International Journal of Mathematical, Engineering and Management Sciences
Vol. 8, No. 4, 672-687, 2023
https://doi.org/10.33889/IJMEMS.2023.8.4.039

Approximation of the Time-Fractional Klein-Gordon Equation using the Integral and Projected Differential Transform Methods

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(Received on January 07, 2023; Accepted on April 17, 2023)

Abstract
In the present investigation, a new integral transform method (NITM) and the projected differential transform method (PDTM) are used to give an analytical solution to the time-fractional Klein-Gordon (TFKG) equation. The time-fractional derivative is used in the Caputo sense. The huge advantage of the suggested approach is the ease with which the nonlinear term can be effortlessly treated by projected differential transform without using Adomian's and He's polynomials. The solution of fractional partial differential equations using the aforementioned method is very simple and straightforward. The efficiency and accuracy of the proposed method are demonstrated by three examples, and the effects of various fractional Brownian motions are demonstrated graphically.

Keywords- TFKG equation, Caputo time-derivative, NITM, Projected DTM.

1. Introduction
Due to its substantial potential applications in several subjects, including physics, mathematics, chemistry, biology, fluid dynamics, and nonlinear optics, fractional calculus has gained more recognition in various scientific and engineering fields (Miller and Ross, 1993; Baleanu et al., 2012). Since fractional differential equations (FDEs) are extensively utilized to describe complex phenomena. Thus, it is noticed that derivatives of non-integer orders are particularly successful in illuminating a variety of natural phenomena, including damping laws, rheology, diffusion process, etc. There is an instant turning in interest from scientists and engineers to analyze fractional calculus in quite a few areas of mathematical biology, fluid mechanics, electrochemistry, etc. (Oldham and Spanier, 1974; Carpinteri and Mainardi, 1997; Podlubny, 1999; Tarasov, 2010). Gejji and Jafari (2005) used the Adomian decomposition method to solve a system of FDEs. He (1998) used Hadamard product and vector extraction operators to obtain analytical solutions for Caputo FDEs. Fractional PDEs are a modern tool in calculus that can be utilized for the simulation of wide-ranging problems (Ara et al., 2018; Owolabi et al., 2020).

In addition, nonlinear differential equations with a variety of fractional derivative operators, such as Caputo, Hilfer, Riemann-Liouville, Caputo-Fabrizio, Antangana-Baleanu in the sense of Caputo, etc. play a vital role in solving real-world problems. Moreover, Nisar et al. (2021) analyze the mathematical SIRD model of COVID-19 with Caputo fractional derivative based on real data. Due to the complexity of fractional PDEs/ODEs, which ordinary operators cannot handle in order to achieve explicit solutions, these fractional operators are highly helpful in fractional calculus, for instance (Srivastava et al., 2013a, 2013b; Shukla et al., 2014; Salahshour et al., 2015; Agarwal and El-Sayed, 2018; Chaurasia et al., 2018; El-Sayed and Agarwal, 2019; Sheikh et al., 2021).

The fractional Klien-Gordan equation is one of the important wave equations in mathematical physics. Numerous investigations have been done on this problem e.g., Golmankhaneh and Baleanu (2011) used a

Homotopy perturbation method to find exact solutions of the nonlinear fractional K-G equation. Also, Saelao and Yokchoo (2020) used the Adomian decomposition method to solve the K-G equation whereas Kumar and Baleanu (2021) used the homotopy analysis transform method with the fractional-derivative of the Mittag-Leffler type of kernel. In Saifullah et al (2022), the general series solution of the nonlinear time-fractional Klien Gordan equation with power law kernel is established by the composition of double Laplace transform with the decomposition method. The fractional Klein–Gordon equation under the fractal fractional operator with the Riemann–Liouville and with the Mittag-Leffler kernel has also been studied numerically by Partohaghighi et al. (2022). Mohammadizadeh et al. (2021) extended the Chebyshev spectral collocation method while Kurulay (2012) and Gepreel and Mohamad (2013) utilized the homotopy analysis method for constructing an approximation of the fractional K-G equation. Khan et al. (2019) developed a numerical technique with the advantage of the Sumudu decomposition method to solve the Caputo fractional K-G equation. Singh et al. (2020) presented a computational technique by combining of collocation method with orthogonal polynomial matrices while Bansu and Kumar (2021) presented a novel collocation method to solve the space-time fractional K-G equation. Khader and Adel (2016) used variational iteration methods with fractional complex transform to solve the fractional K-G equation.

Recently, Liu et al. (2022) developed an approach “Yang homotopy perturbation transform method” to solve the TFKG equations while Karaagac (2019) adopted a numerical method based on the Adams-Bashforth method utilizing the Atangana-Baleanu fractional-derivative. Tamsir and Srivastava (2016a, 2016b) presented the Fractional reduced differential transform method (FRDTM) to solve the TFKG equation and fractional-order gas dynamics equation. Amin et al. (2020) and Ganji et al. (2021) developed new techniques using extended cubic B-spline and clique polynomial functions as basis functions to find the numerical solutions of the TFKG equation. We consider the TFKG equation

$$\frac{\partial^\alpha v}{\partial \tau^\alpha} = \frac{\partial^2 v}{\partial \xi^2} + \beta_1 v + \beta_2 v^2 + \beta_3 v^3, \xi \in R \tag{1}$$

With the initial condition

$$v(\xi, 0) = \varphi(\xi), \xi \in R \tag{2}$$

where, β_1, β_2, β_3 are real constants. For $\alpha = 1$, Equation (1) eases to a classical nonlinear K-G equation.

Recently, Shah et al. (2018) presented a method by combining integral and PDT methods to solve the time fractional gas dynamics equation. In this work, we present the aforementioned method to solve the TFKG equation.

The rest of the paper is organized as follows: Some fundamental definitions of the theory of fractional calculus are highlighted in Section 2. The method's process is described in section 3. Section 4 presents the outcomes of the proposed method. Finally, in section 5 the conclusion of our study is presented.

2. Preliminaries

This section looks over some basic definitions.

Definition 2.1 A real-valued $g(\tau) \in \mathbb{R}$, $t > 0$ is in $C_{\widetilde{\omega}}$, $\widetilde{\omega} \in \mathbb{R}$ if $\exists$ a real number $\hat{s}\ (> \widetilde{\omega})$ such that $g(\tau) = t^{\hat{s}} \hat{g}(\tau)$, where$\hat{g}(\tau) \in C[0, \infty)$, and is in the space $C_{\widetilde{\omega}}^{\hat{l}}$ if $g^{(\hat{l})}(\tau) \in C_{\widetilde{\omega}}$, $\hat{l} \in \mathbb{N}$.

