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Distribution-Free Control Charts Based on Multiple Runs: Advances and Applications in Supply Chain Management

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Abstract
In this article, we improve the behavior of nonparametric Shewhart-type control charts, which employ order statistics and multiple runs-type rules. The proposed class of monitoring schemes includes some existing control charts. In addition, new distribution-free monitoring schemes that pertain to the class, are set up and examined extensively. Explicit expressions for determining the variability and the mean of the run length distribution for the enhanced control charts are additionally delivered. As an example, a real-life managerial application is considered, where the proposed framework is implemented in order to enhance the provided services of a company under a supply chain management environment. Based on numerical comparisons, we draw the conclusion that the new charts outperform their competitors in identifying potential changes in the fundamental distribution in almost all cases considered.

Keywords- Average run length, Nonparametric statistical process control, Order statistics, Multiple runs, Supply chain management application.

1. Introduction

Statistical process control is extensively employed to monitor the quality of a production process, where no matter how diligently it is nurtured, inherent variability inevitably arises. Monitoring schemes help the professionals tasked with identifying identifiable origins of variability. Whenever the production has been changed toward an alternative (undesirable) state, a control chart should quickly identify it and issue an out-of-control signal as soon as possible.

a) It is evident that the preponderance of the monitoring frameworks is based on specific distributional assumptions. However, this presumption is not consistently fulfilled in real-life applications. To overcome this obstacle while still maintaining the traditional arrangement of the usual monitoring framework, several control charts without distributional assumptions have been documented in scholarly works and are called distribution-free. They all employ a suitably selected plotting statistic and pursue the structure of either *Shewhart*-type, *Cumulative* (*CUSUM*, hereafter) or *Exponentially Weighted Moving Average* (*EWMA*, hereafter) schemes.
b) During the previous decade, a few EWMA-type nonparametric monitoring schemes have been presented in scholarly works. As an example, Perdikis et al. (2023) introduced an adjusted Phase II EWMA scheme, which employs the well-known sign statistic, while Perdikis et al. (2021) suggested a nonparametric EWMA-type framework, which utilizes the Wilcoxon signed rank statistic. Moreover, Godase et al. (2022) established deciles-based monitoring schemes, which follow the aforementioned structure and seem to be quite capable for identifying changes in the dispersion of the underlying process. In addition, Xue et al. (2023) and Xue et al. (2024) proposed EWMA control charts for monitoring mixed continuous and count data or mixed continuous and categorical data respectively. A comprehensive and current overview of nonparametric EWMA monitoring schemes can be found in the work of Triantafyllou and Ram (2021a).

c) Moreover, CUSUM-type monitoring schemes have also sparked significant research interest in recent times. For instance, Xue and Qiu (2021) proposed a distribution-free CUSUM-type chart, which is suitable for controlling multidimensional correlated datasets. On the other hand, Wang et al. (2023) introduced nonparametric CUSUM-type control charts, whose aim is to track the dispersion of count data by the aid of categorical data analysis tools. In a bivariate framework, Erem and Mahmood (2023), Koutras and Sofikitou (2017) proposed nonparametric and semi-parametric control charts based on exceedance statistics and order statistics respectively. Some up-to-date advances on the topic have been provided by Abbas et al. (2024) and Tang and Li (2024). A comprehensive and current overview of nonparametric CUSUM monitoring schemes can be found in the work of Triantafyllou and Ram (2021b).
d) Clearly, the Shewhart-type monitoring framework achieves a high performance, particularly in the presence of substantial shifts in the fundamental distribution. In this regard, a few nonparametric charts, which employ order statistics have been documented in scholarly works. For instance, Janacek and Meikle (1997) proposed two-sided distribution-free charts, whose control limits are calculated based on reference data which have been drawn during the in-control phase of the process. Within their structure, determining whether the process stays at an in-control state, or it has been shifted to an out-of-control situation, depends on the median values of test samples which are collected from the underlying production. The aforementioned structure has been also investigated and generalized by Chakraborti et al. (2004), where several quantiles (apart from the median) has been also considered. Advancing this line of research, Balakrishnan et al. (2010) and Triantafyllou and Panayiotou (2020) factored in not only where suitably selected order statistics from the available test data are placed, but took also into consideration the quantity of test observations that lie between the lower and the upper control limit in order to characterize the production as in- or out-of-control. Koutras and Sofikitou (2020) established semiparametric bivariate monitoring schemes, which employ order statistics and have been proven to be quite efficacious in simultaneously controlling the production for plausible changes in either mean or/and variability.
e) On the other hand, Malela-Majika et al. (2022a) proposed an efficacious and powerful one-sided framework by the aid of precedence statistics and advanced runs-type rules, wherein no distributional presumptions in either zero- states or steady-states modes are needed to be made. Additionally, Panayiotou and Triantafyllou (2023) proposed a monitoring framework which employs order statistics and simple runs-type rules. Some up-to-date advances on the topic have been provided by Perdikis et al. (2024), and Triantafyllou (2024). For an in-depth research and several intriguing viewpoints on distribution-free monitoring structures, we touch on the works of Chakraborti and Graham (2019), or Qiu (2018, 2019).
f) Additionally, it is of some interest to point out that several approaches, which have been already introduced in the existing literature, combine the Shewhart framework with the CUSUM or the EWMA one. Some advances on the specific topic can be found in Tyagi (2019), Abujiya et al. (2013), Malela-Majika et al. (2022b), Capizzi and Masarotto (2010), and Wu et al. (2008).

In the present article, we study distribution-free Shewhart-type monitoring schemes, which employ order statistics along with signaling multiple runs-type rules. In other words, we implement the structure, which has been implemented in the work of Triantafyllou and Panayiotou (2020) and the proposed nonparametric chart seems to be improved by adding multiple runs-rules. The implementation of more sophisticated runs-type decision rules is expected (and finally is confirmed) to result in a more efficient monitoring scheme for detecting both smaller or larger shifts in the underlying process distribution. In Section 2, the framework of the advanced monitoring schemes is investigated extensively. Explicit expressions for determining the mean and variance of the corresponding run length distribution are proved in Section 3. These expressions are of high importance since they provide the opportunity to the practitioner to calculate the exact in- and out-of- control performance of the resulting monitoring scheme. In Section 4, an extensive numerical experimentation confirms that the suggested charts outmatch their nonparametric contenders in all challenges considered. Lastly, the Conclusion section summarizes the novelty of the current article, while a few perspectives for forthcoming endeavors are also pointed, while some concluding remarks and a few perspectives for forthcoming endeavors are pointed in conclusion section.

2. The General Framework for Constructing the Improved Distribution-Free Structures

In the present section, we study a class of a modified controlling framework, which employs order statistics, wherein multiple runs are also taken into consideration for characterizing the process as in-control or out-of-control. The control limits of the suggested control schemes utilize reference data, which are collected from the production when it is assumed to be in-control. The proposed schemes are born by implementing the plotting statistics used by Triantafyllou and Panayiotou (2020) and improving their performance by applying additional multiple runs-type rules.

For building up the proposed control charts, a reference data set of *m* observations, e.g. $X_1, X_2, \ldots, X_m$ with cumulative distribution function (c.d.f) F, is collected from the production during its in-control phase. In other words, $X_1, X_2, \ldots, X_m$ are random variables, which share a common c.d.f. If we denote by $X_{i:m}$ the $i-$th ordered statistic of the reference sample, the control limits of the resulting schemes correspond to suitably selected data of the respective ordered sample $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$.

