\alpha^2(1-\alpha)\ell^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\alpha^3\ell^{3\alpha}}{\Gamma(3\alpha+1)}\right]+\ldots \end{aligned}$$

The exact solution of Equation (35) at $\alpha=1$ and $q=0$.

$$\mu(\wp,\varepsilon,\ell) = -e^{\wp+\varepsilon+2\rho\ell}$$
$$\nu(\wp,\varepsilon,\ell) = e^{\wp+\varepsilon+2\rho\ell} \tag{48}$$

To ensure that the suggested procedures are correct and appropriate, the graphical interpretation of various examples is shown in both fractional and integer order.

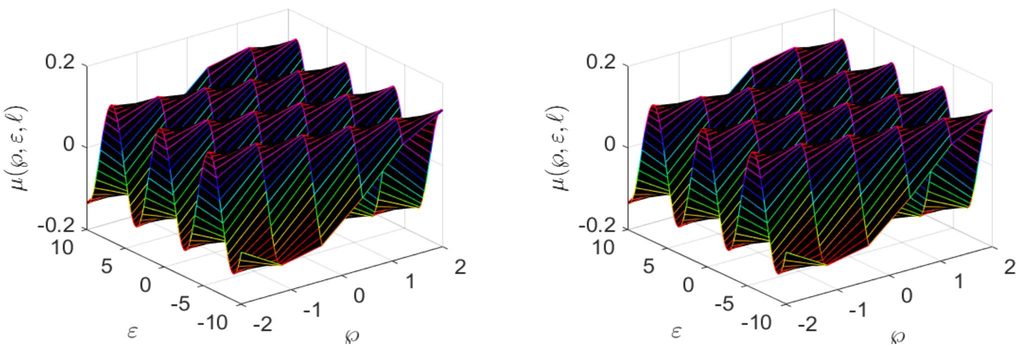

Figure 1. Exact and approximate solution of $\mu(\wp,\varepsilon,\ell)$ at $\alpha = 1,\ \rho = 1$ and $\ell = 1$ of Example 1 by NITM.

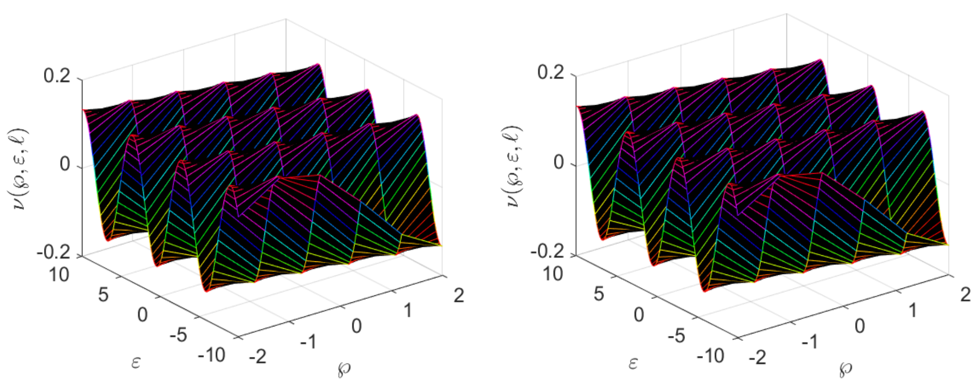

Figure 2. Exact and approximate solution of $\nu(\wp,\varepsilon,\ell)$ at $\alpha = 1,\ \rho = 1$ and $\ell = 1$ of Example 1 by NITM.

