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An Extended Kannan Contraction Mapping and Applications

R. P. Pant
Dalhousie Villa Compound,
Ayarpata, Nainital, Uttarakhand, India.
E-mail: pant_rp@rediffmail.com

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Abstract
We extend the Kannan contraction principle and obtain a result that holds for both contractive and non-expansive mappings. Such mappings admit multiple fixed-points and the fixed-point sets and domains of these mappings display interesting algebraic, geometric and dynamical features. Since contraction mappings admit only one fixed-point, almost all the existing results on contractive mappings can be generalized in the light of our theorem. As an application of our main theorem, we obtain the integral solutions of a nonlinear Diophantine equation; the solutions are Pythagorean triples, which represent right angled triangles, and each integer of the triple belongs to a Fibonacci type sequence. These results can be generalised to obtain integral solutions of Diophantine equations of the type $(n+k)^2 - n^2 = p^2$, $k > 1$, and to check whether the related sequences are Fibonacci sequences.

Keywords- Contraction mappings, Eventual fixed points, Fibonacci sequence, Fixed points, Pythagorean triple.

1. Introduction

Banach (1922) proved that if a self-mapping T of a complete metric space (X, d) satisfies:

$$d(Tx, Ty) \le \lambda\, d(x, y),\ 0 \le \lambda < 1 \quad (1)$$

then T has a unique fixed point. Various useful applications and generalizations of this theorem have been obtained e.g. Boyd and Wong (1969), Chatterjea (1972), Ciric (1971, 1974), Meir-Keeler (1969), Reich (1971a, 1971b, 1972), Suzuki (2008) and Wardowski (2012, 2018). Kannan (1968, 1969) proved that if a self-mapping T of a complete metric space (X, d) satisfies:

$$d(Tx, Ty) \le a[d(x, Tx) + d(y, Ty)],\ 0 \le a < ½ \quad (2)$$

then T has a unique fixed point. The Kannan theorem is an important result since it characterizes metric completeness (see Subrahmanyam, 1975) and is the genesis of the Rhoades problem (Rhoades, 1988) on continuity of contractive mappings at the fixed point. Pant (1999) resolved the Rhoades problem and proved:

Theorem 1.1 Let f be a self-mapping of a complete metric space (X, d) such that for any x, y in X
(i) $d(fx, fy) \le \phi(\max\{d(x, fx), d(y, fy)\})$, $\phi: R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.
(ii) Given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon < \max\{d(x, fx), d(y, fy)\} < \varepsilon + \delta \Longrightarrow d(fx, fy) \le \varepsilon$.

Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if
$\lim_{x\to z} \max\{d(x, fx), d(z, fz)\} = 0$.

Pant and Pant (2017, Theorem 2.9) have shown that the (ε, δ) condition (ii) applies to non-expansive type mappings as well and named such mappings as (ε - δ) non-expansive mappings. Condition (ii) or its variants have been employed by researchers to find new solutions of Rhoades' problem (1988) on continuity of contractive mappings at the fixed point, e.g. Bisht and Pant (2017), Bisht and Rakocevic (2018, 2020), Celik and Ozgur (2020), Pant (2001), Pant et al. (2020), Tas and Ozgur (2019), Zheng and Wang (2017).

In this paper, we extend the Kannan contraction condition (2) and obtain a result that holds for both contraction and non-expansive mappings. Our result is more general than every known theorem for contractive mappings since contractive mappings possess a unique fixed point while our theorem admits a unique fixed point as well as multiple fixed points. All the known fixed-point theorems for contraction mappings can be generalized along the lines of our theorem e.g. Banach (1922), Boyd and Wong (1969), Meir and Keeler (1969), Chatterjea (1972), Ciric (1971, 1972), Hardy and Rogers (1973), Pant (2002), Pasicki (2016), Petrov (2023), Reich (1971a, 1971b, 1972), Savaliya et al. (2024), Suzuki (2008). Besides metric spaces these theorems can be generalized in Fuzzy metric space, Menger PM-space, b-metric space, ordered metric space, cone metric space, etc. by using our approach. Therefore, our result provides a vast scope of obtaining new and significant results. The examples of our main theorem include a mapping on a Fibonacci sequence. Each element of this sequence yields a Pythagorean triple which provides an integral solution of the Diophantine equation $X^2 + (X+1)^2 = Y^2$. Since Pythagorean triples are right angled triangles, our solutions of the Diophantine equation provide a nice combination of fixed-point results, Fibonacci sequences, right angled triangles, Diophantine equation etc. This also opens up scope for studying Diophantine equations of the type $X^2 + (X + k)^2 = Y^2$, k = 2, 3, 4,…

We now give some relevant definitions.