To determine if the production remains in-control or has shifted to out-of-control state, *n* test observations, e.g. $Y_1^h, Y_2^h, \ldots, Y_n^h$ $(h = 1,2, \ldots)$ with c.d.f G, are afterwards drawn independently of each other (and also of the reference data). Within a stricter statistical terminology, we aim at tracking down plausible changes in the production, e.g. to detect possible distributional shifts from F to G.

The main idea of the suggested controlling framework is related to the ordered test data collected from the production. Let $Y_{1:n}^h, Y_{2:n}^h, \ldots, Y_{n:n}^h$ $(h = 1,2, \ldots)$ be the ordered statistics of the $h-th$ test sample, the monitoring statistics of the proposed charts are actually some data from the ordered $Y-$ sample. To be more precise, we should first choose four ordered reference observations, say $X_{a:m}, X_{b:m}, X_{c:m}, X_{d:m}$ with $1 \leq a < b < c < d \leq m$. We next collect two suitably chosen order statistics, say $Y_{i:n}^h, Y_{j:n}^h$ with $i < j$, from the $h-th$ test sample $Y_1^h, Y_2^h, \ldots, Y_n^h$ $(h = 1,2, \ldots)$ and these order statistics are actually the first two plotting statistics of the proposed monitoring scheme. Apart from them, we also focus on the amount of observations from each test sample, which are placed in the intervals $(X_{a:m}, X_{b:m})$ and $(X_{c:m}, X_{d:m})$ respectively. If we denote by $R_1^h = R\left(Y_1^h, Y_2^h, \ldots, Y_n^h; X_{a:m}, X_{b:m}\right)$ and $R_2^w = R\left(Y_1^h, Y_2^h, \ldots, Y_n^h; X_{c:m}, X_{d:m}\right)$ the number of $Y-$observations that belong to the territories $(X_{a:m}, X_{b:m})$ and $(X_{c:m}, X_{d:m})$ correspondingly, then the random variables R_1^h and R_2^h are two additional monitoring statistics of the proposed scheme, which shall be used along with $Y_{i:n}^h, Y_{j:n}^h$ mentioned earlier.

Moreover, we propose that the process shall be characterized as in-control or out-of-control after observing a specified number of multiple runs. Since each group of test observations, which is drawn from the process, gives either a good (in-control, IC hereafter) or bad (out-of-control, OOC hereafter) signal, the testing procedure can be viewed as a series of binary trials with two plausible results, e.g. IC or OOC. Under this framework, a multiple run $MR_{r,k}$ is occurred upon the $r-$th appearance of a run of OOC signals with length k, where, r, k are positive integer-valued parameters. In practical terms, the production shall be named as out-of-control, when k consecutive test samples give an OOC signal not for the first but for the $r-$th time (Balakrishnan and Koutras, 2002). It is straightforward that the proposed monitoring framework follows the traditional outlet of a distribution-free control chart, e.g. there is no distributional assumption therein. In addition, the implementation of some sophisticated runs-type rules seem that it can boost the performance of the new charts and make them more efficient in detecting small changes in the underlying distribution process.

The random variable that holds paramount significance is the waiting time for the occurrence of a multiple run $MR_{r,k}$. In the literature, there exist different enumeration approaches while dealing with the distribution of the above-mentioned waiting time random variable. The traditional structure for tasks related to runs of specified length is the framework provided by Feller (1968). Within the aforementioned argumentation, once k consecutive OOC signals take place, a OOC run of length k is recorded, while the enumeration procedure begins a new run. It is straightforward that under this enumeration framework there is no overlapping counting. Within the framework of the current work, we tend to follow the aforementioned enumeration scheme.

The overall structure of the suggested class of modified monitoring schemes is described below.
Stage A. Collect *m* reference observations from the in-control production.
Stage B. Calculate the control limits of the controlling structure via the respective reference ordered data.
Stage C. Collect independent test groups of *n* observations from the production.
Stage D. Compute the observed values of monitoring statistics $Y_{i:n}^h, Y_{j:n}^h, R_1^h$ and R_2^h based on the observations of each test sample.
Stage E. Activate a multiple run $MR_{r,k}$ rule.
Stage F. Characterize the production as in-control or out-of control with the help of the monitoring statistics and the $MR_{r,k}$ rule, which has been determined previously.

Before we proceed to prove the main theoretical outcomes for the suggested family of controlling schemes, we should succinctly provide some information regarding the plotting statistics, which shall be used and how an OOC signal is produced by the new schemes. As mentioned before, the proposed control charts utilize four different monitoring statistics, e.g. $Y_{i:n}^h, Y_{j:n}^h$, R_1^h and R_2^h with $i < j$. All these statistics are calculated for all test samples, which are collected from the production. Under the proposed framework, the h −th test sample does not produce a bad signal, whenever the next constraints are met

$$X_{a:m} \leq Y_{i:n}^h \leq X_{b:m},\ X_{c:m} \leq Y_{j:n}^h \leq X_{d:m},\ R_1^h \geq r_1 \text{ and } R_2^h \geq r_2 \qquad (1)$$

where, r_1, r_2 are positive integer-valued parameters. Within the suggested controlling framework which is presented in Equation (1) ($NMCC_{r,k,a,b,c,d,i,j}$ −chart, hereafter), the production is declared as out-of-control once we notice r subsequences of k bad signals, namely every time that we witness r subsequences of test groups of length k, wherein at least one condition stated in Equation (1) has been violated. In simpler words, the parameters $a, b, c, d, m, n, r_1, r_2$ are design parameters of the proposed monitoring framework providing the suitable set of conditions that are needed to be satisfied in order not to produce an out-of-control alarm.

It is worth mentioning that certain nonparametric monitoring schemes, previously presented in scholarly works, can be regarded as constituents of the novel distribution-free class introduced in this work. For instance, the control charts introduced by

- Janacek and Meikle (1997) can be viewed as a $NMCC_{1,1,a,b,a,b,med,med}$ −chart where med correspond to the median of the underlying test sample.
- Chakraborti et al. (2004) can be viewed as a $NMCC_{1,1,a,b,a,b,i,i}$ −chart with $r_1 = r_2 = 0$.
- Balakrishnan et al. (2010) can be viewed as a $NMCC_{1,1,a,b,a,b,i,i}$ −chart.

- Triantafyllou and Panayiotou (2020) can be viewed as a $NMCC_{1,1,a,b,c,d,i,j}$ −chart.
- Panayiotou and Triantafyllou (2023) can be viewed as a $NMCC_{1,k,a,b,c,d,i,j}$ −chart.

As a concluding remark for the discussion that has unfolded so far, we may readily deduce that the proposed class of monitoring schemes is a generalization of some existing nonparametric frameworks, which employ order statistics from the test samples to clarify if the production remains in an in-control state, or a shift has been occurred.

3. Main Results for the $NMCC_{r,k,a,b,c,d,i,j}$ − Monitoring Schemes

In the current section, we prove some theoretical outcomes for the suggested nonparametric monitoring schemes. We study two important features of the run length distribution for the suggested class of $NMCC_{r,k,a,b,c,d,i,j}$ − charts. To be more accurate we prove closed expressions for determining the average run length and the respective variance of $NMCC_{r,k,a,b,c,d,i,j}$ − charts. It is noticeable that, given the great quantity of design parameters of the suggested schemes, we offer to the professional the opportunity to satisfy any pre-specified requirements by determining appropriately the values of the parameters.