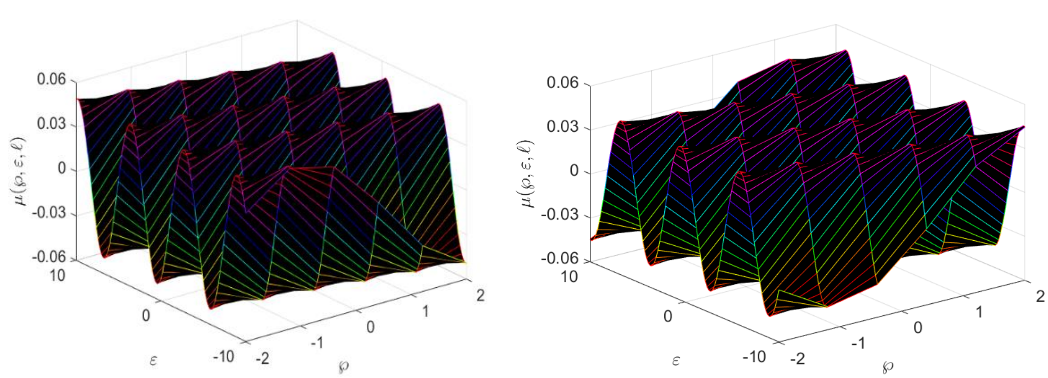

Figure 3. Approximate solution of $\mu(\wp,\varepsilon,\ell)$ at $\alpha = 0.3, 0.6$ and $\rho = 0.5$ of Example 1 by NITM.

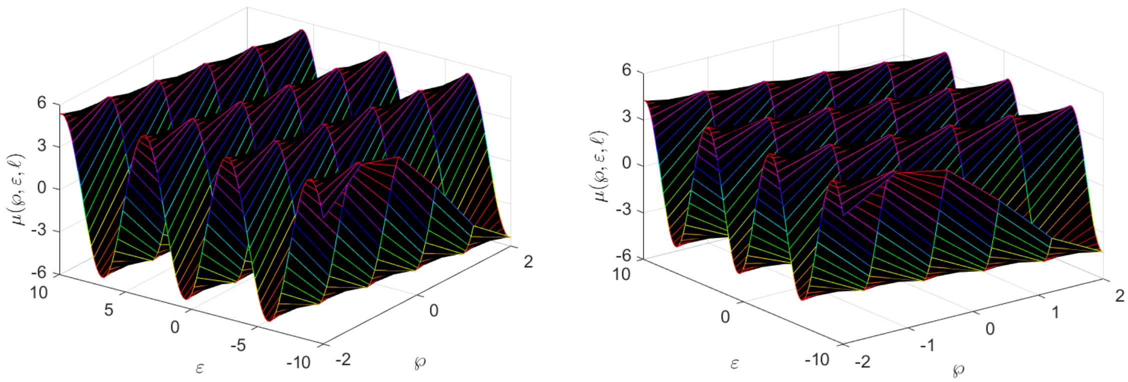

Figure 4. Approximate solution of $\nu(\wp, \varepsilon, \ell)$ at $\alpha = 0.3, 0.6$ and $\rho = 1$ of Example 1 by NITM.

