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Optimal Pricing and Advertisement Policy for an Advance Order Booking Inventory System with Order Cancellation under Inflationary Condition

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Abstract
The e-commerce business has grown significantly over the past few years, mainly due to providing a better customer experience through advance booking and order cancellation options with a full refund. In this paper, the inventory control problem of a firm that offers advance booking for the product and cancellation (with a full refund) before the due delivery date is addressed. A profit maximization model is formulated to find the optimum inventory cycle length, the selling price of the product, and advertisement expenditure. The discounted cash flow approach is used to take into account cash flows at different time points. Advance booking is made at the beginning of the inventory cycle at a discounted price until the time of stock arrival, followed by usual spot sales. The number of order cancellations during the advance booking period is dependent on the waiting time for receiving the order. The impact of revenue collected through advance sales is considered by including interest earned. Product demand is assumed to be the function of advertisement expenditure and the selling price of the product. A solution procedure is suggested, and the model is illustrated through numerical analysis, thereby providing valuable managerial insights based on the results obtained.

Keywords- Joint decision, Inventory-pricing, Inventory-advertisement, Advance sales, Discount, Order-cancellation.

1. Introduction

In recent years, the e-commerce industry has witnessed a significant surge, resulting in a highly competitive market. To maintain a competitive edge, companies must devise innovative customer service strategies. One such approach is the implementation of Advance booking discount (ABD), which involves offering discounted prices to customers who commit to and pay for orders in advance while also allowing refunds for order cancellations. ABD can foster customer loyalty and drive up shopping frequency, ultimately leading to increased sales. ABD is a widely adopted strategy in various industries, particularly those with high demand for their products or services. Retailers, for instance, often offer advance booking for limited-edition products such as video games, electronics, and fashion collections. The travel industry also frequently utilizes advance booking strategies, with airlines, hotels, and tour companies offering discounts to customers who book their services well in advance. This enables companies to better forecast their demand and optimize their inventory levels, while also providing customers with an incentive to commit to their services early. Other industries that can benefit from ABD include food and beverage, event planning, and subscription-based services, all of which rely on inventory management and demand forecasting.

This paper focuses on ABD as a strategy, employed by retailers to offer discounted prices to customers who commit to, and pay for orders in advance. Many lifestyle product retailers, such as those selling mobile phones, televisions, and other electronics, utilize ABD to attract and retain customers. With advancements in information technology, companies can now provide an online platform for advance order booking and payment transactions, making the process more convenient for customers. Moreover, ABD often includes order cancellation facilities, which are becoming increasingly prevalent during stockout situations. This feature helps decrease demand estimation errors and increases market share. Given the increased competition in the marketplace, companies are now measuring their investments in marketing policies, such as advertising across various mediums and offering incentives like coupons, price reductions, and trade credit, to influence product demand. Employing such strategies can help companies generate higher sales and sustain their competitiveness.

Effective inventory planning is essential in today's complex and competitive e-commerce industry. Companies must consider multiple factors that affect demand, such as advertising effort, order cancellation rate, and advance booking discounts, when planning their inventory. Order cancellations can decrease demand, while advance payment discounts and advertising efforts can increase it. Thus, companies need to accurately estimate demand by considering these factors to optimize their inventory planning. Moreover, the inventory system's cash flows under advance payment discounts have different timing, and companies need to recognize the exact timing of cash flows explicitly. The Discounted cash flow (DCF) approach can help companies account for the timing of cash flows associated with the inventory system. By using this approach, companies can make better inventory planning and pricing decisions, ultimately leading to increased profits and a competitive edge in the e-commerce industry.

The practice of advance order booking with order cancellation has become increasingly popular in many industries due to its potential benefits for both companies and customers. However, the added complexity of inflationary conditions requires companies to consider additional factors when implementing pricing and advertising policies for their advance order booking inventory systems. This paper addresses this challenge by developing an optimal pricing and advertising policy for an advance order booking inventory system with order cancellation under inflationary conditions.

The paper proposes a profit maximization model that determines the optimal inventory cycle length, the selling price of the product, and advertising expenditure, while taking into account the impact of inflation on cash flows. The novelty of the paper lies in the use of a DCF approach that explicitly recognizes the timing of cash flows associated with the inventory system, particularly in an inflationary condition. The paper also incorporates the impact of interest earned on revenue collected through advance sales, and the impact of advertising expenditure and selling price on product demand. This makes it a comprehensive model for pricing and advertising policy in advance order booking inventory systems.

The motivation behind this paper is to provide practical solutions for firms managing advance order booking inventory systems while optimizing their profits. The paper's findings provide valuable insights for companies operating in industries that use advance order booking and order cancellation, such as the retail and travel industries. By considering the unique features of advance order booking inventory systems, such as order cancellations and advance payment discounts, the proposed model can help firms make informed decisions about pricing and advertising policies, ultimately leading to increased profits and a competitive edge in the e-commerce industry.

This paper is structured as follows: Section 2 presents a comprehensive literature review, while Section 3 identifies the research gap. Section 4 introduces the notations and assumptions; followed by Section 5, which presents the mathematical model. Section 6 delves into optimality and the solution procedure, while Section 7 discusses numerical and sensitivity analysis. Section 8 offers managerial insights, and finally, Section 9 concludes the paper.