Definition 2.2 The Caputo integral of the function $g(\tau) \in \mathbb{R}$ of order $\alpha \geq 0$ is defined by Podlubny (1999).

$$\begin{cases} {}^cJ^{\alpha}g(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\chi)^{\alpha-1}g(\chi)d\chi, \alpha>0, \tau>0, \\ {}^cJ^{\alpha}g(\tau) = g(\tau). \end{cases} \quad (3)$$

Definition 2.3 The Caputo derivative of $g(\tau) \in \mathbb{R}$ is defined by Podlubny (1999).

$${}^cD_t^{\alpha}g(\tau) = {}^cJ_{\tau}^{\hat{l}-\alpha}D_{\tau}^{\hat{l}}g(\tau) = \frac{1}{\Gamma(\hat{l}-\alpha)}\int_0^{\tau}(\tau-\chi)^{\hat{l}-\alpha-1}g^{(\hat{l})}(\chi)d\chi \quad (4)$$

For $\hat{l} < \alpha \leq \hat{l}+1$, $t>0$, $g \in C_{\widetilde{\omega}}^{\hat{l}}$, $\widetilde{\omega} \geq$ -1, $\hat{l} \in \mathbb{N}$.

Lemma 2.1 If $\hat{l} < \alpha \leq \hat{l}+1$, $\hat{l} \in \mathbb{N}$ and $g \in C_{\widetilde{\omega}}^{\hat{l}}$, $\widetilde{\omega} \geq$ -1, then

$$\begin{cases} {}^cD_{\tau}^{\alpha}\ {}^cJ_{\tau}^{\alpha}g(\tau) = g(\tau), \tau>0, \\ {}^cJ_{\tau}^{\alpha}\ {}^cD_{\tau}^{\alpha}g(\tau) = g(\tau) - \sum_{k=0}^{\hat{l}} f^{(k)}(0^+)\frac{\tau^k}{k!},\ \tau>0. \end{cases} \quad (5)$$

3. New Integral and Projected Differential Transforms

The new integral transform is given by Kasuri and Fundu (2013).

$$K[v(\tau)] = \frac{1}{\omega}\int_0^{\infty} e^{-\frac{\tau}{\omega^2}}v(\tau)d\tau, \qquad \tau \geq 0, -k_1 < \omega < k_2. \quad (6)$$

where, k_1, k_2 may be finite or infinite.

Theorem 1. The new integral transform of $\frac{\partial^m v}{\partial \tau^m}$ is given by

$$K\left[\frac{\partial^m v}{\partial \tau^m}\right] = \frac{K[v(\xi,\tau)]}{\omega^{2m}} - \sum_{j=0}^{m-1}\frac{1}{\omega^{2(m-j)-1}}\frac{\partial^j}{\partial \tau^j}v(\xi,0) \text{ for } m \geq 1 \quad (7)$$

Remark 1. The new integral transform of the Caputo fractional derivative $\frac{\partial^{m\alpha} v}{\partial \tau^{m\alpha}}$ is given by

$$K\left[\frac{\partial^{m\alpha} v}{\partial \tau^{m\alpha}}\right] = \frac{K[v(\xi,\tau)]}{\omega^{2m\alpha}} - \sum_{j=0}^{m-1}\frac{1}{\omega^{2(m\alpha-j)-1}}\frac{\partial^j}{\partial \tau^j}v(\xi,0), \text{ where } \alpha>0, m-1<\alpha \leq m \in \mathbb{N} \quad (8)$$

Definition 1. The basic definition of projected DTM of $g(\xi_1,\xi_2,\xi_3,\cdots,\xi_m)$ is given as

$$g(\xi_1,\xi_2,\xi_3,\cdots,\xi_m) = \frac{1}{\eta!}\left[\frac{\partial^{\eta}(\xi_1,\xi_2,\xi_3,\cdots,\xi_m)}{\partial \xi_m^{\eta}}\right]_{\xi_m=0},$$

such that $g(\xi_1,\xi_2,\xi_3,\cdots,\xi_{m-1},\eta)$ is the projected DTM of $g(\xi_1,\xi_2,\xi_3,\cdots,\xi_m)$, and differential inverse transform of $g(\xi_1,\xi_2,\xi_3,\cdots,\xi_{m-1},\eta)$ is

$$g(\xi_1,\xi_2,\xi_3,\cdots,\xi_m) = \sum_{\eta=0}^{\infty} g(\xi_1,\xi_2,\xi_3,\cdots,\xi_{m-1},\eta)(\xi-\xi_0)^{\eta}.$$

3.1 Basic Idea of the Proposed Method

Now, we consider the following time fractional PDE,

$$\frac{\partial^{\alpha} v}{\partial \tau^{\alpha}} + \chi_1(\xi)\frac{\partial^m v}{\partial \xi^m} + \chi_2(\xi)v\frac{\partial v}{\partial \xi} + \chi_3(\xi)v + \chi_4(\xi)v^n = g(\xi,\tau), \xi \in R, \tau \geq 0, 0<\alpha \leq 1 \quad (9)$$

with the initial solution $v(\xi,0) = \varphi(\xi)$.

Now applying the new integral transform method, we have

$$\frac{K[v(\xi,\tau)]}{\omega^{2\alpha}} - \sum_{j=0}^{\alpha-1}\frac{1}{\omega^{2(\alpha-j)-1}}\frac{\partial^j}{\partial \tau^j}v(\xi,0) = K\left[-\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m} - \chi_2(\xi)v\frac{\partial v}{\partial \xi} - \chi_3(\xi)v - \chi_4(\xi)v^n + g(\xi,\tau)\right],$$

To simplify it, we have

$$K[v(\xi,\tau)] = \sum_{j=0}^{\alpha-1} \frac{1}{\omega^{-2j-1}} \frac{\partial^j}{\partial \tau^j} v(\xi,0) + \omega^{2\alpha} K[g(\xi,\tau)] - \omega^{2\alpha} K\left[-\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m} - \chi_2(\xi) v \frac{\partial v}{\partial \xi} - \chi_3(\xi) v - \chi_4(\xi) v^n\right] \tag{10}$$

Now, using the inverse of NITM, we have

$$v(\xi,\tau) = H(\xi,\tau) + K^{-1}\left\{\omega^{2\alpha} K\left[-\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m} - \chi_2(\xi) v \frac{\partial v}{\partial \xi} - \chi_3(\xi) v - \chi_4(\xi) v^n\right]\right\} \tag{11}$$

where, $H(\xi,\tau)$ signifies the term arises from the source term and given initial condition. Now applying projected DTM, we get,

$$v(\xi,\eta+1) = K^{-1}\{\omega^{2\alpha} K[P_\eta + Q_\eta + R_\eta + S_\eta]\} \text{ and } H(\xi,\tau) = v(\xi,0) = \varphi(\xi) \tag{12}$$

where $P_\eta = -\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m}$, $Q_\eta = -\chi_2(\xi)\sum_{i=0}^{\eta} v(\xi,i)\frac{\partial v(\xi,\eta-i)}{\partial \xi}$, $R_\eta = -\chi_3(\xi) v(\xi,\eta)$,
and
$S_\eta = -\chi_4(\xi)\sum_{i_{m-1}}^{\eta} \cdots \sum_{i_2}^{i_3}\sum_{i_1}^{i_2} v(\xi,i_1)v(\xi,i_2-i_1)v(\xi,i_3-i_2)\cdots v(\xi,\eta-i_{m-1})$ are projected DTM of $-\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m}$, $-\chi_2(\xi) v\frac{\partial v}{\partial \xi}$, $-\chi_3(\xi) v$, and $-\chi_4(\xi) v^n$, respectively.