Definition 1.1 If T is a self-mapping of a set X then a point x in X is called an eventually fixed point of T if there exists a natural number N such that $T^{n+1}(x) = T^n(x)$ for $n \geq N$. If Tx = x then x is called a fixed point of T. A point x in X is called a periodic point of period n if $T^n x = x$. The least positive integer n for which $T^n x = x$ is called the prime period of x (Devaney, 1986; Holmgren, 1994).

Definition 1.2 The set $\{x \in X: Tx = x\}$ is called the fixed-point set of the mapping $T: X \rightarrow X$.

Definition 1.3 The function $T: (-\infty, \infty) \rightarrow (-\infty, \infty)$, such that T(x) is the least integer not less than x, is called the least integer function or the ceiling function and is denoted by $T(x) = \lceil x \rceil$.

Definition 1.4 A triplet (a, b, c) of natural numbers, $a < b < c$, is called a Pythagorean triple if $a^2 + b^2 = c^2$.

Definition 1.5 For a Fibonacci sequence $\{u_i\}$the limiting value of the ratio u_{i+1}/u_i as $i \rightarrow \infty$ is called the golden ratio of $\{u_i\}$.

Properties of the golden ratio have been widely studied. Renowned architect Le Corbusier and famous artist Salvador Dali have used the golden ratio to design their works.

In Section 2 we prove our main result and give five examples to substantiate it. In Remark 2.2 we show that the fixed-point sets and domains of the mappings satisfying our theorems display interesting algebraic, geometric and dynamical features. In Theorems 3.1 and 3.2 in Section 3, as an application of mappings satisfying Theorem 2.1, we obtain $2\sum m = \sum n$ as the generating equation for the integral solutions of the nonlinear Diophantine equation $X^2 + (X+1)^2 = Y^2$. The solutions are Pythagorean triples and the values of each of X, X+1, Y yield a Fibonacci type sequence. The value of the golden ratio of each Fibonacci sequences is $3+2\sqrt{2}$. Section 4 summarizes the work done in Sections 2 and 3. It also outlines the vast scope of possible work along the lines of Theorems 2.1, 3.1, 3.2 and Remark 3.1.

The main results are given in Section 2 using the Picard iteration method to obtain the fixed point as the limit of the sequence of iterates. Application of our results is given in Section 3 and the method consists of using the generating equation $2\sum m = \sum n$ obtained by us.

2. Main Results

Theorem 2.1 Let (X, d) be a complete metric space and T: X → X be such that for each x, y in X with $x \neq Tx$ or $y \neq Ty$ we have,
(iii) $d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)],\ 0 \leq a < ½$.

Then T has a fixed point. T has a unique fixed point ⟺ (iii) is satisfied for each $x \neq y$ in X.

Proof. Let y_0 be any point in X and $\{y_n\}$ be the sequence defined by $y_n = Ty_{n-1}$, that is, $y_n = T^n y_0$. If $y_n = y_{n+1}$ for some n, then y_n is a fixed point of T and the theorem holds. Therefore, assume that $y_n \neq y_{n+1}$ for each $n \geq 0$. Then using (iii) we have,

$$d(y_n, y_{n+1}) = d(Ty_{n-1}, Ty_n) \leq a[d(y_{n-1}, Ty_{n-1}) + d(y_n, Ty_n)] = a[d(y_{n-1}, y_n) + d(y_n, y_{n+1})].$$

This implies that,

$$d(y_n, y_{n+1}) \leq (a/(1-a))d(y_{n-1}, y_n) \leq (a/(1-a))^2 d(y_{n-2}, y_{n-1}) \leq \ldots \leq (a/(1-a))^n d(y_0, y_1).$$

From this it follows that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ and $\lim_{n\to\infty} d(y_n, y_{n+p}) = 0$, that is, $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists z in X such that $\lim_{n\to\infty} y_n = z$ and $\lim_{n\to\infty} Ty_n = z$.