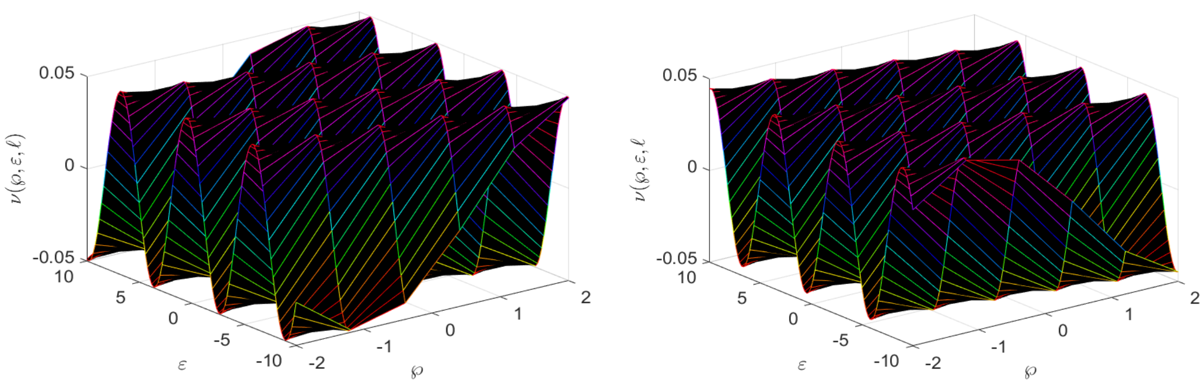

Figure 5. Approximate solution of $\nu(\wp, \varepsilon, \ell)$ at $\alpha = 0.3, 0.6$ and $\rho = 0.5$ of Example 1 by NITM.

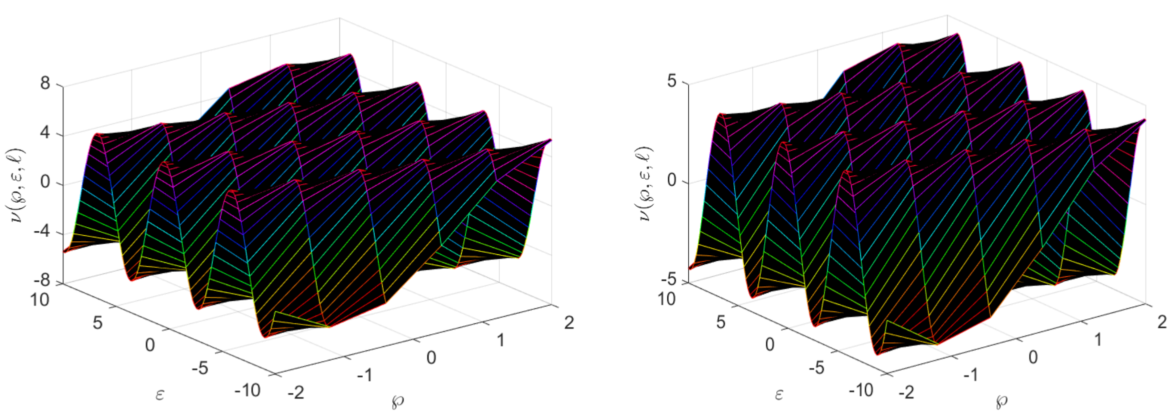

Figure 6. Approximate solution of $\nu(\wp, \varepsilon, \ell)$ at $\alpha = 0.3, 0.6$ and $\rho = 1$ of Example 1 by NITM.

Figures 1-2 show the behavior of the exact and analytical solutions of Example 1 of $\mu(\wp,\varepsilon,\ell)$ and $\nu(\wp,\varepsilon,\ell)$ at $\alpha=1$, demonstrating that the NITM solution figures are identical and close to the exact solution. Figures 3-4 show the physical features of $\mu(\wp,\varepsilon,\ell)$ for the different fractional orders $\alpha=0.3,0.6$ at $\rho=0.5,1$ in Example 1. Figures 5-6 illustrate the graphical solution of $\nu(\wp,\varepsilon,\ell)$ for different fractional orders $\alpha=0.3,0.6$ at $\rho=0.5,1$ in Example 1. The NITM solution is strongly aligned with the precise solution and has a high rate of convergence.