Hence, the approximate solution of TFKG equation (1) is given by

$$v(\xi,\tau) = v(\xi,0) + v(\xi,1) + v(\xi,2) + v(\xi,3) + \ldots = \sum_{\tau=0}^{\infty} v(\xi,\tau) \tag{13}$$

where,

$$v(\xi,1) = K^{-1}\{\omega^{2\alpha} K[P_0 + Q_0 + R_0 + S_0]\},$$
$$v(\xi,2) = K^{-1}\{\omega^{2\alpha} K[P_1 + Q_1 + R_1 + S_1]\},$$
$$v(\xi,3) = K^{-1}\{\omega^{2\alpha} K[P_2 + Q_2 + R_2 + S_2]\},$$
$$\vdots$$

4. Convergence Analysis

In this section, we demonstrate the existence of unique solution in the theorem (4.1) and convergence in the theorem (4.2) of the NITM.

Theorem 4.1 The solution derived with the aid of PDTM of the equation (9) is unique, whenever $0 < (L_1 + L_2)\frac{\tau^\alpha}{\Gamma(\alpha+1)} < 1$.

Proof: Consider Xbe the Banach space of all continuous functions on $I = [0,T]$ with the norm $\|v(t)\| = max|v(t)|$.

Define a mapping $H: X \to X$, where,

$$v(\xi,\tau) = v(\xi,0) + K^{-1}\left\{\omega^{2\alpha} K\left[-\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m} - \chi_2(\xi) v \frac{\partial v}{\partial \xi} - \chi_3(\xi) v - \chi_4(\xi) v^n\right]\right\},$$

Now assume $|Fv - F\overline{v}| < L_1|v - \overline{v}|$ and $|Mv - M\overline{v}| < L_2|v - \overline{v}|$, where $F = -\chi_1(\xi)\frac{\partial^m v}{\partial \xi^m} - \chi_2(\xi) v\frac{\partial v}{\partial \xi}$, $M = -\chi_3(\xi) v - \chi_4(\xi) v^n$ and the function value v and $\overline{v}$ are distinct.

$$\|Fv - F\overline{v}\| = max|Fv - F\overline{v}|$$
$$= |v(\xi,0) + K^{-1}\{\omega^{2\alpha}K[Fv(\xi,\tau) + Mv(\xi,\tau)]\} - v(\xi,0) - K^{-1}\{\omega^{2\alpha}K[F\overline{v}(\xi,\tau) + M\overline{v}(\xi,\tau)]\}|$$
$$= |K^{-1}\{\omega^{2\alpha}K[Fv(\xi,\tau) - F\overline{v}(\xi,\tau)]\} + K^{-1}\{\omega^{2\alpha}K[Mv(\xi,\tau) - M\overline{v}(\xi,\tau)]\}|.$$

Let us consider $F(v)$ and $M(v)$ satisfy Lipschitz condition with constant L_1 and L_2.

$$\|Fv - F\overline{v}\| \le max[K^{-1}\{\omega^{2\alpha}K[|Fv(\xi,\tau) - F\overline{v}(\xi,\tau)| + |Mv(\xi,\tau) - M\overline{v}(\xi,\tau)|]\}]$$
$$\le max(L_1 + L_2)[K^{-1}\{\omega^{2\alpha}K[|v(\xi,\tau) - \overline{v}(\xi,\tau)|]\}]$$
$$\le (L_1 + L_2)[K^{-1}\{\omega^{2\alpha}K[\|v(\xi,\tau) - \overline{v}(\xi,\tau)\|]\}]$$
$$= (L_1 + L_2)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\|v(\xi,\tau) - \overline{v}(\xi,\tau)\|.$$

Hence equation (9) has unique solution.

Theorem 4.2 The solution of equation (6), converges if $0 < L < 1$ and $\|v_i\| < \infty$ where $L = (L_1 + L_2)\frac{\tau^\alpha}{\Gamma(\alpha+1)}$.

Proof. Let $v_n = \sum_{\tau=0}^{n} v(\xi,\tau)$ is the partial sum of the series. To prove that $\{v_n\}$ is a Cauchy sequence in Banach space X. Consider,

$$\|v_m - v_n\| = \max\left|\sum_{\tau=n+1}^{m} v(\xi,\tau)\right|, \qquad n = 1,2,3,\ldots$$
$$\le max\left|K^{-1}\left\{\omega^{2\alpha}K\left[\sum_{\tau=n+1}^{m}\left(F(v(\xi,\tau-1)) + M(v(\xi,\tau-1))\right)\right]\right\}\right|$$
$$= max\left|K^{-1}\left\{\omega^{2\alpha}K\left[\sum_{\tau=n}^{m-1}\left(F(v(\xi,\tau)) + M(v(\xi,\tau))\right)\right]\right\}\right|$$
$$\le max|K^{-1}\{\omega^{2\alpha}K[F(v_{m-1}) - F(v_{n-1})]\}| + max|K^{-1}\{\omega^{2\alpha}K[M(v_{m-1}) - M(v_{n-1})]\}|.$$

By Lipschitz condition

$$\|v_m - v_n\| \le L_1\left|K^{-1}\left\{\omega^{2\alpha}K\left[F(v_{m-1}) - F(v_{n-1})\right]\right\}\right| + L_2\left|K^{-1}\left\{\omega^{2\alpha}K\left[M(v_{m-1}) - M(v_{n-1})\right]\right\}\right|$$
$$\le (L_1 + L_2)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\|(v_{m-1}) - (v_{n-1})\|.$$

If $m = n + 1$

$$\|v_{n+1} - v_n\| \le L\|v_n - v_{n-1}\| \le L^2\|v_{n-1} - v_{n-2}\| \le \ldots \le L^n\|v_1 - v_0\|.$$

where, $L = (L_1 + L_2)\frac{\tau^\alpha}{\Gamma(\alpha+1)}$. In the similar way,

$$\|v_m - v_n\| \le \|v_{n+1} - v_n\| + \|v_{n+2} - v_{n+1}\| + \ldots + \|v_m - v_{m-1}\|$$
$$\le (L^n + L^{n+1} + \ldots + L^{m-1})\|v_m - v_{m-1}\|$$
$$\le L^n\left(\frac{1-L^{m-n}}{1-L}\right)\|v_1\|.$$

We see that, $1 - L^{m-n} < 1$as $0 < L < 1$. Thus,

$$\|v_m - v_n\| \le \frac{L^n}{1-L}\max\|v_1\|.$$

Since, $\|v_1\| < \infty$. So, $\|v_m - v_n\| \to 0$ as $n \to \infty$. Hence v_m is a Cauchy sequence in X. Therefore, the series is convergent.

5. Results and Discussions

This section involves three problems of the TFKG equation to check the accuracy of the method.