Using (iii) we get,

$$d(Ty_n, Tz) \leq a[d(y_n, Ty_n) + d(z, Tz)].$$

Taking the limit as $n\to\infty$ we get $z = Tz$, that is, z is a fixed point of T. It follows easily that z is the unique fixed point. Further, let u be any point in X. Then, $T^n u = T^{n+1} u = z$ for some n or $\lim_{n\to\infty} d(T^n u, Tz) = \lim_{n\to\infty} ad(T^{n-1}u, T^n u) = 0$, that is, $\lim_{n\to\infty} T^n u = z$. Thus, if there exists a point y_0 such that $T^{n+1} y_0 \neq T^n y_0$ for each n, then for each u in X the sequence of iterates $\{T^n u\}$ converges to z and z is the unique fixed point. Therefore, $T^{n+1} y_0 \neq T^n y_0$, $n \geq 0$, for some y_0 implies the uniqueness of the fixed point.

Now, assume that condition (iii) is satisfied for all x, y in X. Then T can have only one fixed point and we obtain the Kannan theorem (1968, 1969) as a particular case of Theorem 2.1. Conversely, suppose that T has a unique fixed point. Then for distinct x, y we have $x \neq Tx$ or $y \neq Ty$ which implies that condition (iii) holds for each $x \neq y$. This proves the theorem.

Example 2.1 Let $X = [1, \infty)$ and d be the Euclidean metric. Let T: X → X be the signum function Tx = sgn x defined as,

$$Tx = -1 \text{ if } x < 0,\ T0 = 0,\ Tx = 1 \text{ if } x > 0.$$

Then Tx = 1 for each x, T satisfies (iii) and T has a unique fixed-point x = 1. If $x \neq 1$ then $T^2 x = Tx$ and x is an eventually fixed point.

Example 2.2 Let $F = \{re^{i\theta}: 0 \leq \theta \leq 2\pi, r = 1, 6, 6^2, \ldots\}$ be the self-similar family of concentric circles, each lying within larger circles having radii in a geometric progression, in the xy-plane. Let X be the set of points of intersection of F with the N rays beginning at the origin and respectively making angles 0, 2π/N, 2(2π/N), 3(2π/N), …, (N-1)(2π/N) measured counter clockwise with the positive x-axis and let d be the usual metric on X. Define T: X→ X by,

$$T(re^{i\theta}) = \lceil r/6 \rceil e^{i\theta}.$$

where, ⌈x⌉ denotes the least integer not less than x. Then T satisfies Theorem 2.1 with a = 2/5 and has N fixed points $e^{i0}, e^{i2\pi/N}, e^{i2(2\pi/N)}, e^{i3(2\pi/N)}, \ldots, e^{i(N-1)(2\pi/N)}$. The mapping T does not satisfy Kannan theorem (1968,

1969) if $N > 1$. However, if $N = 1$ then T is a Kannan contraction mapping and has a unique fixed point $e^{i0} = 1$ or if we take the restriction of T along one of the N rays then the restriction of T is a Kannan contraction.

Example 2.3 Let $X = \{z = re^{i\theta}: 0 \leq \theta \leq 2\pi, r = 1, 6, 6^2, \ldots\}$ be the self-similar family of concentric circles, each lying within larger circles having radii in a geometric progression, in the xy-plane and let d be the usual metric on X. Define $T: X \rightarrow X$ by $T(z) = z/|z| = z/r$.

Then T satisfies (iii) with $a = 2/5$ and each point on the unit circle $z = e^{i\theta}$ is a fixed point while every other point is an eventually fixed point. In this example, the unit circle is a fixed circle. Fixed circles are presently an active area of study (Hussain et al., 2020; Ozgur and Tas, 2018, 2019a, 2019b; Ozgur, 2019; Tas et al., 2018; Saleh et al., 2020).

Example 2.4 Let (X, d) be a metric space and T be the identity mapping on X. Then each point is a fixed point and condition (iii) holds since there is no pair of points (x, y) in X that violates (iii). Therefore, T satisfies Theorem 2.1. However, T does not satisfy Kannan Contraction Theorem.

Example 2.5 Let $\{m_i, i \geq 1\}$ be the Fibonacci type sequence given by,

$$m_1 = 0,\ m_2 = 2,\ m_i = 6m_{i-1} - (m_{i-2} - 2) \text{ for } i \geq 3.$$

This gives $m_3 = 14$, $m_4 = 84$, $m_5 = 492$, $m_6 = 2870$, $m_7 = 16730$, $m_8 = 97512$, $m_9 = 568344$, …

Let $X = [0, \infty)$ equipped with the Euclidean metric. Define $T: X \rightarrow X$ by,

$$Tm_i = (m_i + m_{i-2} - 2)/6 = m_{i-1} \text{ if } i \geq 2, \qquad Tx = 0 \text{ otherwise.}$$

Then T is a nonlinear mapping that satisfies the conditions of Theorem 2.1 with $a = 1/3$ and has a unique fixed point at 0. If we take $x = 14$ and $y = 70$ then $T(x + y) = 14$ while $T(x) + T(y) = 2$ showing that T is a nonlinear mapping. If $x = 492$ and $y = 490$ then $d(Tx, Ty) = 84$ and $d(x, y) = 2$. This shows that T does not satisfy the fixed-point theorems due to Banach (1922), Boyd and Wong (1969), Meir and Keeler (1969), Pant (2002), Pasicki (2016), Reich (1971, 1972), Suzuki (2008), Wardowski (2012, 2018).