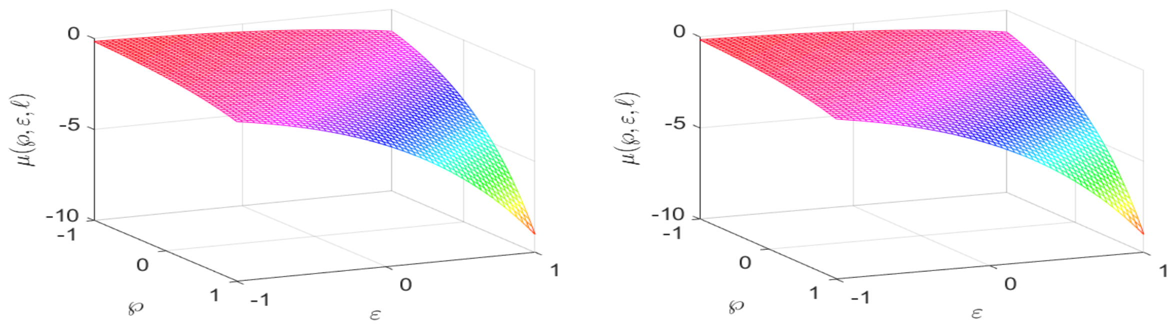

Figure 7. Exact and approximate solution of $\mu(\wp,\varepsilon,\ell)$ at $\alpha=1$, $\rho=0.5$ and $\ell=1$ of Example 2 by NITM.

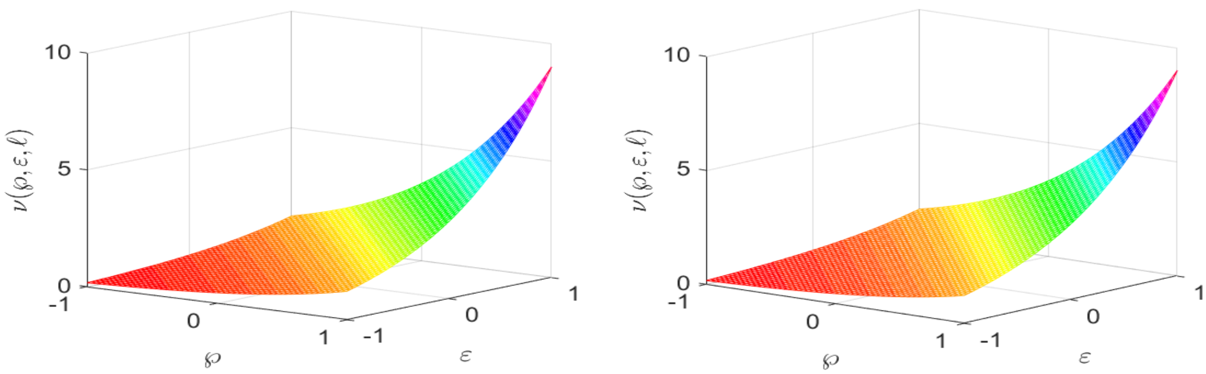

Figure 8. Exact and approximate solution of $\nu(\wp,\varepsilon,\ell)$ at $\alpha=1$, $\rho=0.5$ and $\ell=1$ of Example 2 by NITM.

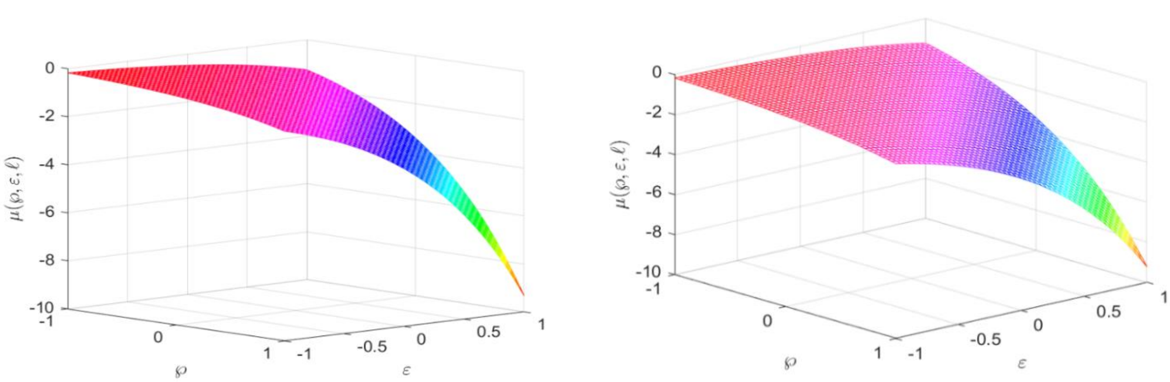

Figure 9. Approximate solution of $\mu(\wp,\varepsilon,\ell)$ at $\alpha=0.5,\ 1$ and $\rho=1$ and $\ell=1$ of Example 2 by NITM.