The next proposition offers integral expressions for computing the average run length (*ARL*) and the respective variance (*VRL*) of the suggested $NMCC_{r,k,a,b,c,d,i,j}$ − charts.

***Proposition* 1.** (i) The unconditional *ARL* and the unconditional *VRL* of the $NMCC_{r,k,a,b,c,d,i,j}$ − scheme are determined as

$$ARL_{r,k} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{r(1-p^k)}{(1-p)p^k} f_{a,b,c,d}(s_1,t_1,s_2,t_2) ds_1 dt_1 ds_2 dt_2 \tag{2}$$

and

$$VRL_{r,k} = \int_0^1 \int_0^{t_2} \int_0^{s_2} \int_0^{t_1} \frac{r(1-(2k+1)(1-p)p^k - p^{2k+1})}{((1-p)p^k)^2} \times f_{a,b,c,d}(s_1,t_1,s_2,t_2) ds_1 dt_1 ds_2 dt_2 \tag{3}$$

respectively, while

$$p = 1 - q(GF^{-1}(s_1), GF^{-1}(t_1), GF^{-1}(s_2), GF^{-1}(t_2); r_1, r_2)$$

and

$$q(v,w,t,z;r_1,r_2) = \sum_{c_1=0}^{n-2} \sum_{c_2=\max(0,r_1-c_1-1)}^{\min(n-c_1-2,j-i-1)} \sum_{c_3=0}^{\min(n-c_1-c_2-2,j-i-1-c_2)} \sum_{c_4=\max(0,r_2+c_2+c_3-j+i)}^{n-c_1-j+i-1} n! \times \frac{1}{(i-c_1-1)!(c_1+c_2+1)!c_3!(j-i-c_2-c_3+c_4)!(n-j-c_4)!} \times v^{i-c_1-1}(w-v)^{c_1+c_2+1}(t-w)^{c_3}(z-t)^{j-i-c_2-c_3+c_4}(1-z)^{n-j-c_4} \tag{4}$$

and

$$f_{a,b,c,d}(s_1,t_1,s_2,t_2) = \frac{m!}{(a-1)!(b-a-1)!(c-b-1)!(d-c-1)!(m-d)!} \times s_1^{a-1}(t_1-s_1)^{b-a-1}(s_2-t_1)^{c-b-1}(t_2-s_2)^{d-c-1}(1-t_2)^{m-d}, \; 0 < s_1 < t_1 < s_2 < t_2 < 1 \tag{5}$$

calls for the joint density function of the random variables s_1, t_1, s_2, t_2.

***Proof*:** Let $T_{r,k}$ be the waiting time till the r −appearance of a run of k consecutive bad signals, namely till an out-of-control signal is provided by the $NMCC_{r,k,a,b,c,d,i,j}$ − scheme. It is quite clear that $T_{r,k}$ coincides with the run length of the resulting control chart. Since the test samples are assumed to be independent from each other and from the reference sample, the random variable $T_{r,k}$ follows the negative binomial distribution of order k. The particular terminology was initially coined by Philippou (1984) and since then it has garnered significant research attention. Recent progress on the subject is presented by Georghiou et al. (2021) and De Souza and Diniz (2022).

Given $X_{a:m} = x, X_{b:m} = y, X_{c:m} = z, X_{d:m} = w$, the random variable $T_{r,k}$ can be considered as the r −th convolution of the geometric distribution of order k. The probability generating function of $T_{r,k}$ is determined as follows

$$E(z^{T_{r,k}}) = (G(z))^r \tag{6}$$

where, $G(z)$ expresses the probability generating function of the waiting time till the occurrence of the first run, e.g. if p is the success probability of the waiting distribution, then $G(z)$ can be computed as

$$G(z) = \frac{(pz)^k}{1-(1-p)zA(z)} \tag{7}$$

where,

$$A(z) = \frac{1-(pz)^k}{1-pz} \tag{8}$$

Therefore, if $G'(z)$ and $G''(z)$ denote the first and second derivate of $G(z)$, subsequently the conditional expected value and variance of the random variable $T_{r,k}$ are given via the next expressions

$$E\left(T_{r,k} | X_{a:m} = x_a, X_{b:m} = x_b, X_{c:m} = x_c, X_{d:m} = x_d\right) = r \cdot G'(1) = r \cdot \frac{(pz)^{rk}}{(1-(1-p)zA(z))^r} \tag{9}$$

and

$$Var\left(T_{r,k} | X_{a:m} = x_a, X_{b:m} = x_b, X_{c:m} = x_c, X_{d:m} = x_d\right) = r \cdot (G''(1) + G'(1) - (G'(1))^2) = \frac{r(1-(2k+1)(1-p)p^k - p^{2k+1})}{((1-p)p^k)^2} \tag{10}$$

respectively. Under the $NMCC_{r,k,a,b,c,d,i,j}-$ monitoring scheme, the success probability p of the latter geometric distribution of order k expresses actually the possibility that the group of constraints declared in Equation (1) is not met. On the other hand, the particular possibility can be expressed via the next multiple sum (Triantafyllou and Panayiotou, 2020).

$$\sum_{c_1=0}^{n-2} \sum_{c_2=\max(0, r_1-c_1-1)}^{\min(n-c_1-2, j-i-1)} \sum_{c_3=0}^{\min(n-c_1-c_2-2, j-i-1-c_2)} \sum_{c_4=\max(0, r_2+c_2+c_3-j+i)}^{n-c_1-j+i-1} n! \times \frac{v^{i-c_1-1}(w-v)^{c_1+c_2+1}(t-w)^{c_3}(z-t)^{j-i-c_2-c_3+c_4}(1-z)^{n-j-c_4}}{(i-c_1-1)!(c_1+c_2+1)!c_3!(j-i-c_2-c_3+c_4)!(n-j-c_4)!} \tag{11}$$

Combining Equations (9), (10) and (11), the desirable results are readily obtained.

With the results proved in Proposition 1 readily available, the unconditional in-control *ARL* and *VRL* of the suggested monitoring framework are easily acquired by switching $F = G$ in the expressions proved previously. **Table 1** displays the in-control *ARL's* of $NMCC_{r,k,a,b,c,d,i,j}-$ control charts under different choices of design parameters.

Table 1. In-control *ARL* of $NMCC_{r,k,a,b,c,d,i,j}-$ charts for several designs.