Remark 2.1 We observe that the nonlinear mapping defined in Example 2.5 satisfies Theorem 2.1 but does not satisfy the well-known theorems due to Banach (1922), Boyd and Wong (1969), Meir and Keeler (1969), Pant (2002), Pasicki (2016), Reich (1971, 1972), Suzuki (2008), Wardowski (2012, 2018) and the extensions and generalizations of these and various other theorems. Examples 2.2, 2.3, 2.4 admit multiple fixed points and, therefore, Theorem 2.1 is more general than every contraction mapping theorem.

Remark 2.2 The N fixed points $e^{i0}, e^{i2\pi/N}, e^{i2(2\pi/N)}, e^{i3(2\pi/N)}, \ldots, e^{i(N-1)(2\pi/N)}$ in Example 2.2 are:
a) The N^{th} roots of unity and these lie on the unit circle and form a cyclic group under multiplication,
b) Vertices of a regular polygon of N sides.

If $N = 2^n - 1$ then the fixed-point set is identical with the set of periodic points of period n for the doubling map which is important in dynamics of complex functions (Devaney, 1986; Holmgren, 1994).

Also, the domain of the mapping in Example 2.3 is a self-similar family of circles. The domain in Example 2.5 is a Fibonacci sequence which, as shown in Theorem 3.2, yields right angled triangles as integral solutions of the equation $X^2 + (X+1)^2 = Y^2$. Thus, the domain and the fixed-point sets of the mappings satisfying Theorem 2.1 may possess interesting algebraic, geometric and dynamical features. If we replace the self-similar family of circles by a self-similar family of spheres then the domain and the fixed-point set will be more intricate and interesting sets.

3. Applications

In this section, we give an application of Theorem 2.1 and Example 2.5 in obtaining the generating equation of: (a) the Fibonacci type sequence $\{m_i\}$ introduced in Example 2.5 and (b) the integral solutions of the nonlinear Diophantine equation $X^2+(X+1)^2 = Y^2$.

Fibonacci numbers are very interesting numbers and these are of particular interest to biologists and physicists because they are observed in various objects and phenomena. Because of the importance and applications of Fibonacci sequences the journal Fibonacci Quarterly is devoted to the studies on Fibonacci sequences. Fibonacci numbers were introduced in 1202 by Italian mathematician Leonardo, also known as Fibonacci. In India Fibonacci numbers were first described as early as 200 BC in the work by Grammarian Pingala (see Singh, 1985) for enumerating the possible patterns of Sanskrit poetry.

The Pythagorean triples are interesting triplets of natural numbers that find a number of applications. The integral solutions of the equation $X^2+(X+1)^2 = Y^2$ are, obviously, Pythagorean triples which represent right angled triangles having hypotenuse of length Y and the other sides of length X and X+1 respectively.

The sum $1 + 2 + 3 + \ldots + n$ of first n natural numbers is generally denoted by $\sum n$ and $\sum n = n(n+1)/2$. Let us define $\sum 0 = 0$. We can treat $\sum n$ as a function F on nonnegative integers defined as $F(n) = \sum n$. In the following, we shall treat 0, 1, 1 as a Pythagorean triple.

In Remark 3.1 we will discuss some simple but interesting properties of odd numbers and will see that the sum function $\sum n$ is inevitably required in expressing odd powers of integers as sum of consecutive odd numbers and in finding the sums of odd powers of numbers. Here we discuss a property of odd numbers that yields the generating equation for the solutions of the nonlinear Diophantine equation $X^2+(X+1)^2 = Y^2$. It is easy to see that the square of every odd number is a difference of squares of two consecutive integers, that is,

$$(2m+1)^2 = (1+2m(m+1))^2 - (2m(m+1))^2 = (1+4\sum m)^2 - (4\sum m)^2 \quad (3)$$