Example 1. Consider the linear TFKG equation

$$\frac{\partial^\alpha v}{\partial \tau^\alpha} - \frac{\partial^2 v}{\partial \xi^2} - v = 0, \tau \geq 0 \tag{14}$$

with the initial condition

$$u(\xi, 0) = 1 + \sin \xi \tag{15}$$

Using new integral transform in (14), we get

$$\frac{K[v(\xi,\tau)]}{\omega^{2\alpha}} - \sum_{j=0}^{\alpha-1} \frac{1}{\omega^{2(\alpha-j)-1}} \frac{\partial^j}{\partial \tau^j} v(\xi, 0) = K\left[\frac{\partial^2 v}{\partial \xi^2} + v\right].$$

Simplifying above equation, we get

$$K[v(\xi, \tau)] = \omega v(\xi, \tau) + \omega^{2\alpha} K\left[\frac{\partial^2 v}{\partial \xi^2} + v\right] \tag{16}$$

Taking the inverse of the new integral transform in above equation, we get

$$v(\xi, \tau) = H(\xi, \tau) + K^{-1}\left\{\omega^{2\alpha} K\left[\frac{\partial^2 v}{\partial \xi^2} + v\right]\right\} \tag{17}$$

Now applying Projected DTM, we get

$$v(\xi, \eta + 1) = K^{-1}\{\omega^{2\alpha} K[P_\eta + Q_\eta]\}, H(\xi, \tau) = v(\xi, 0) = 1 + \sin \xi \tag{18}$$

where, $P_\eta = \frac{\partial^2 v(\xi,\tau)}{\partial \xi^2}$ and $Q_\eta = v(\xi, \eta)$ are projected DTM of $\frac{\partial^2 v}{\partial \xi^2}$ and v, respectively.

Now,

$$P_0 = \frac{\partial^2 v(\xi,0)}{\partial \xi^2} = -\sin \xi \text{ and } Q_0 = v(\xi, 0) = 1 + \sin \xi,$$
$$\Rightarrow v(\xi, 1) = K^{-1}\{\omega^{2\alpha} K[-\sin \xi + 1 + \sin \xi]\} = \frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

$$P_1 = \frac{\partial^2 v(\xi,1)}{\partial \xi^2} = 0 \text{ and } Q_1 = v(\xi, 1) = \frac{\tau^\alpha}{\Gamma(\alpha+1)},$$
$$\Rightarrow v(\xi, 2) = K^{-1}\left\{\omega^{2\alpha} K\left[\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right\} = \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$P_2 = \frac{\partial^2 v(\xi,2)}{\partial \xi^2} = 0 \text{ and } Q_2 = v(\xi, 2) = \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$\Rightarrow v(\xi, 2) = K^{-1}\left\{\omega^{2\alpha} K\left[\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right]\right\} = \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}.$$

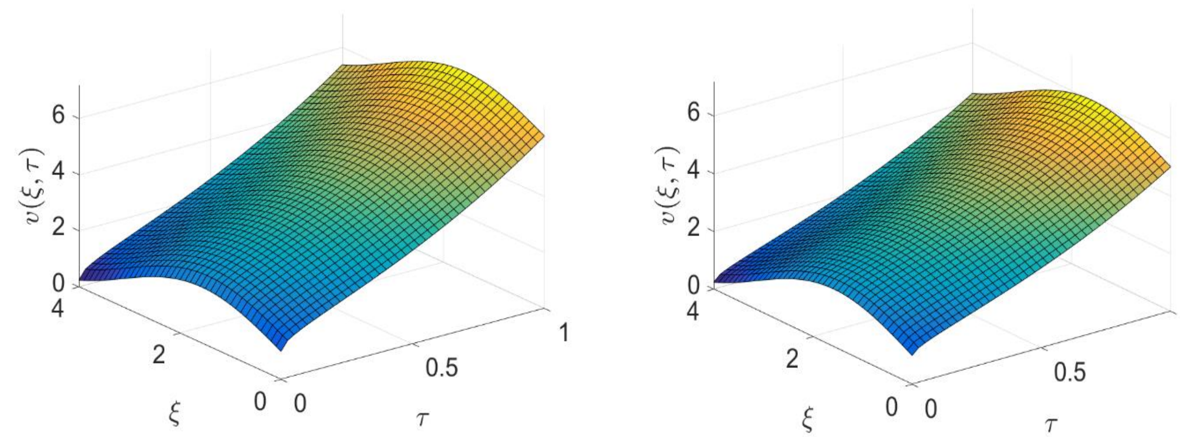
Using in equation (13), we get

$$v(\xi, \tau) = 1 + \sin \xi + \frac{\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots = 1 + \sin(\xi) + \sum_{r=1}^{\infty} \frac{\tau^{r\alpha}}{\Gamma(1+r\alpha)} \tag{19}$$

The solution (19) is the analytical solution of (14). The obtained solution is same as those given in (Golmankhaneh and Baleanu, 2011; Tamsir and Srivastava, 2016a). Especially, when $\alpha \rightarrow 1$, we have

$$v(\xi,\tau)=1+\sin(\xi)+\sum_{r=1}^{\infty}\frac{\tau^{r}}{\Gamma(1+r)} \quad (20)$$

The solution (20) is the exact solution of the classical KG equation. One can notice that the obtained solution is in comprehensive agreement with those given in (Golmankhaneh and Baleanu, 2011; Tamsir and Srivastava, 2016a). Figure 1 demonstrates the physical performance of $v(\xi,\tau)$ for α =0.4, 0.5, 0.7, and 0.9 whereas Figure 2 demonstrates the contour plots of $v(\xi,\tau)$ for various fractional Brownian motion α =0.4, 0.5, 0.7, and 0.9. Figure 3 demonstrates the approximated solutions $v(\xi,\tau)$ for various fractional Brownian motions α =0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1. One can notice that a monotonically decrease in the fractional Brownian motions tend to zero i.e., as the values of α tend to integer order, there is a decay in the solution influence of $v(\xi,\tau)$.



(a) (b) (c) (d)

Figure 1. Physical performance of $v(\xi,\tau)$ relating to α (a) 0.4 (b) 0.5 (c) 0.7 (d) 0.9.