Reference sample of size *m*							
		50		100		200	
ARL₀	***n***	**(*r,k,a, b,c, d, i, j, r₁, r₂*)**	***Exact ARL_in***	**(*r,k,a, b,c, d, i, j, r₁, r₂*)**	***Exact ARL_in***	**(*r,k,a, b,c, d, i, j, r₁, r₂*)**	***ARL_in***
370	5	(3,11,5,14,24,42,1,3,2,3)	370.06	(3,14,6,27,40,70,1,3,2,3)	371.39	(2,14,9,52,121,170,1,3,2,3)	370.05
	7	(3,9,9,16,25,33,1,4,2,2)	368.15	(3,8,9,16,45,72,1,4,2,2)	372.73	(3,5,9,31,92,135,1,4,2,3)	376.34
	11	(3,1,9,17,25,31,2,6,2,3)	373.59	(4,7,9,30,65,90,2,5,2,3)	370.97	(4,16,9,33,121,150,1,6,2,3)	368.55
	15	(4,8,10,19,25,41,3,9,2,4,)	370.99	(3,14,10,19,46,64,3,9,3,4)	368.13	(3,6,15,33,96,126,3,9,3,4)	366.19
500	5	(3,2,5,14,24,40,1,3,2,3)	504.99	(3,14,9,27,40,70,1,3,2,3)	499.45	(3,7,9,52,90,140,1,3,2,3)	501.17
	7	(3,9,9,16,26,33,1,4,2,2)	497.82	(3,14,9,16,42,75,1,4,2,3)	497.17	(3,5,9,28,92,135,1,4,2,3)	501.45
	11	(4,9,9,20,25,31,2,5,2,3)	497.09	(4,13,9,27,65,90,2,5,2,3)	501.09	(4,10,9,33,125,150,1,6,2,3)	502.29
	15	(3,10,10,19,26,40,3,9,2,4)	502.04	(3,8,10,19,46,62,3,9,3,4)	493.48	(3,19,15,33,96,124,3,9,3,4)	497.96

In accordance with the numerical results given at **Table 1**, the professional could pick up the most suitable designs for building up a nonparametric monitoring scheme that fulfills the pre-specified requirements, namely the desired in-control *ARL*-performance. For example, we next contemplate the scenario, where a reference data set of $m = 200$ observations is drawn from the in-control process. We next collect independently succeeding samples of n test observations. Our objective is to build a control chart that brings about an in-control *ARL* in the region of 500. In accordance with **Table 1**, the latter requirement is fulfilled by building up

- A $NMCC_{r,k,a,b,c,d,i,j}-$ chart with design parameters $n = 5, r = 3, k = 7, a = 9, b = 52, c = 90, d = 140, i = 1, j = 3, r_1 = 2, r_2 = 3$. In other words, the practitioner should collect test samples of 5 observations, while the 9th, 52nd, 90th and 140th ordered reference observation should be chosen as the control limits. The resulting $NMCC_{r,k,a,b,c,d,i,j}-$scheme brings about an in-control *ARL* equal to 501.17, or

- A $NMCC_{r,k,a,b,c,d,i,j}-$ chart with design parameters $n = 7, r = 3, k = 5, a = 9, b = 28, c = 92, d = 135, i = 1, j = 4, r_1 = 2, r_2 = 3$. In other words, the practitioner should collect test samples of 7 observations, while the 9th, 28th, 92nd and 135th ordered reference observation should be chosen as the control limits. The resulting $NMCC_{r,k,a,b,c,d,i,j}-$chart brings about an in-control *ARL* equal to 501.45, or

- A $NMCC_{r,k,a,b,c,d,i,j}-$ chart with design parameters $n = 11, r = 4, k = 10, a = 9, b = 33, c = 125, d = 150, i = 1, j = 6, r_1 = 2, r_2 = 3$. In other words, the practitioner should collect test samples of 5 observations, while the 9th, 33rd, 125th and 150th ordered reference observation should be chosen as the control limits. The resulting $NMCC_{r,k,a,b,c,d,i,j}-$chart brings about an in-control *ARL* equal to 502.29, or

- A $NMCC_{r,k,a,b,c,d,i,j}-$ chart with design parameters $n = 15, r = 3, k = 19, a = 15, b = 33, c = 96, d = 124, i = 3, j = 9, r_1 = 3, r_2 = 4$. In other words, the practitioner should collect test samples of 5 observations, while the 15th, 33rd, 96th and 124th ordered reference observation should be chosen as the control limits. The resulting $NMCC_{r,k,a,b,c,d,i,j}-$chart brings about an in-control *ARL* equal to 497.96.

4. Numerical Comparisons

We next conduct thorough numerical experiments to illuminate the effectiveness of the new monitoring schemes and their resilience attributes across various out-of-control scenarios. The calculations leverage the theoretical findings established in section 3.

A conventional method for comparing two alternative control schemes involves defining a shared in–control *ARL* and afterwards to look at the corresponding out–control *ARL's*. It is evident that once a new monitoring framework is introduced as a generalization of charts already established, the direct contrasting against them is strongly advised. Consequently, we next contrast the behavior of the $NMCC_{r,k,a,b,c,d,i,j} -$ chart to the one proposed by Triantafyllou and Panayiotou (2020), Panayiotou and Triantafyllou (2023) and Balakrishnan et al. (2010).

Table 2 offers several numerical comparisons between the suggested framework and the nonparametric schemes established by Triantafyllou and Panayiotou (2020) and Panayiotou and Triantafyllou (2023) (*Competitive scheme* 1 and *Competitive scheme* 2, respectively).

We next consider the scenario that $m = 100$ reference observations are within reach, while test samples of $n = 25$ observations are afterwards collected from the production to determine if it remains at an in-control state, or it has shifted to an out-of- control situation. All competitive charts are constructed in a way that an in-control *ARL* in the region of 500 is observed. Throughout the lines of **Table 2**, the in-control distribution of the production is presumed to be the Exponential distribution with parameter $\lambda = 1$. The out–of–control behavior of the competitive charts is appraised via the respective *ARL's* for different changes of the distribution.

Table 2. *ARL* values of the $NMCC_{r,k,a,b,c,d,i,j} -$ schemes against competitors under exponential distribution for different shifts θ (m = 100, n = 25).

Exponential distribution **(λ)**			
Shift	$NMCC_{r,k,a,b,c,d,i,j}$ ***−chart*** $r = 1, k = 1, a = 10, b = 44, c = 56, d = 83,$ $i = 5, j = 20, \boldsymbol{r}_1^{\square} = 2\ , \boldsymbol{r}_2^{\square} = 2$	***Competitive scheme* 1**	***Competitive scheme* 2**
0.000	491.91	497.21	492.12
0.025	377.66	485.98	402.57
0.050	286.73	464.96	319.50
0.075	215.71	434.80	246.08
0.100	161.47	396.84	184.13
0.125	120.76	353.09	134.15
0.150	90.59	306.01	95.51
0.175	62.44	258.20	66.80
0.200	46.17	212.12	46.23

For instance, we next contemplate the scenario where the production mean has been changed $\theta = 0.05$ units. Within the latter scenario, the suggested $NMCC_{r,k,a,b,c,d,i,j} -$ framework attains an out-of-control *ARL* equal to 286.73 (see **Table 2**), while the corresponding *ARL*- values for both competitors are larger and more specifically are equal to 464.96 and 319.50 for *Competitive scheme* 1 and *Competitive scheme* 2 respectively. In simpler words, the proposed scheme detects the underlying shift after approximately 287 test samples, while its competitors are experiencing a delay in identifying the change since Competitive Scheme 1 calls for about 465 test samples and Competitive Scheme 2 calls for about 320 test samples in order to produce an alarm.

In the next lines, we explore the out-of-control behavior of the family of $NMCC_{r,k,a,b,c,d,i,j}-$ framework by offering numerical contrasting versus *Competitive scheme* 1 and *Competitive scheme* 2. As previously stated, since the suggested $NMCC_{r,k,a,b,c,d,i,j}-$ framework constitutes a generalization of the monitoring schemes introduced by Triantafyllou and Panayiotou (2020) and Panayiotou and Triantafyllou (2023), making a direct comparison between them is of significant interest.