This shows that for each odd number 2m+1 both terms on the righthand side of (3) contain $4\sum m$, a multiple of four. On the other hand, the multiples of four are the only integers that can be written as sum of two consecutive odd numbers:

$$4 = 1+3,\ 8 = 3+5,\ 12 = 5+7,\ 16 = 7+9,\ 20 = 9+11,\ 24 = 11+13,\ 28 = 13+15, \ldots \quad (4)$$

Equations (3) and (4) indicate some interesting relationship between odd numbers and the multiples of four and immediately draw our attention to hymns 24 and 25 of Chapter 18 of Shukla-Yajurved. In hymn 24 the prayer has been expressed in the form of pairs of odd numbers while in hymn 25 the prayer has been expressed through multiples of four. In Remark 3.1 we will discuss the relationship between odd numbers and multiples of four in a little more detail.

Now, for each integer $k \geq 0$ we have $k^2 + (k+1)^2 = (1+k(k+1))^2 - (k(k+1))^2$. This means that if $(2m+1)^2$ is the sum of squares of two consecutive integers n, n+1 then using (3) we get $n(n+1) = 4\sum m$, that is, $\sum n = 2\sum m$.

In the next two theorems we show that the equation $2\sum m = \sum n$ generates the Fibonacci sequence of Example 2.5 and the solutions of the Diophantine equation $X^2 + (X+1)^2 = Y^2$.

Theorem 3.1 There exist pairs (m, n) of nonnegative integers such that $2\sum m = \sum n$. The values of m form a Fibonacci type sequence and the values of n also form a Fibonacci type sequence.

Proof. It is easy to verify that $2\sum 0 = \sum 0$, $2\sum 2 = \sum 3$, $2\sum 14 = \sum 20$, $2\sum 84 = \sum 119$, $2\sum 492 = \sum 696$, $2\sum 2870 = \sum 4059$, $2\sum 16730 = \sum 23660$, $2\sum 97512 = \sum 137903$, that is, the first eight vales of m in increasing order are 0, 2, 14, 84, 492, 2870, 16730, 97512 and the corresponding values of n are 0, 3, 20, 119, 696, 4059, 23660, 137903. We see that the initial eight values of m are the same as the corresponding values of Fibonacci type sequence $\{m_i, i \geq 1\}$ of Example 2.5 above. We see that each element of the sequence $\{m_i, i \geq 1\}$ satisfies the equation $2\sum m = \sum n$. Similarly, we find that the values of n form a Fibonacci type sequence $\{n_i, i \geq 1\}$ such that $2\sum m_i = \sum n_i$ and $n_1 = 0$, $n_2 = 3$, $n_i = 6n_{i-1} – (n_{i-2} – 2)$ for $i \geq 3$.
This proves the theorem.

Using the above theorem, we now generate the integral solutions of the Diophantine equation $X^2 + (X+1)^2 = Y^2$. It is easy to see that Y is an odd number if $Y^2 = X^2 + (X+1)^2$.

Theorem 3.2 The equation $2\sum m = \sum n$ generates the integral solutions of the nonlinear Diophantine equation $X^2 + (X+1)^2 = Y^2$.

Proof. Let m, n be nonnegative integers such that $2\sum m = \sum n$. Then

$$2\sum m = \sum n \Leftrightarrow m(m + 1) = n(n + 1)/2 \Leftrightarrow 4m^2 + 4m = 2n^2 + 2n$$
$$\Leftrightarrow (2m + 1)^2 = n^2 + (n+1)^2 \qquad (5)$$

Thus, the equation $2\sum m = \sum n$ is equivalent to the integral solution $n^2 + (n+1)^2 = (2m + 1)^2$ of the Diophantine equation $X^2 + (X+1)^2 = Y^2$ and, therefore, yields all the integral solution of $X^2 + (X+1)^2 = Y^2$. Using the value of m and n from Theorem 3.1 some initial solutions of the equation $X^2 + (X+1)^2 = Y^2$ are:

$n_1^2 + (n_1+1)^2 = 0^2 + 1^2 = 1^2 = (2m_1+1)^2$;
$n_2^2 + (n_2+1)^2 = 3^2 + 4^2 = 5^2 = (2m_2+1)^2$;
$n_3^2 + (n_3+1)^2 = 20^2 + 21^2 = 29^2 = (2m_3+1)^2$;
$n_4^2 + (n_4+1)^2 = 119^2 + 120^2 = 169^2 = (2m_4+1)^2$;
$n_5^2 + (n_5+1)^2 = 696^2 + 697^2 = 985^2 = (2m_5+1)^2$;
$n_6^2 + (n_6+1)^2 = 4059^2 + 4060^2 = 5741^2 = (2m_6+1)^2$;
$n_7^2 + (n_7+1)^2 = 23660^2 + 23661^2 = 33461^2 = (2m_7+1)^2$;…