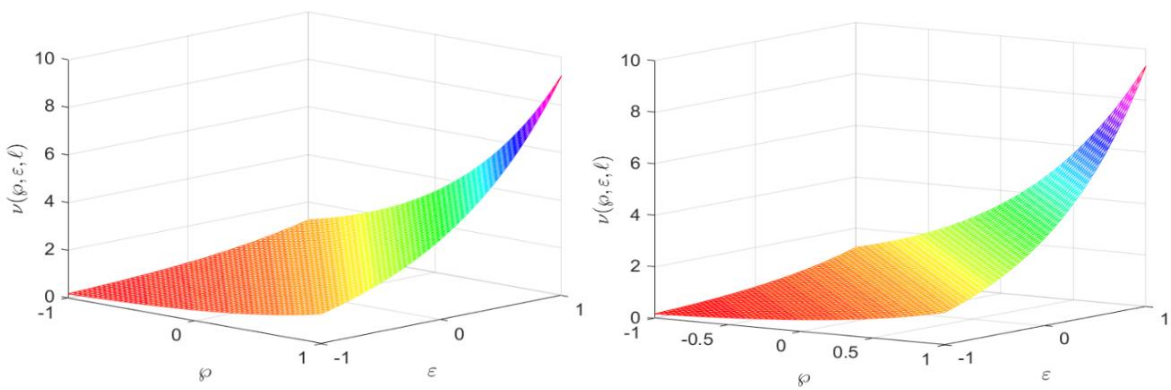

Figure 10. Approximate solution of $\nu(\wp,\varepsilon,\ell)$ at $\alpha = 0.5,\ 1$ and $\rho = 1$ and $\ell = 1$ of Example 2 by NITM.

Figures 7-8 demonstrate the analytical and exact solutions of $\mu(\wp,\varepsilon,\ell)$ and $\nu(\wp,\varepsilon,\ell)$ for $\alpha = 1$ and $\rho = 0.5$ in Example 2. The proposed ways to NITM solution yielded identical graphs, confirming the current technique's applicability. In Figures 9-10, the values of $\mu(\wp,\varepsilon,\ell)$ and $\nu(\wp,\varepsilon,\ell)$ are visually interpreted to correspond with the various fractional orders $\alpha = 0.5,1$ at $\rho = 1$ and indicate the analytical behavior of the solution.

Table 1. Exact and approximate results of $\mu(\wp,\varepsilon,\ell)$ of Example 1 by NITM with parameters $\ell = 0.01$, $\rho = 1$ and $\alpha = 1$ of Example 1 by NITM.

$\wp$	ε	Exact Solution	Numerical Solution	Absolute error
0.1	0.1	-0.194735414472060	-0.194735413152878	$1.319181930359292\times 10^{-9}$
0.3	0.3	-0.553461803314459	-0.553461799565183	$3.749275889752823\times 10^{-9}$
0.5	0.5	-0.824808742934829	-0.824808737347387	$5.587441842536123\times 10^{-9}$
0.7	0.7	-0.965936517945189	-0.965936511401715	$6.543473540787659\times 10^{-9}$
0.9	0.9	-0.954564155789734	-0.954564149323299	$6.466434609997407\times 10^{-9}$

Table 2. Exact and approximate results of $\nu(\wp,\varepsilon,\ell)$ of Example 1 by NITM with parameters $\ell = 0.01$, $\rho = 1$ and $\alpha = 1$ of Example 1 by NITM.