Table 3 illustrates a numerical contrasting between the $NMCC_{r,k,a,b,c,d,i,j}-$ scheme and the nonparametric charts introduced by Triantafyllou and Panayiotou (2023) and Balakrishnan et al. (2010) (*Competitive scheme* 1 and *Competitive scheme* 3, respectively).

We next consider the scenario where, $m = 100$ reference data are within reach, while the competitive structures are built up such that an in-control *ARL* in the region of 500 is attained. Throughout the lines of **Table 3**, two distinct cases for the distribution of the production are studied, namely Normal distribution and Laplace distribution.

Let us begin with the scenario of a production having normality, e.g. the in-control distribution to be the well-known standard Normal distribution with parameters 0 and 1. W suppose that the out-of-control distribution is still normal distribution, but plausible changes in mean and/or standard deviation (equal to θ units and δ units respectively) are present.

The initial segment of **Table 3** distinctly indicates that, assuming the process follows a normal distribution, the $NMCC_{r,k,a,b,c,d,i,j}-$ chart performs better than *Competitive scheme* 1 and *Competitive scheme* 3, with respect to the out-of-control *ARL* values, in all scenarios studied. For instance, we next focus on the case where the production mean has been changed θ=0.25 units. As it is readily concluded with the assistance of **Table 3**, the $NMCC_{r,k,a,b,c,d,i,j}-$ framework achieves an out-of-control ARL equal to 57.31, while the respective *ARL*-value for *Competitive scheme* 1 and *Competitive scheme* 3 are equal to 176.43 and 248.92 respectively.

Table 3. *ARL* values of the $NMCC_{r,k,a,b,c,d,i,j}-$ schemes versus competitors under normal distribution and Laplace distribution for given shifts θ, δ (m = 100).

		Normal distribution **(θ, 1+δ)**			***Laplace distribution*** **(θ, 1+δ)**		
θ	δ	$NMCC_{r,k,a,b,c,d,i,j}-$ ***chart***	***Competitive scheme* 1**	***Competitive scheme* 3**	$NMCC_{r,k,a,b,c,d,i,j}-$***chart***	***Competitive scheme* 1**	***Competitive scheme* 3**
0	0	490.90	475.84	458.07	490.90	475.84	458.07
0.25	0	57.31	176.43	248.92	84.16	263.59	374.63
0.5	0	6.78	45.77	81.88	18.2	108.07	257.35
1	0	4.02	6.30	10.00	9.38	13.82	84.08
0.25	0.05	28.57	124.01	160.15	44.16	192.43	268.20
0.5	0.05	6.17	37.91	59.08	23.44	84.65	187.85
1	0.05	4.02	6.23	8.81	8.71	12.68	65.12
0.25	0.10	17.32	91.21	109.40	37.55	145.26	198.46
0.5	0.10	5.73	32.11	44.65	14.92	68.14	141.62
1	0.10	4.02	6.17	7.90	8.09	11.76	51.92
0.25	0.15	12.16	69.60	78.44	29.04	112.82	151.09
0.5	0.15	5.40	27.67	35.01	9.52	56.10	109.75
1	0.15	4.02	6.10	7.19	7.37	10.99	42.42
0.25	0.20	9.44	54.74	58.51	14.40	89.79	117.88
0.5	0.20	5.15	24.19	28.27	10.29	47.09	87.09
1	0.20	4.02	6.04	6.61	7.07	10.36	35.39

On the other hand, at the right part of **Table 3**, the in-control production distribution is assumed to be Laplace distribution with parameters 0 and 1, while the changes we aim to detect are induced by a modification in either in the mean or the scale parameter. All competitors are built up such that an in-control *ARL* in the region of 500 is attained. For providing areas comparison between the suggested framework and the ones introduced by Triantafyllou and Panayiotou (2020) and Panayiotou and Triantafyllou (2023) (*Competitive scheme* 1 and *Competitive scheme* 3 respectively), we used a design given by the authors themselves.

As it is readily deduced, the $NMCC_{r,k,a,b,c,d,i,j}-$ framework is, under Laplace distribution, better than its competitors in all scenarios considered. As an example, we assume that the production mean of the in-control distribution has been changed 0.5 units and the corresponding scale parameter has also shifted 0.05 units, $NMCC_{r,k,a,b,c,d,i,j}-$ monitoring scheme achieves (**Table 3**) an out-of-control *ARL* equal to 23.44, while the respective *ARL*-values for *Competitive scheme* 1 and *Competitive scheme* 3 are worse and more specifically equal to 84.65 and 187.85 respectively.

5. An Illustration Example of the New Monitoring Framework in Supply Chain Management Environment

We next implement the suggested monitoring scheme to a real-life application. Generally speaking, supply chain management oversees and controls all the processes involved in transforming raw materials into finished products, which are subsequently sold to end-users. Its primary objective is to supervise the planning, design, manufacturing, inventory management, and distribution stages necessary for the production and sale of a company's products. One of the key aims of supply chain management is to enhance efficiency by synchronizing the activities of the different entities within the supply chain. This can ultimately improve the quality of the products, potentially leading to increased sales. Recent progression on the field is presented in the works provided by Kafeel et al. (2023), Gupta et al. (2023), Das et al. (2021), Mishra et al. (2023).

Let us next consider a supply chain management example involving a company that produces and sells smartphones. The particular application is related to an international company, which engages in production, transportation and exports of mobile phones. The available data refer to a production period 15 years ago. The main steps of the process are given briefly as follows:

***Stage* 1.** Raw material sourcing: The supply chain starts with the procurement of raw materials, such as metals, plastics, and electronic components.
***Stage* 2.** Manufacturing: The raw materials are transported to the manufacturing facility where they are assembled into smartphones.
***Stage* 3.** Distribution: Once the smartphones are manufactured, they are distributed to warehouses located in different regions.
***Stage* 4.** Retailers: Retailers place orders based on demand forecasts.
***Stage* 5.** Customer sales: Customers purchase smartphones from retailers or through online channels.
***Stage* 6.** Reverse logistics: The supply chain also includes a system for handling returns and defective products.
***Stage* 7.** Information flow: Throughout this process, there is a constant flow of information facilitated by technology. This includes real-time tracking of inventory, demand forecasting, and communication between suppliers, manufacturers, distributors, and retailers.

Efficient supply chain management in this example involves optimizing each stage to minimize costs, reduce lead times, and improve overall responsiveness to changes in demand. We shall next investigate how the proposed monitoring scheme can contribute to the above-mentioned supply chain management scheme. Generally speaking, the supply chain management environment refers to the complex network of activities, resources, technologies, and stakeholders involved in the production and distribution of goods and services from suppliers to end customers. The proposed monitoring framework aims at enhancing the logistics and transportation part of the procedure, but also it is expected to result in some improvements concerning the manufacturing and production facilities.

As it concerns the aforementioned case study, we are aware that there are 3 eight-hour daily work shifts at the factory. **Table 4** displays the average number of devices (per work shift) whose assembly is completed. The reporting time period corresponds to 160 working days. These data refer actually to the manufacturing stage, e.g. Stage 2.

Table 4. The average number of devices (per work shift) whose assembly is completed for 160 consecutive days.