The values obtained for (2m + 1) are 1, 5, 29, 169, 985, 5741, 33461, …, and these form a Fibonacci type sequence, say $\{k_i, i \geq 1\}$ defined as,

$$k_1 = 1,\ k_2 = 5,\ k_i = 6k_{i-1} – k_{i-2} \text{ if } i \geq 3.$$

The golden ratio of $\{k_i\}$ is $\lim_{i\to\infty}(6k_i – k_{i-1})/k_i$. Since $k_i \to \infty$ as $i \to \infty$ and $5 \leq (6k_i – k_{i-1})/k_i < 6$, the golden ratio, say r, of $\{k_i\}$ is given by $r = 6 – (1/r)$, that is, $r = 3+2\sqrt{2}$. Similarly, it follows that the golden ratio of each of $\{m_i\}$ and $\{n_i\}$ equals $3+2\sqrt{2}$.

Remark 3.1 We have seen above that if the square of an odd number (2m+1) is sum of squares of two consecutive numbers n, n+1 then $\sum n = 2\sum m$. In Theorems 3.1 and 3.2 we have seen that the equation $\sum n = 2\sum m$ yields three Fibonacci sequences and solutions of the Diophantine equation $X^2 + (X+1)^2 = Y^2$.

Besides yielding the solutions of $X^2 + (X+1)^2 = Y^2$, the sum function $\sum n$ is very useful in writing the odd powers of integers as sum of consecutive odd numbers and in finding the sum of odd powers of integers.

We know that the sum of first n odd numbers equals n^2, that is, $1 + 3 + 5 + \ldots + (2n-1) = n^2$. This means that the sum of first $\sum n$ odd numbers will be $(\sum n)^2 = (n(n+1)/2)^2$. We show that cubes, 5^{th} powers, and 7^{th} powers of natural numbers can also be expressed as sum of consecutive odd numbers. Now,

$1 = 1^3$,
$3+5 = 2^3$,
$7+9+11 = 3^3$,
$13+ 15+ 17+19 = 4^3$,
$5^3 = 21+23+25+27+29$,
$6^3 = 31+33+35+37+39+41$,
$7^3 = 43+45+ \ldots+54+55$, …

This implies that

a) $1^3 + 2^3 + 3^3 + \ldots+n^3$ = sum of first $\sum n$ odd numbers = $(n(n+1)/2)^2$, which is the well-known formula for sum of cubes of consecutive integers,
b) n^3 = sum n consecutive odd numbers after the first $\sum(n-1)$ odd numbers.

Let us now consider the sums,

$1 = 1^5$,
$5+7+9+11 = 32 = 2^5$,
$19+21+23+25+27+29+31+33+35 = 243 = 3^5$,
$49+51+53+\ldots+75 +77+79 = 1024 = 4^5$,
$101+103+105+\ldots+147+149 = 3125 = 5^5$,
$181+183+187+\ldots+249+251 = 7776 = 6^5$,
$295+297+\ldots+389+391 = 16807=7^5$, …

Here we leave $\sum 1$ odd numbers after 1 and add the next 2^2 odd numbers to get 2^5; leave the next $\sum 2$ odd numbers and add the following 3^2 odd numbers to get 3^5; leave the next $\sum 3$ odd numbers and add the following 4^2 odd numbers to get 4^5, leave the next $\sum 4$ odd numbers and add the following 5^2 odd numbers to get 5^5, … This gives a recurring pattern involving $\sum n$ for expressing 5^{th} powers as sum of consecutive odd numbers. In writing n^5 as sum of odd numbers the last odd number used will be $((n^3+n^2)/2)$th odd number that is, n^3+n^2-1.