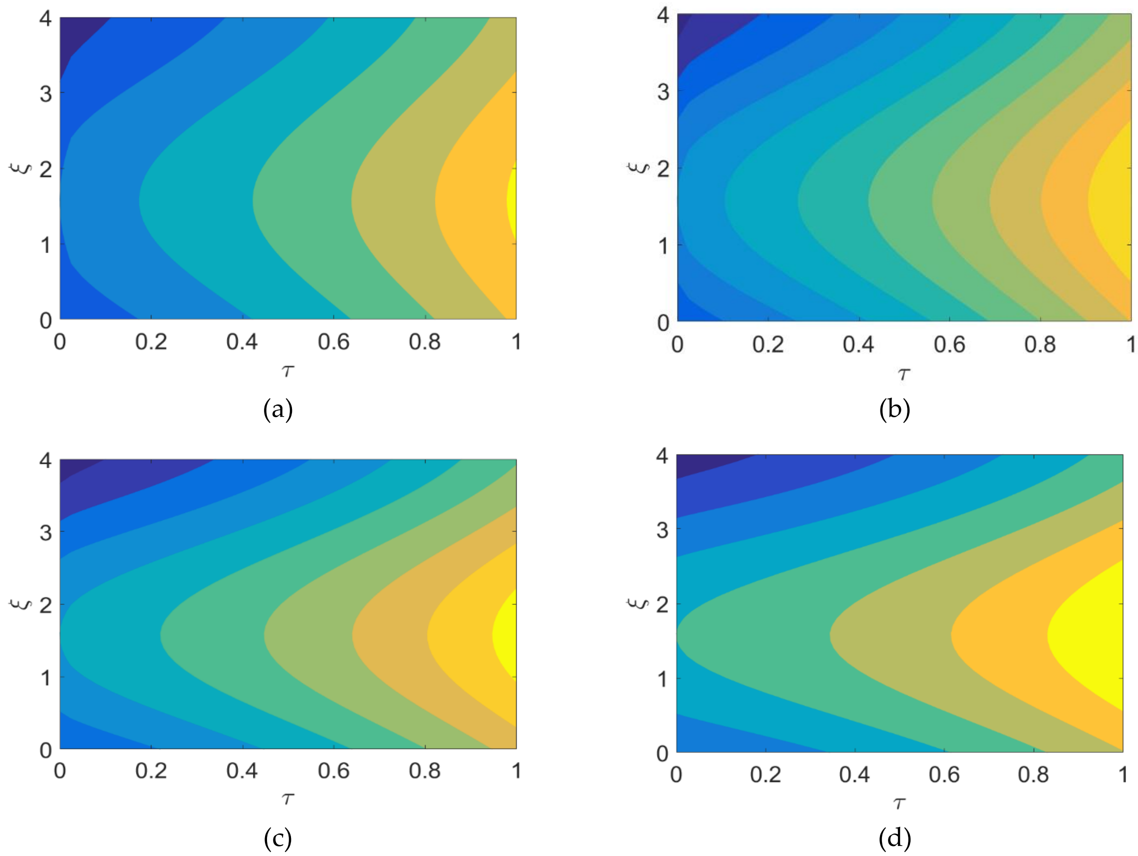


Figure 2. The contour plot of $v(\xi,\tau)$ relating to α (a) 0.4 (b) 0.5 (c) 0.7 (d) 0.9.

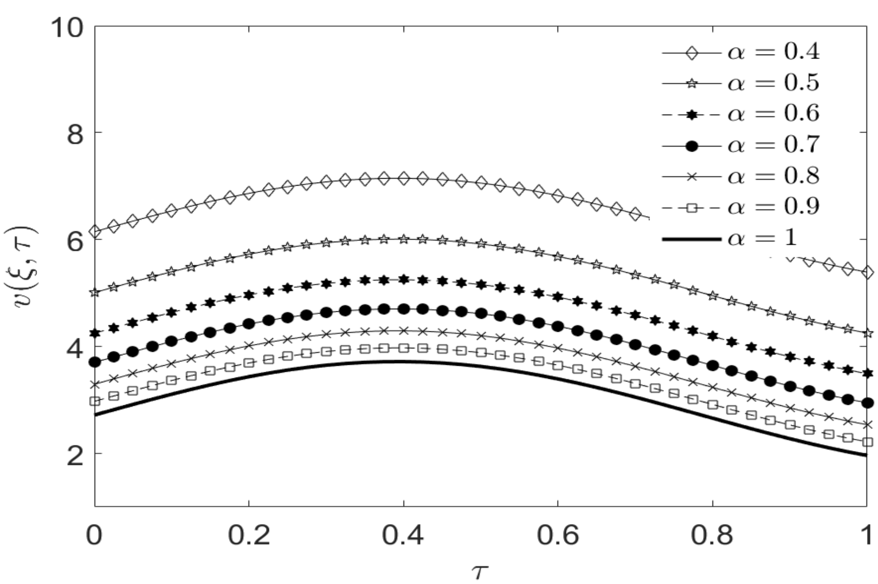


Figure 3. Approximated solutions $v(\xi,\tau)$ for various values of α.

Example 2. Consider the nonlinear TFKG equation

$$\frac{\partial^\alpha v}{\partial \tau^\alpha} - \frac{\partial^2 v}{\partial \xi^2} + v^2 = 0, \tau \geq 0 \tag{21}$$

with the initial condition

$$v(\xi, 0) = 1 + sin\,\xi \tag{22}$$

Using a new integral transform in (21), we get

$$\frac{K[v(\xi,\tau)]}{\omega^{2\alpha}} - \sum_{j=0}^{\alpha-1} \frac{1}{\omega^{2(\alpha-j)-1}} \frac{\partial^j}{\partial \tau^j} v(\xi, 0) = K\left[\frac{\partial^2 v}{\partial \xi^2} - v^2\right].$$

Simplifying the above equation, we get

$$K[v(\xi,\tau)] = \omega v(\xi,\tau) + \omega^{2\alpha} K\left[\frac{\partial^2 v}{\partial \xi^2} - v^2\right] \tag{23}$$

Taking the inverse of the new integral transform in the above equation, we get

$$v(\xi,\tau) = H(\xi,\tau) + K^{-1}\left\{\omega^{2\alpha} K\left[\frac{\partial^2 v}{\partial \xi^2} - v^2\right]\right\} \tag{24}$$

Now applying the projected differential transform method, we get

$$v(\xi, \eta + 1) = K^{-1}\{\omega^{2\alpha} K[P_\eta - Q_\eta]\}, H(\xi,\tau) = v(\xi, 0) = 1 + sin\,\xi \tag{25}$$

where, $P_\eta = \frac{\partial^2 v(\xi,\tau)}{\partial \xi^2}$ and $Q_\eta = \sum_{i=0}^{\eta} v(\xi, i)\, v(\xi, \eta - i)$ are projected DTM of $\frac{\partial^2 v}{\partial \xi^2}$ and v^2, respectively.

Now,

$$P_0 = \frac{\partial^2 v(\xi,0)}{\partial \xi^2} = -\,sin\,\xi \text{ and } Q_0 = v(\xi,0)v(\xi,0) = 1 + sin^2\,\xi + 2\,sin\,\xi,$$

$$\Rightarrow v(\xi,1) = K^{-1}\{\omega^{2\alpha} K[-1 - sin^2\,\xi - 3\,sin\,\xi]\} = -(1 + sin^2\,\xi + 3\,sin\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)},$$

$$\Rightarrow v(\xi,1) = -(1 + sin^2\,\xi + 3\,sin\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

$$P_1 = \frac{\partial^2 v(\xi,1)}{\partial \xi^2} = (-2 + 4\,sin^2\,\xi + 3\,sin\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

and

$$Q_1 = \sum_{i=0}^{1} v(\xi,i)v(\xi,1-i) = 2v(\xi,0)v(\xi,1),$$

$$\Rightarrow Q_1 = \frac{\partial^2 v(\xi,1)}{\partial \xi^2} = -(2 + 8\,sin\,\xi + 8\,sin^2\,\xi + 2\,sin^3\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

$$\Rightarrow v(\xi,2) = K^{-1}\{\omega^{2\alpha} K[P_1 - Q_1]\},$$

$$\Rightarrow v(\xi,2) = K^{-1}\left\{\omega^{2\alpha} K\left[(11\,sin\,\xi + 12\,sin^2\,\xi + 2\,sin^3\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right]\right\}, \Rightarrow v(\xi,2) = (11\,sin\,\xi + 12\,sin^2\,\xi + 2\,sin^3\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