Day	Average number of devices (per work shift)	Day	Average number of devices (per work shift)	Day	Average number of devices (per work shift)	Day	Average number of devices (per work shift)
1	109.1	41	94.4	81	102.2	121	90.2
2	90.2	42	102.2	82	107.3	122	93.3
3	102.2	43	94.4	83	103.6	123	103.3
4	107.2	44	101.1	84	80.4	124	107.9
5	89.2	45	100.7	85	70.9	125	111.2
6	99.7	46	95.6	86	78.9	126	82.2
7	108.4	47	94.4	87	83.5	127	109.5
8	110.2	48	102.2	88	100.2	128	110.1
9	90.7	49	99.9	89	92.2	129	80.9
10	102.2	50	95.5	90	95.5	130	90.9
11	107.2	51	94.4	91	105.5	131	100.8
12	90.2	52	104.4	92	103.2	132	110.9
13	96.5	53	108.8	93	109.1	133	105.6
14	110.1	54	101.2	94	100.2	134	95.6
15	108.8	55	80.2	95	105.5	135	85.6
16	109.9	56	85.4	96	108.3	136	95.2
17	108.3	57	90.2	97	102.7	137	88.8
18	107.7	58	103.5	98	108.1	138	98.7
19	109.1	59	104.5	99	92.9	139	89.1
20	109.1	60	107.7	100	91.1	140	102.2
21	108.8	61	80.9	101	101.2	141	109.3
22	98.6	62	90.9	102	103.8	142	111.2
23	99.2	63	87.5	103	108.8	143	90.2
24	100.2	64	81.1	104	111.2	144	104.4
25	94.4	65	92.2	105	104.4	145	99.9
26	90.2	66	94.4	106	109.4	146	100.2
27	106.2	67	88.8	107	89.7	147	88.2
28	102.2	68	101.1	108	93.9	148	78.9
29	98.8	69	102.1	109	106.4	149	95.2
30	107.2	70	100.9	110	105.2	150	105.7
31	108.1	71	80.2	111	112.2	151	112.4
32	107.2	72	78.7	112	113.9	152	105.2
33	94.7	73	85.5	113	107.5	153	99.9
34	94.4	74	90.9	114	80.5	154	103.2
35	94.4	75	100.0	115	90.2	155	90.1
36	104.5	76	109.8	116	89.6	156	78.9
37	97.7	77	102.1	117	85.2	157	95.2
38	107.2	78	101.8	118	80.6	158	105.7
39	107.9	79	103.9	119	78.4	159	112.4
40	107.2	80	104.2	120	84.1	160	105.2

We know that, during the first 48 days of the aforementioned period, the assembly process was operating satisfactorily and in compliance with factory specifications. However, from Day 49 onwards, some changes in the workforce of the factory caused a disruption in the production process, and the quality manager observed that the assembly process no longer meets his requirements. Based on these facts, data reported in Days 1 to 48 can be viewed as $m = 48$ reference data.

We tend to apply the proposed monitoring scheme in order to figure out whether the disruption that has occurred from Day 49 onwards can be detected by the control chart or not. We first determine the design parameters of the suggested monitoring framework such as its in-control *ARL* to be in the region of 500. Indeed, if we specify the values of the remaining parameters as given below
$a = 3, b = 7, c = 27, d = 31, n = 8, i = 2, j = 5, r_1 = 1, r_2 = 2, r = 2, k = 3$

the resulting scheme, which includes four different control charts, attains an in-control *ARL*-value equal to 499.36. The abovementioned design indicates practically the following:

- $m = 48$ reference observations are drawn when the production is said to be in-control.
- the ordered observations at the 3rd, 7th, 27th and 31st position of the reference, e.g. the observed values $X_{3:48} = 91.1, X_{7:48} = 94.3, X_{27:48} = 102.6, X_{31:48} = 103.7$ are utilized as control limits for the schemes with plotting statistics $Y_{i:n}^h, Y_{j:n}^h$ (see also (1)).
- $n = 8$ test observations are collected from the production.
- the 2nd and the 5th ordered observation (e.g. $Y_{2:8}^h, Y_{5:8}^h$) of each test sample is detected ($h = 1,2, \dots ,20$).
- the observed values of all monitoring statistics are computed for the test h −th sample ($h = 1,2, \dots ,20$), namely the values of $Y_{2:8}^h, Y_{5:8}^h, R_1^h, R_2^h$ are calculated for all available test data.

Afterwards, we construct four monitoring schemes, which compose the suggested framework and the results are displayed at **Figures 1-4**.

Under the proposed framework, the h −th test sample does not produce a bad signal, if the conditions hold true
$X_{3:48} \leq Y_{2:8}^h \leq X_{7:48}, X_{27:48} \leq Y_{5:8}^h \leq X_{31:48}, R_1^h \geq r_1$ and $R_2^h \geq r_2$

or equivalently

$$91.1 \leq Y_{2:8}^h \leq 94.3,\ \ 102.6 \leq Y_{5:8}^h \leq 103.7, R_1^h \geq 1 \text{ and } R_2^h \geq 2 \qquad (12)$$

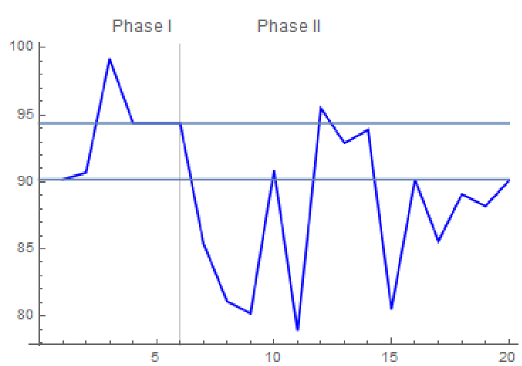

Figure 1. The monitoring statistic $Y_{2:8}^h$ for device assembly data.

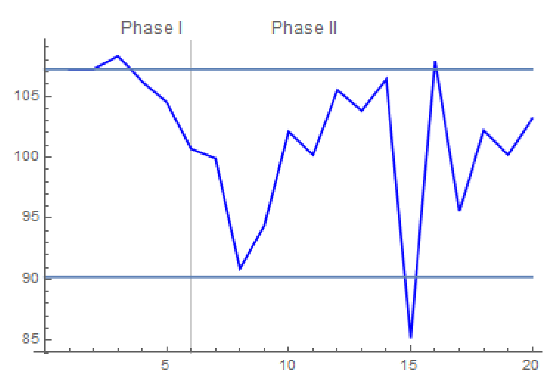

Figure 2. The monitoring statistic $Y_{5:8}^h$ for the device assembly data.

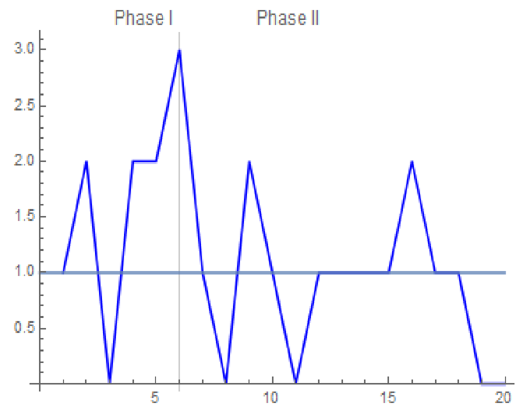

Figure 3. The monitoring statistic R_1^h for the device assembly data.