We can find the sum of odd numbers left out at each stage and also the sum of fifth powers of integers using $\sum n$. The sum of the odd numbers left out is,

$$3\,(\sum 1+\sum 2+\sum 3+\ldots+\sum(n-1))^2 = 3[(n-1)\,n(n+1)/6]^2.$$

Therefore,

$$\begin{aligned}1^5+2^5+3^5+\ldots+n^5 &= \text{Sum of first } (n^3+n^2)/2 \text{ odd numbers - sum of odd numbers left out}\\ &= ((n^3+n^2)/2)^2 - 3\,(\sum 1+\sum 2+\sum 3+\ldots+\sum(n-1))^2\\ &= ((n^3+n^2)/2)^2 - 3\,[(n-1)\,n(n+1)/6]^2 = n^2(n+1)^2(2n^2+2n-1)/12.\end{aligned}$$

Similarly, we can write the 7^{th} powers of integers as sum of consecutive odd numbers as:

$1^7 = 1$,
$2^7 = 9+11+\ldots+21+23 = 128$,
$3^7 = 55+57+\ldots+105+107 = 2187$,
$4^7 = 193+195+\ldots+317+319 = 16384$,
$5^7 = 501+503+\ldots+747+749 = 78125$,
$6^7 = 1081+1083+\ldots+1509+1511 = 279936$, …

Here we leave (2.1+1) ∑1 = 3 odd numbers after 1 and add the next 2^3 odd numbers to get 2^7,
leave next (2.2+1) ∑2 = 15 odd numbers and add the next 3^3 odd numbers to get 3^7,
leave next (2.3+1) ∑3 = 42, odd numbers and add the next 4^3 odd numbers to get 4^7,
leave next (2.4+1) ∑4 = 90 odd numbers and add the next 5^3 odd numbers to get 5^7,
leave next (2.5+1) ∑5 =165 odd numbers and add the next 6^3 odd numbers to get 6^7, …

Again, we get a recurring pattern involving ∑n for expressing 7th powers as sum of consecutive odd numbers. To sum up, having obtained $(k-1)^5$ we leave ∑(k-1) odd number and add next k^2 odd numbers to get k^5 and having obtained $(k-1)^7$ leave (2(k-1) +1) ∑(k-1) odd numbers and add next k^3 odd numbers to get k^7. Similarly, we add k consecutive odd numbers after (∑(k-1))th odd number to get k^3. Now, the sum of odd numbers left out,

$$= 15\,(\sum 1+\sum 2+\sum 3+\ldots+\sum(n-1))\;[(\sum 1)^2+(\sum 2)^2+(\sum 3)^2+\ldots+(\sum(n-1))^2].$$
$$= (n-1)^2 n^2 (n+1)^2[3n^2-2]/24.$$

Therefore,
$1^7+2^7+\ldots+n^7$ = Sum of first $(n^4+n^3)/2$ odd numbers - sum of odd numbers left out.

That is,
$$1^7+2^7+\ldots+n^7 = (n^6(n+1)^2)/4 - (n-1)^2n^2(n+1)^2[3n^2-2]/24.$$

In a similar manner, other odd powers of numbers can be expressed as sums of consecutive odd numbers and sums of 9th and 11th powers may possibly be obtained; this can be a good exercise for school and college students besides finding the values of sums like ∑1+∑2+∑3+…+∑(n-1) and $(\sum 1)^2+(\sum 2)^2+(\sum 3)^2+\ldots+(\sum(n-1))^2$.

The above computations for odd numbers involve the sum function ∑n at every stage.

In Equation (3) we see that squares of odd numbers can be written as the difference of squares of two consecutive integers as $(2m+1)^2 = (1+2m(m+1))^2 - (2m(m+1))^2$, that is,
$1^2 = 1^2 - 0^2$,
$3^2 = 5^2 - 4^2$,
$5^2 = 13^2 - 12^2$,
$7^2 = 25^2 - 24^2$,
$9^2 = 41^2 - 40^2$,
$11^2 = 61^2 - 60^2$,
$13^2 = 85^2 - 84^2$,
$15^2 = 113^2 - 112^2$,
$17^2 = 145^2 - 144^2$, …

Here, the numbers occurring at the second place on the right-hand side are 0, 4, 12, 24, 40, 60, 84, 112, 144, … or equivalently,
4∑0, 4∑1, 4∑2, 4∑3, ∑4, 4∑5, 4∑6, 4∑7, 4∑8, … (6)

In the set of integers 0, 4, 12, 24, 40, 60, 84, 112, …, the sums of two consecutive integers are respectively 4, 16, 36, 64, 100, 144, … (4.1^2, 4.2^2, 4.3^2, 4.4^2, 4.5^4, 4.6^2, …) while the respective differences between two consecutive integers are 4, 8, 12, 16, 20, 24, … These differences are multiples of four and are the only numbers that can be written as sum of two consecutive odd numbers and, as in (4), all the odd numbers can be obtained as follows,

4 = 1+3,
8 = 3+5,
12 = 5+7,
16 = 7+9,
20 = 9+11,
24 = 11+13,
28 = 13+15, …

The above computations are reversible since beginning with multiples of four we can obtain all the odd numbers by using (4) and their squares in the form $(1+4\sum n)^2 - (4\sum n)^2 = (2n+1)^2$ while multiples of four are obtained from odd numbers by using (4).