$\wp$	ε	Exact Solution	Numerical Solution	Absolute error
0.1	0.1	0.381707342492255	0.381707339906483	$2.585772318486335\times 10^{-9}$
0.3	0.3	0.703151488588232	0.703151483824924	$4.763307970279129\times 10^{-9}$
0.5	0.5	0.913583475535116	0.913583469346294	$6.188822121444559\times 10^{-9}$
0.7	0.7	0.979780719573737	0.979780712936479	$6.637257299324517\times 10^{-9}$
0.9	0.9	0.891292131415780	0.891292125377964	$6.037815580839379\times 10^{-9}$

Table 3. Exact and approximate results of $\mu(\wp,\varepsilon,\ell)$ of Example 2 by NITM with parameters $\ell = 0.01$, $\rho = 1$ and $\alpha = 1$ of Example 1 by NITM.

$\wp$	ε	Exact Solution	Numerical Solution	Absolute error
0.1	0.1	-1.246076730587381	-1.246076722412016	$8.175364607510005\times 10^{-9}$
0.3	0.3	-1.858928041846342	-1.858928029650131	$1.219621115211567\times 10^{-8}$
0.5	0.5	-2.773194763964298	-2.773194745769689	$1.819460893059954\times 10^{-8}$
0.7	0.7	-4.137120440251392	-4.137120413108227	$2.714316504892622\times 10^{-8}$
0.9	0.9	-6.171858449883554	-6.171858409390707	$4.049284640927908\times 10^{-8}$

Table 4. Exact and approximate results of $\nu(\wp, \varepsilon, \ell)$ of Example 2 by NITM with parameters $\ell = 0.01$, $\rho = 1$ and $\alpha = 1$ of Example 1 by NITM.

$\wp$	ε	Exact Solution	Numerical Solution	Absolute error
0.1	0.1	1.521961555618634	1.521961545633221	$9.985412940949345\times 10^{-9}$
0.3	0.3	2.270499837532406	2.270499822635921	$1.489648537855715\times 10^{-8}$
0.5	0.5	3.387187733621334	3.387187711398389	$2.222294526177393\times 10^{-8}$
0.7	0.7	5.053090316563868	5.053090283411128	$3.315273922055440\times 10^{-8}$
0.9	0.9	7.538324884203848	7.538324933661922	$4.945807408063274\times 10^{-8}$

Table 1 and 2 represents that exact solution is analogous to approximate solution obtained by NITM of example 1, and have large level of accuracy between the numerical results. The absolute error is approximately 10^{-9} between numerical results for $\mu(\wp, \varepsilon, \ell)$ and $\nu(\wp, \varepsilon, \ell)$.

Table 3 and 4 represents that exact solution is analogous to approximate solution obtained by NITM of example 2, and have sufficient of accuracy between the numerical results. The absolute error is approximately 10^{-8} order between numerical results of $\mu(\wp, \varepsilon, \ell)$ and of $\nu(\wp, \varepsilon, \ell)$.

5. Conclusion

In this research, we have presented a comprehensive analysis of two-dimensional fractional N-S equations analytically under the umbrella of the AB fractional derivative. Two methods, NITM and HPTM, are used along with the Aboodh transform, which shows these fractional operators are very simple and powerful when used with initial conditions. This has demonstrated the effectiveness and versatility of these methodologies in solving such complex systems. The study successfully offered new solutions that are very informative of the intriguing dynamics and patterns within the AB fractional derivative operator framework. The derived results are graphed and analyzed to demonstrate the validity and efficacy of the proposed methodologies. Moreover, the fractional results obtained closely resemble the exact solutions, confirming the accuracy and reliability of these methods. Additionally, the numerical simulations also demonstrated the resilience and efficiency of the approaches used. This study's findings have promising implications for nonlinear physics and related domains, fractional dynamics, and applied mathematics, leading to additional research and applications in a variety of scientific subjects.

Conflict of Interest

There is no competing interest among the authors regarding the publication of the article.

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