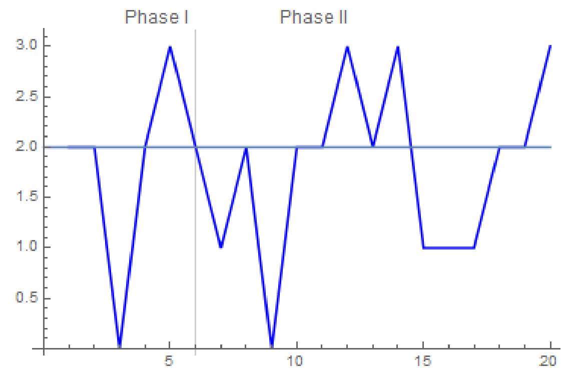

Figure 4. The monitoring statistic R_2^h for the device assembly data.

Figure 1 is related to the first mentioned condition provided in Equation (12), while **Figure 2** illustrates the statistic $Y_{5:8}^h$. In simpler words, **Figure 1** displays the 2nd ordered observation of each test sample, which is drawn from the process. It is easily observed that there exist plenty test samples that exceed the control limits of the particular chart, namely there are a few test samples, where the 2nd ordered test observation is greater than the upper control limit or less than the lower control limit of the chart.

In a similar way, **Figure 2** depicts the 5th ordered observation of each test sample. It is readily deduced that there exist three test samples that exceed the control limits of the particular chart. In addition, **Figure 3** displays the third restriction, while **Figure 4** is connected to monitoring statistic R_2^h. In both figures, it seems that in many cases the random variables R_1^h, R_2^h take on values out of limits. This is evident since the sampling values of the aforementioned monitoring statistics are greater than 1 and 2 respectively, e.g. exceed the control limit of the corresponding chart.

Thereof, under the proposed monitoring scheme the production is described as out-of-control once we detect $r = 2$ subsequences of $k = 3$ bad signals, namely once we detect $r = 2$ subsequences of test samples of length $k = 3$, wherein at least one condition stated in Equation (12) has been violated. As it is readily observed, while the Phase I samples (Days 1 to 48) are not producing an out-of-control event (as predicted since we know that this period was a good one), the new nonparametric scheme gives an alarm in the prospective phase (Days 49 onwards). More precisely, we scrutinize that an out-of-control signal appears upon the 6th test sample (or 11th overall) since at that time 2 subsequences of 3 bad signals have already been reported. Based on the above results, the quality manager may deduce that the underlying changes in the workforce of the factory really caused a disruption in the production process and the assembly process no longer meets his requirements. It is noticeable that the proposed scheme, which has been applied for monitoring the process, detects promptly the undesirable alteration of the process status.

In addition, the proposed framework can be proved useful at Stage 3 of the aforementioned supply chain management scenario. More precisely, our interest focuses now on the days spent till the devices arrive at their ultimate destination, namely at warehouses located in different regions. For this reason, the company collaborates with two carriers A and B. We know that carrier A is a reliable one and its collaboration with the company has always been harmonic. On the other hand, the collaboration with carrier B has just started and the manager needs to supervise the distribution procedure in order to verify whether the time required for the devices to reach their destination (by the aid of carrier B) is comparable with the one reported for carrier A.

Table 5 displays time (in hours) required for the device batches to reach their destination. The available data correspond to 112 product batches, which have been drawn from the process. These data refer actually to the distribution stage, e.g. are related to Stage 3. Note that batches 1 to 35 have been transferred with carrier A, while the remaining ones, e.g. batches 36 to 112 have been transported with carrier B.

Table 5. Time (in hours) required for 112 device batches to reach the warehouses.

Device batch	Time (in hours)	Device batch	Time (in hours)	Device batch	Time (in hours)	Device batch	Time (in hours)
1	40.5	29	40.5	57	25.5	85	30.2
2	43.2	30	70.2	58	79.2	86	40.9
3	75.0	31	65.6	59	72.2	87	93.3
4	92.9	32	50.5	60	63.5	88	77.5
5	58.2	33	70.2	61	65.4	89	71.2
6	60.2	34	70.2	62	69.9	90	49.6
7	87.6	35	5.4	63	55.2	91	60.1
8	70.2	36	22.3	64	90.9	92	55.5
9	95.2	37	47.0	65	81.5	93	90.2
10	90.5	38	49.2	66	77.7	94	70.3
11	15.0	39	55.4	67	66.4	95	73.3
12	40.5	40	66.8	68	54.3	96	75.6
13	50.5	41	79.9	69	44.4	97	56.3
14	68.8	42	83.3	70	31.1	98	40.1
15	54.4	43	50.0	71	33.3	99	88.2
16	18.6	44	40.0	72	43.4	100	72.2
17	74.3	45	30.4	73	45.5	101	43.4
18	62.2	46	20.5	74	92.2	102	47.8
19	40.5	47	71.2	75	77.2	103	49.2
20	54.5	48	77.7	76	70.9	104	62.3
21	63.7	49	78.2	77	69.6	105	67.1
22	10.0	50	33.3	78	60.6	106	60.2
23	47.5	51	47.2	79	50.1	107	70.9
24	56.2	52	50.1	80	40.1	108	65.3
25	120.0	53	62.2	81	30.9	109	55.1
26	78.4	54	71.5	82	72.2	110	30.1
27	54.4	55	71.1	83	70.1	111	20.9
28	80.0	56	82.5	84	90.0	112	100.5

Since the first 35 batches have been transferred by carrier A, which is supposed to be reliable and trusty, we face these data as a reference sample of size $m = 35$. We next apply the suggested framework for clarifying whether the fact that we changed the carrier has affected time required for the device batches to reach their destination or not. In other words, we shall encounter batches 36 to 112 as test data and our goal shall be the timely and accurate identification of possible undesired changes in the process.

We first determine the design parameters of the suggested framework such as its in-control *ARL* to be in the region of 500. Indeed, if we specify the values of the remaining parameters as given below,
$a = 8, b = 12, c = 26, d = 29, n = 7, i = 2, j = 5, r_1 = 2, r_2 = 1, r = 2, k = 2$

The resulting scheme, which includes four different control charts based on statistics, attains an in-control *ARL*-value equal to 490.07. The abovementioned design indicates practically the following:

- $m = 35$ reference observations are drawn when the process is said to be in-control.

- The ordered observations at the 8th, 12th, 26th and 29th position of the reference, e.g. the observed values $X_{8:35} = 40.5, X_{12:35} = 50.5, X_{26:35} = 70.2, X_{29:35} = 78.4$ are utilized as control limits for the schemes with plotting statistics $Y_{i:n}^h, Y_{j:n}^h$ (see also Equation (1)).
- $n = 7$ test observations are collected from the production.
- The 2nd and the 5th ordered observation (e.g. $Y_{2:7}^h, Y_{5:7}^h$) of each test sample is detected ($h = 1,2, \dots ,16$)
- The observed values of all monitoring statistics are computed for the test h −th sample ($h = 1,2, \dots ,16$), namely the values of $Y_{2:7}^h, Y_{5:7}^h, R_1^h, R_2^h$ are calculated for all available test data.

Afterwards, we construct four schemes, which compose the suggested monitoring framework and the results are displayed in **Figures 5-8**.