Now, hymn 25 of Chapter 18 of Shukla-Yajurved contains multiples of four and multiples of four are the only numbers that can be expressed as sum of consecutive odd numbers as done in Equation (4) and the order of occurrence of pairs of consecutive odd numbers in Equation (4) is exactly the same as the order of occurrence of pairs of consecutive odd numbers in hymn 24. These two hymns are routinely recited in many religious ceremonies of the Hindus. It is most unlikely that the occurrence of multiples of four in hymn 25 and the occurrence of pairs of consecutive odd numbers in hymn 24 in the same order as in Equation (4) is a mere coincidence because the decimal system of counting, one of the greatest contributions to mathematics, was already in existence in the era of Rig-Ved, the oldest Ved. In Vedic era teaching-learning followed the listening-memorising (Shruti-Smriti) mode in which instructions were imparted orally and, therefore, hymns 24 and 25 had the potential of working as a formula for orally explaining the arithmetic of odd numbers and the multiples of four.

Remark 3.2 Theorems 3.1, 3.2 and the results on cubes, 5^{th} and 7^{th} powers of integers in Remark 3.1 may be helpful in motivating the students at school and college level to explore properties of numbers on their own on the lines of Remark 3.1 by using the sum function $\sum n$. We discuss this aspect in some detail in the next section.

4. Conclusion

In this paper we obtained a proper extension of the Kannan contraction theorem and the result has been substantiated by various types of examples. The Fibonacci sequence defined in Example 2.5 and the generating equation $2\sum m = \sum n$ defined in Theorems 3.1 and 3.2 have been successfully applied to obtain the integral solutions of the nonlinear Diophantine equation $X^2 + (X+1)^2 = Y^2$. From a geometric point of view, the solutions are Pythagorean triples which represent right angled triangles with hypotenuse of length Y and the other two sides of length X, X+1 respectively. In the proofs given in Theorems 3.1, 3.2 and Remark 3.1 the sum function $\sum n$ is an important tool. In Remark 3.1 we have expressed the 5^{th} powers and 7^{th} powers of integers as sums of consecutive odd numbers and have also obtained the sum of 5^{th} powers and 7^{th} powers of integers.

Further, in Theorems 3.1 and 3.2 we see that the equation $2\sum m = \sum n$ generates the Pythagorean triple n, n+1, (2m+1) and three Fibonacci sequences $\{m_i\}$, $\{n_i\}$ and $\{k_i\}=\{2m_i+1\}$. We also proved that the golden ratio of each of these sequences is $3+2\sqrt{2}$. The equation $2\sum m = \sum n$ also generates related Pythagorean triples $(2m_i+1)$, $n_i(n_i+1)$, $1+n_i(n_i+1)$ and sequences $\{n_i(n_i+1)\}$ and $\{1+n_i(n_i+1)\}$. This opens up a vast scope, for researchers interested in Fibonacci sequences and even students at school and college level, for doing some work along the following lines:

(i) Generalization of various theorems for contractive mappings along the lines of Theorem 2.1 and finding applications of the results obtained. Besides metric spaces such results can be attempted in fuzzy metric spaces, Menger PM-spaces, symmetric spaces, b-metric spaces, ordered metric spaces, cone metric spaces and a large number of interesting and significant results can be established.

(ii) To check whether the sequences $\{n_i(n_i+1)\}$ and $\{1+n_i(n_i+1)\}$, $\sum n_i = 2\sum m_i$, are Fibonacci sequence and find the golden ratio if these are Fibonacci sequences.

(iii) To find Pythagorean triples of the form $n^2 + (n+k)^2 = p^2$, $k > 1$, and to check whether the related sequences are Fibonacci sequences.

(iv) To find Pythagorean triples of the form $(n + k)^2 - n^2 = p^2$, $k > 1$, and to check whether the related sequences are Fibonacci sequences.

The author proposes to take up some of these questions in future. The equation $2\sum m = \sum n$ has large scope of yielding new and interesting results on numbers in a simpler manner. This can motivate students at the pre-university level to explore interesting properties of numbers.

Conflict of Interests
There is no conflict of interest.

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