Using in equation (13), we get

$$v(\xi,\tau) = 1 + sin\,\xi - (1 + sin^2\,\xi + 3\,sin\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + (11\,sin\,\xi + 12\,sin^2\,\xi + 2\,sin^3\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \ldots \tag{26}$$

The solution (26) is a series solution for the nonlinear TFKG equation (21). The obtained solution in a close agreement with the solution given in (Golmankhaneh and Baleanu, 2011; Tamsir and Srivastava, 2016a). Figure 4 demonstrates the physical attributes of $v(\xi,\tau)$ related to $\alpha = 0.3,0.5,0.7 \text{and} 0.9$, respectively whereas Figure 5 demonstrates the performance of $v(\xi,\tau)$ corresponding to various fractional Brownian motions $\alpha = 0.3,0.5,0.7 \text{and}\ 0.9$. Figure 6 demonstrates the approximated solutions $v(\xi,\tau)$ for various fractional Brownian motions α =0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1. From this figure, we noticed a monotonically decrease in the fractional Brownian motions and tend to zero i.e., as the values of α tending to integer order, there is a decay in the solution influence of $v(\xi,\tau)$.

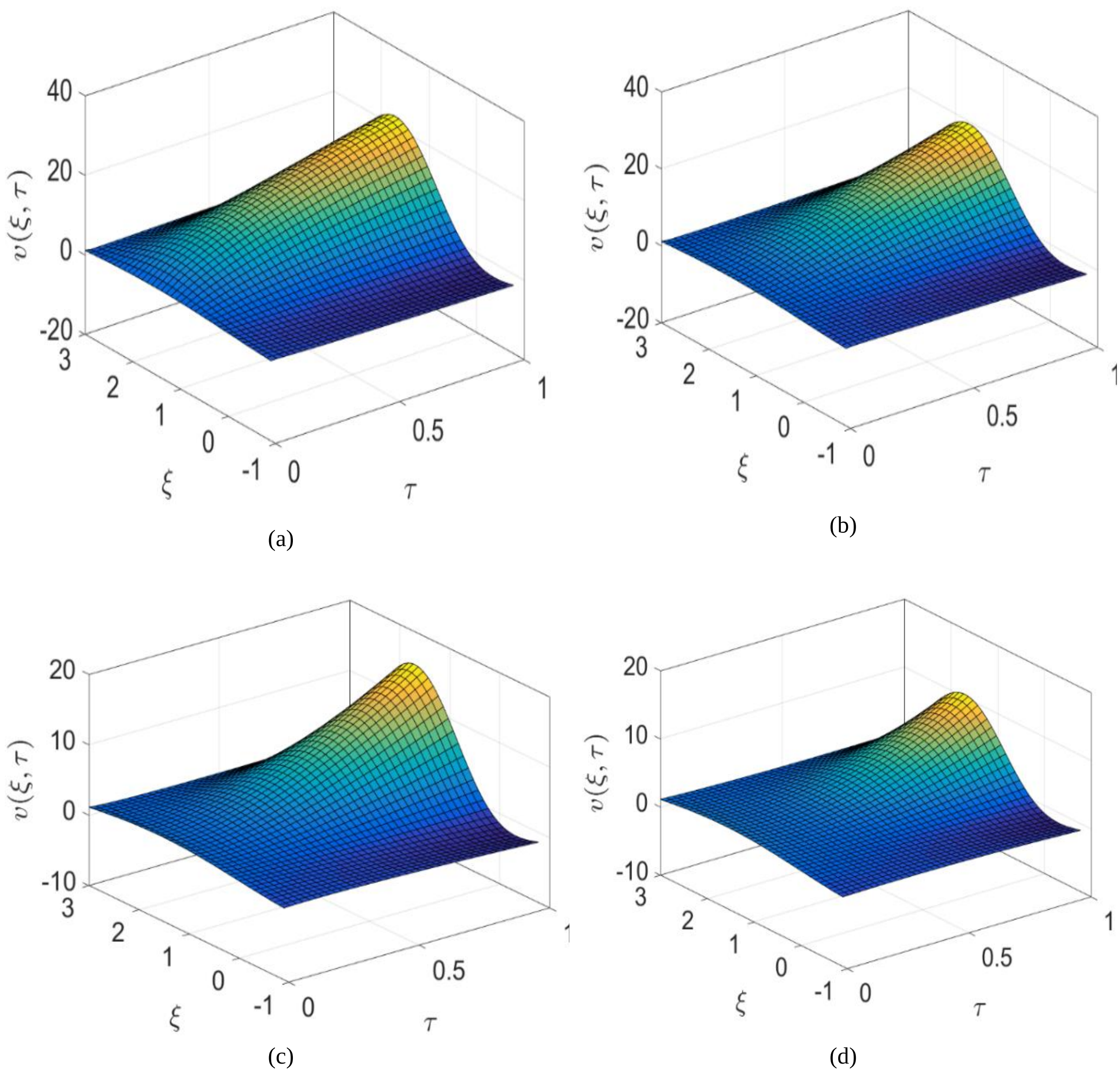


Figure 4. Physical performance of $v(\xi,\tau)$ relating to α (a) 0.3 (b) 0.5 (c) 0.7 (d) 0.9.