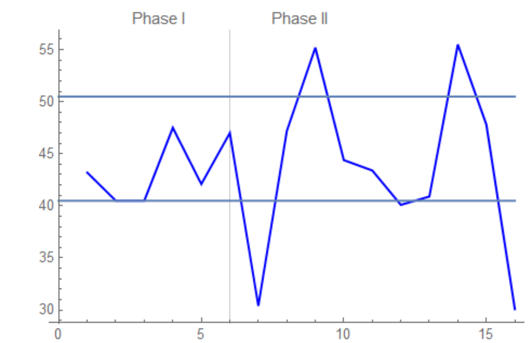

Figure 5. The monitoring statistic $Y_{2:7}^h$ for device assembly data.

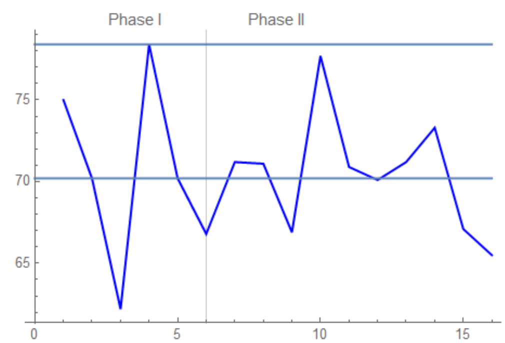

Figure 6. The monitoring statistic $Y_{4:7}^h$ for the device assembly data.

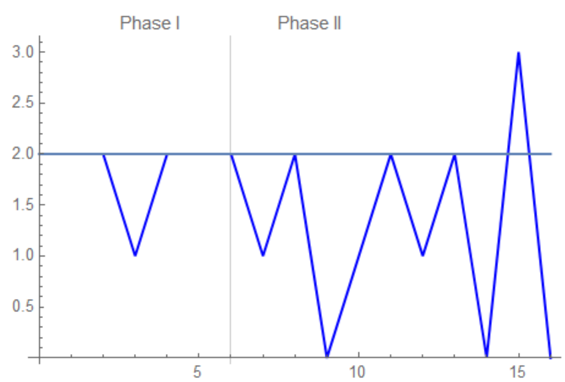

Figure 7. The monitoring statistic R_1^h for the device assembly data.

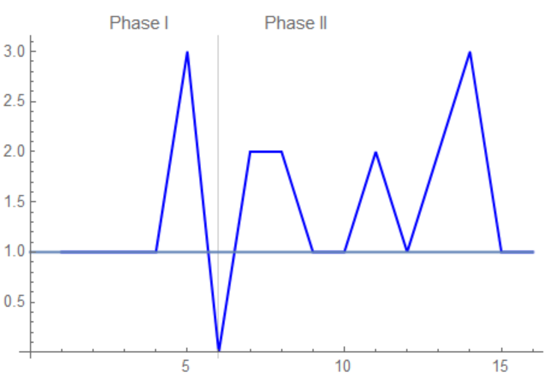

Figure 8. The monitoring statistic R_2^h for the device assembly data.

Under the proposed framework, the h −th test sample does not produce a bad signal, if the conditions hold true
$X_{8:35} \leq Y_{2:7}^h \leq X_{12:35}, X_{26:35} \leq Y_{5:7}^h \leq X_{29:35}$, $R_1^h \geq r_1$ and $R_2^h \geq r_2$

or equivalently
$$40.5 \leq Y_{2:7}^h \leq 50.5,\ 70.2 \leq Y_{5:7}^h \leq 78.4,\ R_1^h \geq 2 \text{ and } R_2^h \geq 1 \quad (13)$$

Under the proposed monitoring scheme the process is described as out-of-control once we detect $r = 2$ subsequences of $k = 2$ bad signals, namely once we detect $r = 2$ subsequences of test samples of length $k = 2$, wherein at least one condition stated in Equation (13) has been violated.

As it is readily observed, while the Phase I samples (batches 1 to 35) are not producing an out-of-control event (as predicted since we know that transferring procedure was reliable during this period), the suggested framework gives signal in the prospective phase (batches 36 onwards).

More precisely, we see that an out-of-control signal appears upon the 5th test sample (or 10th overall) since at that time 2 subsequences of 2 bad signals have just been reported. The above results provide some evidence to the quality manager that the time required for the devices to reach their destination does not meet, by the aid of carrier B, the requirements stated by the company. It is noticeable that the proposed scheme, which has been applied for monitoring the transporting procedure, detects promptly once again the undesirable alteration of the process status.

Figure 5 is related to the first mentioned condition provided in Equation (13), while **Figure 6** displays the statistic $Y_{5:7}^{h}$. In simpler words, **Figure 5** displays the 2nd ordered observation of each test sample, which is drawn from the process. It is easily observed that there exist four test samples that exceed the control limits of the particular chart, namely there are four test samples, where the 2nd ordered test observation is greater than the upper control limit or less than the lower control limit of the chart.

In a similar way, **Figure 6** depicts the 5th ordered observation of each test sample. It is readily deduced that there exist some test samples that exceed the control limits of the particular chart. In addition, **Figure 7** displays the third restriction, while **Figure 8** is connected to monitoring statistic R_2^h. In both figures, it seems that in many cases the random variables R_1^h, R_2^h take on values out of limits. This is evident since the sampling values of the aforementioned monitoring statistics are greater than 2 and 1 respectively, e.g. exceed the control limit of the corresponding chart.

6. Discussion

The present article provides a nonparametric monitoring framework, wherein multiple runs are utilized for the first time in the relative literature. The implemented runs-type rules are combined with a distribution-free monitoring statistic and the results seem to be encouraging. In addition, a nice real life application in the field of supply chain management is also illustrated. More precisely, a new family of nonparametric Shewhart-type framework which relies on order statistics and multiple runs is proposed. The plotting statistics coincide with appropriately chosen order statistics of the reference data, while the decision whether the production remains at an in-control state, or it has shifted to an out-of- control situation is reached by employing additional multiple runs rules.

7. Conclusion

The run length of the suggested distribution-free framework is studied for both in control and out-of- control scenarios. In accordance with the numerical investigation conducted, we deduce that the suggested framework can detect quite fast plausible changes of the production. The practitioner can use the designs provided in the tables of the current work for constructing the most suitable monitoring scheme for the underlying application. For instance, the proposed monitoring scheme performs well under both location and scale parameters shifts. This is readily observed, by looking at the numerical investigation provided in section 3 of the present manuscript. In addition, on a more theoretical basis, we may conclude that the implementation of sophisticated runs-type rules has proved to be efficient in providing a capable monitoring framework under several distributional assumptions. Indeed, the proposed methodology uses multiple runs along with order statistics and the final result seems attain the desired level of in- and out-of-control performance.

Moreover, the numerical investigation, which has been carried out, reveals that the proposed performs better than its competitors. More precisely, the out-of-control behavior of the new class of distribution-free control charts has been compared to the corresponding performance of existing charts based on order statistics. The results confirm that the proposed monitoring scheme detects faster both smaller and larger distribution shifts than the competitive charts under different process distribution.

Throughout the lines of the paper, it is highlighted how the proposed control charts can be proved helpful in the supply chain management environment. Indeed, the new nonparametric monitoring scheme was proved to be quite helpful in several stages of the aforementioned managerial application, such as the underlying logistics or the pure industrial part of the project.

It sounds quite intriguing for forthcoming work, to employ scans- or runs-type rules to other distribution-free schemes for enhancing their ability to detect possible changes in the underlying distribution or alternatively to modify existing distribution-free control charts by adding more advanced runs- or scans-type rules.

Conflict of Interest

The authors declare that there is no conflict of interest for this publication.

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