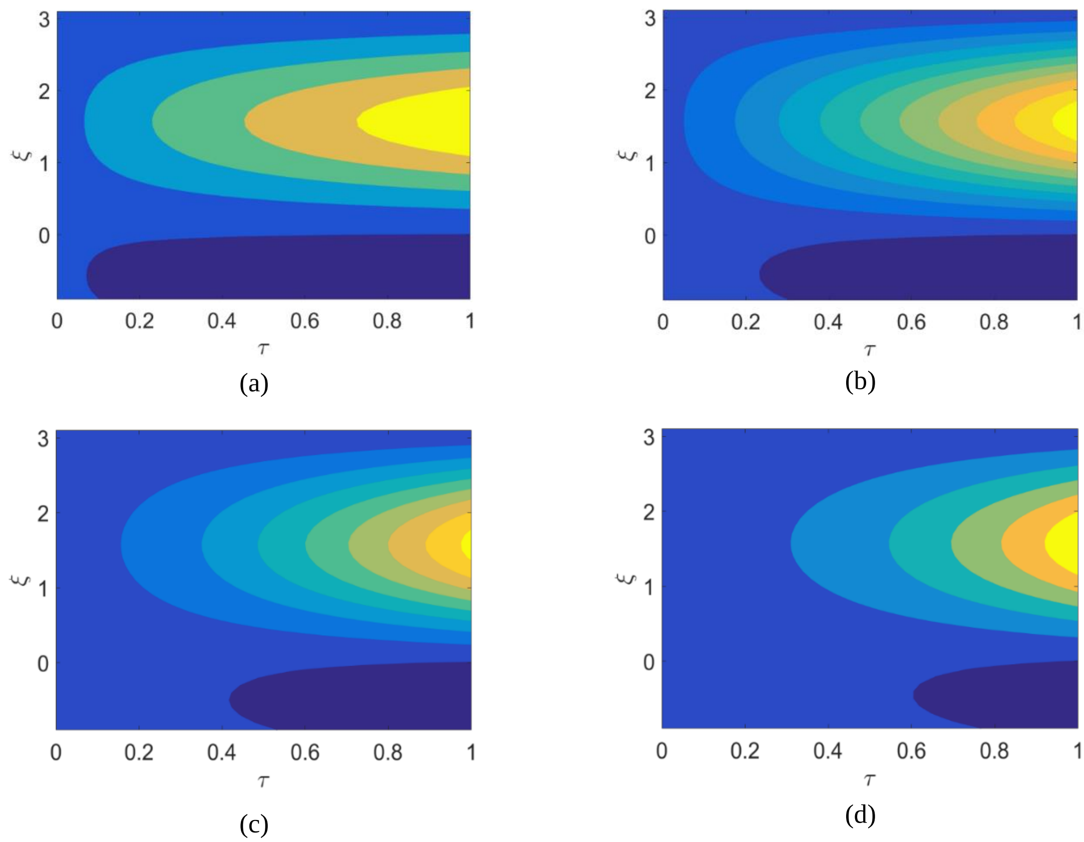


Figure 5. The contour plot of $v(\xi,\tau)$ relating to α (a) 0.3 (b) 0.5 (c) 0.7 (d) 0.9.

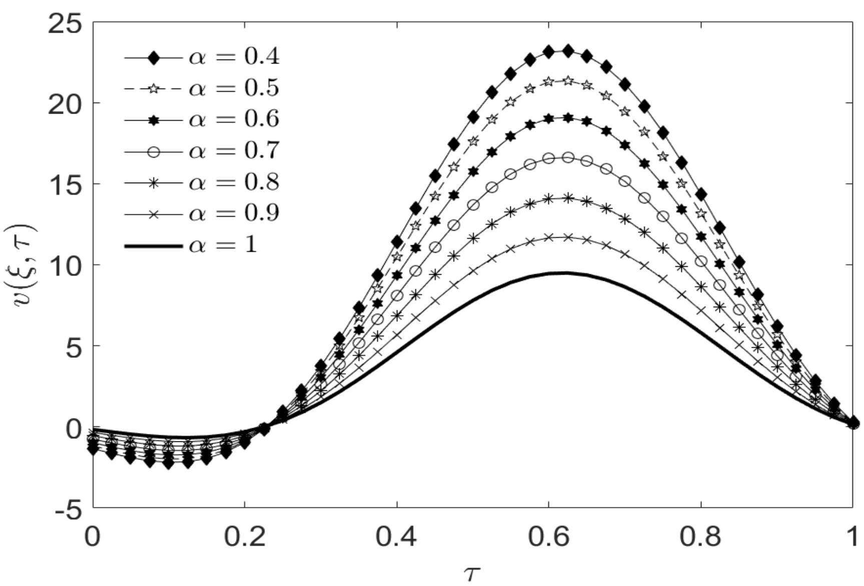


Figure 6. Approximated solutions $v(\xi,\tau)$ for various values of α.

Example 3. Finally, we consider the nonlinear TFKG equation

$$\frac{\partial^\alpha v}{\partial \tau^\alpha} - \frac{\partial^2 v}{\partial \xi^2} + v - v^3 = 0, \tau \geq 0 \tag{27}$$

with the initial condition

$$v(\xi, 0) = -\sec h\,(\xi) \tag{28}$$

Using a new integral transform in (27), we get

$$\frac{K[v(\xi,\tau)]}{\omega^{2\alpha}} - \sum_{j=0}^{\alpha-1} \frac{1}{\omega^{2(\alpha-j)-1}} \frac{\partial^j}{\partial \tau^j} v(\xi, 0) = K\left[\frac{\partial^2 v}{\partial \xi^2} - v + v^3\right].$$

Simplifying the above equation, we get

$$K[v(\xi,\tau)] = \omega v(\xi,\tau) + \omega^{2\alpha} K\left[\frac{\partial^2 v}{\partial \xi^2} - v + v^3\right] \tag{29}$$

Taking the inverse of the new integral transform in the above equation, we get

$$v(\xi,\tau) = H(\xi,\tau) + K^{-1}\left\{\omega^{2\alpha} K\left[\frac{\partial^2 v}{\partial \xi^2} - v + v^3\right]\right\} \tag{30}$$

Now applying the projected differential transform method, we get

$$v(\xi, \eta+1) = K^{-1}\{\omega^{2\alpha} K[P_\eta + Q_\eta]\}, H(\xi,\tau) = v(\xi, 0) = -\sec h\,\xi \tag{31}$$

where, $P_\eta = \frac{\partial^2 v(\xi,\tau)}{\partial \xi^2}$ and $Q_\eta = -v(\xi,0) + \sum_{j=0}^{\eta} \sum_{i=0}^{j} v(\xi,i)v(\xi,j-i)v(\xi,\eta-j)$ are projected DTM of $\frac{\partial^2 v}{\partial \xi^2}$ and $-v + v^3$, respectively. Now,

$$P_0 = \frac{\partial^2 v(\xi,0)}{\partial \xi^2} = -\sec h\,\xi + 2\sec h^3\,\xi,$$

and

$$Q_0 = v(\xi,0) - v(\xi,0)v(\xi,0)v(\xi,0) = -\sec h\,\xi + \sec h^3\,\xi.$$

$$\Rightarrow v(\xi,1) = K^{-1}\{\omega^{2\alpha} K[-2\sec h\,\xi + 3\sec h^3\,\xi]\} = (-2\sec h\,\xi + 3\sec h^3\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)}.$$

Similarly,

$$P_1 = \frac{\partial^2 v(\xi,1)}{\partial \xi^2} = (-2\sec h\,\xi + 31\sec h^3\,\xi - 36\sec h^5\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)},$$

and

$$Q_1 = -v(\xi,1) + \sum_{i_2}^{1} \sum_{i_1=0}^{i_2} v(\xi,i_1)v(\xi,i_2-i_1)v(\xi,\eta-i_2).$$

$$= (-2\sec h\,\xi + 3\sec h^3\,\xi - 9\sec h^5\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)},$$

So,

$$v(\xi,2) = K^{-1}\{\omega^{2\alpha} K[P_1 + Q_1]\} = (-4\sec h\,\xi + 34\sec h^3\,\xi - 45\sec h^5\,\xi)\frac{\tau^\alpha}{\Gamma(2\alpha+1)}.$$

Using in equation (13), we get

$$v(\xi,\tau) = -\sec h\,\xi + (-2\sec h\,\xi + 3\sec h^3\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + (-4\sec h\,\xi + 34\sec h^3\,\xi - 45\sec h^5\,\xi)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \ldots \tag{32}$$

The solution (32) is a series solution for the nonlinear TFKG equation (27). The obtained solution is in close agreement with the solution with those given in (Golmankhaneh and Baleanu, 2011; Tamsir and Srivastava, 2016a). Figure 7 demonstrates the physical attributes of $v(\xi,\tau)$ related to $\alpha = 0.01$, 0.5, 0.7, and 0.9, and Figure 8 demonstrates the approximated solutions $v(\xi,\tau)$ for various fractional Brownian motions α =0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1. It is examined that a monotonically decrease in the fractional Brownian motions and tend to zero as the values of α tending to integer order, there is a decay in the solution influence of $v(\xi,\tau)$.

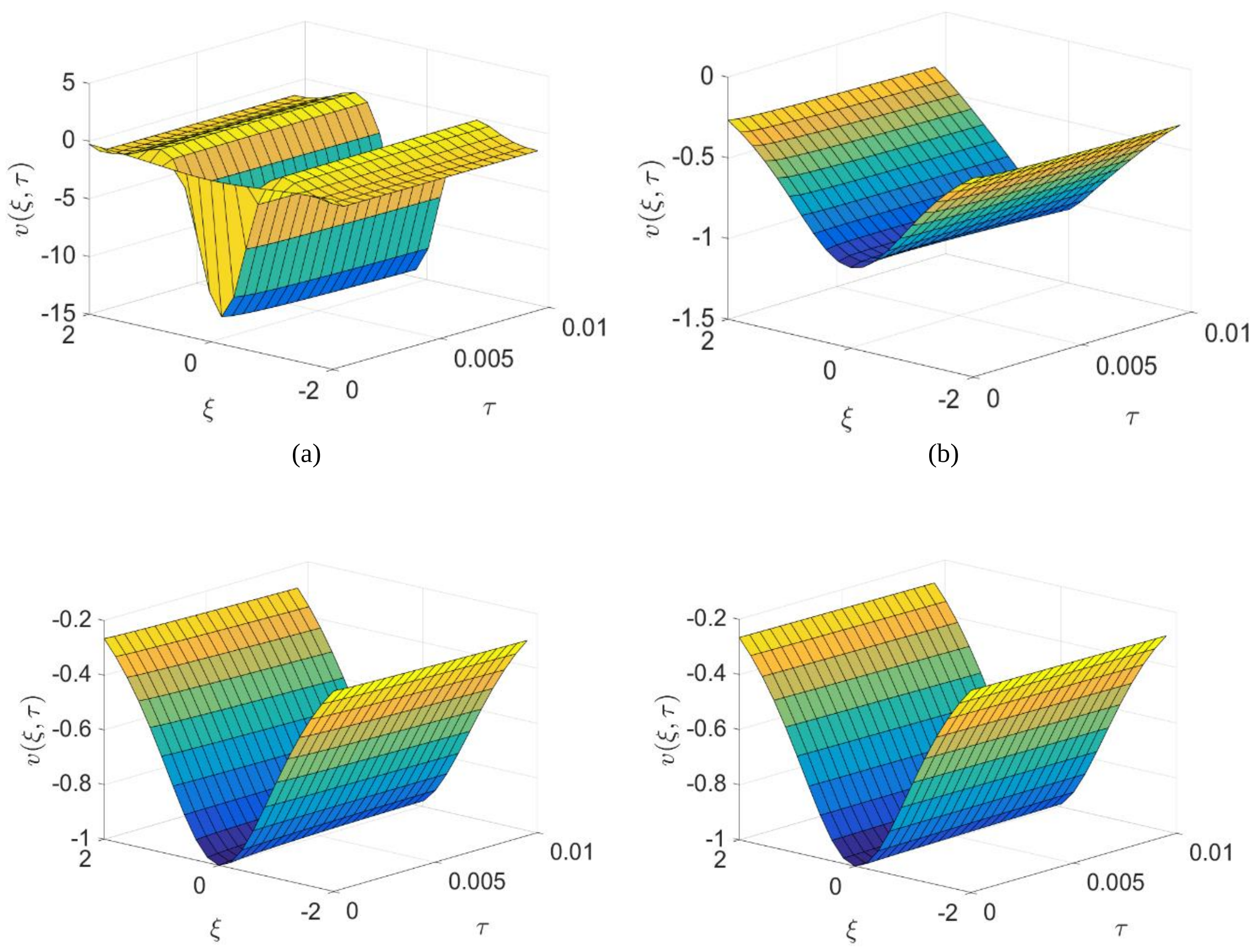


Figure 7. Physical performance of $v(\xi,\tau)$ relating to α (a) 0.01 (b) 0.5 (c) 0.7 (d) 0.9.

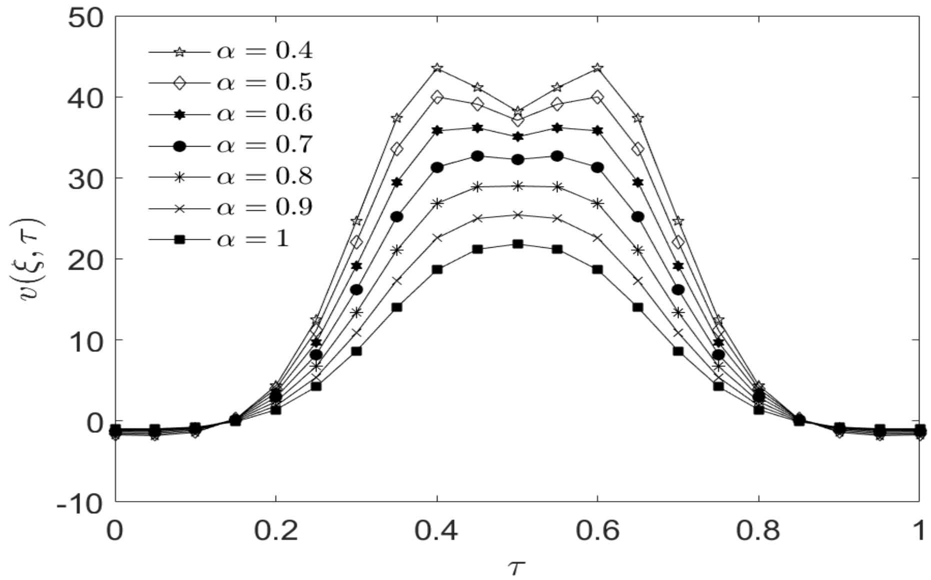


Figure 8. Approximated solutions $v(\xi, \tau)$ for various values of α.

6. Conclusions

In this article, the implementation of a new integral transform method along with the projected DTM has been successfully done to evaluate the Caputo TFKG problem analytically. To corroborate the effectiveness and exactness of the method TFKG equation, three examples have been considered and the results demonstrate how effective, precise, and easy the method is to use. The effects of various fractional Brownian motion are demonstrated graphically. From the investigations, it has been noticed that as the fractional Brownian motions tend toward non-fraction Brownian motions, the solutions describe decay. In Fig. 1 to 7, which depict some intriguing dynamics of the model, the impacts of various values of fractional order, α on the solution profile are shown. We also noted that the purported series solutions are in outstanding agreement with those solutions given in (Golmankhaneh and Baleanu, 2011; Tamsir and Srivastava, 2016a). Additionally, the analysis demonstrates that the suggested method is much simpler to use than the homotopy perturbation method. As for future, this method can be easily extended to solve the higher-order fractional PDEs.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgments

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors would like to thank the editor and anonymous reviewers for their comments that help improve the quality of this work.

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