2. Literature Review

The joint inventory and pricing decision problem has been extensively studied in the literature, with significant contributions from various researchers. One of the earliest works in this field was conducted by Whitin (1955). Chen and Simchi-Levi (2004) provided a comprehensive survey that discussed and analyzed the existing studies on joint inventory and pricing. Building upon this foundation, subsequent researchers have made valuable contributions. Khanna et al. (2017) explored the inventory and pricing problem specifically for imperfect and deteriorating items, considering the impact of credit policies. Shah and Naik (2018) investigated the influence of quantity discounts on inventory and pricing decisions. Khan et al. (2019) addressed the expiration date of deteriorating items in the context of inventory and pricing decisions. Shah et al. (2021) proposed a dynamic demand-based inventory-pricing model. The integration of environmental concerns into inventory pricing models was examined by Maihami et al. (2021), who focused on greening investments. Asghari et al. (2021) discussed an inventory-pricing model that considered the capacity constraints of the warehouse. Mashud et al. (2021) proposed an inventory pricing model that incorporated the product life cycle. To determine the optimal order quantity and selling price for deteriorating items, Das et al. (2021) employed the Stackelberg game method.

The concept of order cancellation in the inventory problem has been the subject of several studies conducted by various researchers. Cheung and Zhang (1999) were among the first to examine the impact of order cancellation on an inventory system and total costs. Koidea and Ishii (2005) delved into the specific context of hotel room allocations, considering the effects of early booking discounts and cancellations. By studying this scenario, they aimed to optimize room allocation strategies and mitigate the impact of cancellations on revenue. You (2006) focused on the joint ordering and pricing problem within an advance sales system, assuming a constant rate of order cancellation. This research sought to find the optimal ordering and pricing strategies that maximize profits, taking into account potential cancellations. Building upon this work, You and Wu (2007) expanded the scope by considering time-dependent order cancellations for an inventory system operating over a finite planning horizon. By incorporating the temporal aspect of cancellations, they aimed to enhance the accuracy of inventory management decisions. Xie and Gerstner (2007) explored the impact of order cancellation policies that offer full refunds within an advance selling system. Their research aimed to understand the implications of such policies on inventory management and customer satisfaction. Tsao (2009) focused on the inventory system with advance sales discounts, examining how discounts offered for advance purchases affect inventory control and profitability. Dye and Hsieh (2013) considered an inventory model that accounts for both advance and spot sales, specifically for deteriorating items. By incorporating these sales types, they aimed to optimize inventory decisions and minimize costs. Zhang et al. (2014) addressed the inventory problem in the context of advance booking, incorporating order cancellation and cash-on-delivery payment options. Kumar (2021) investigated the advance sale system for perishable products, considering the possibility of partial order cancellations.

In addition to the above literature, several studies explored joint inventory and advertisement decisions. Balcer (1980) and Balcer (1983) were among the first to propose papers that assumed a positive relationship between advertisement expenditure and demand. Sogomonian and Tang (1993) developed a model for joint production and promotion decisions within a finite planning horizon. Cho (1996) focused on the joint production and advertising problem in a clear-cut environment. Expanding on this, Cheng and Sethi (1999) investigated the combined production and advertising problem, considering stochastic random demand that depended on the level of advertising through a Markov process. Furthermore, other studies have explored joint inventory, pricing, and advertising decisions, including the works of Yu et al. (2009), Yu and Huang (2010), Li et al. (2013), and Jiang et al. (2015). Khan et al. (2020) assumed that the demand for a perishable product was influenced by both the selling price and advertising efforts. They derived optimal values for the order quantity, selling price, and frequency of advertisement based on this assumption. Martin and Mayan (2022) incorporated digital advertising into an inventory model, considering smart attributes and a linear demand pattern. Bhadoriya et al. (2022) studied the combined effect of carbon emissions and an exchange scheme alongside advertising efforts. Shah et al. (2022) considered an inspection process to assess the quality of old products received through an exchange scheme, determining optimal values for the selling price, order quantity, and advertisement expenditure.

Table 1. Comparison between present study and other published papers.

Authors	Demand type	Advertisement	Sales type	Order cancellation	Discount	DCF	Decision variables
You (2006)	Price dependent	No	Advance	Yes	No	No	Price, Order Cycle Length
You and Wu (2007)	Price dependent	No	Advance	Yes	No	No	Price, Order Cycle Length
Tsao (2009)	Constant	No	Advance	No	Yes	No	Order Cycle Length
Dye and Hsieh (2013)	Price dependent	No	Advance and Spot	Yes	No	No	Price, Order Cycle Length
Zhang et al. (2014)	Price dependent	No	Spot	Yes	No	No	Price and Order Quantity
Jiang et al. (2015)	Advertisement dependent	Yes	No	No	No	No	Advertisement Investment, Order Cycle Length
Shah and Naik (2018)	Price and stock dependent	No	Spot	No	No	No	Price, Order Cycle Length
Khan et al. (2020)	Price and advertisement dependent	Yes	Advance	Yes	No	No	Order Cycle Length
Mashud (2021)	Price dependent	No	Advance	No	Yes	No	Price, Order Cycle Length
Kumar (2021)	Price and advertisement dependent	Yes	Spot	No	No	No	Preservation Cost, Advertisement Frequency, Order Cycle Length
Rathore and Sharma (2021)	Price and advertisement dependent	Yes	Spot	No	No	No	Price, Preservation Cost, Advertisement Frequency, Order Cycle Length
Martin and Mayan (2022)	Constant	Yes	Spot	No	No	No	Order Cycle Length
Shah et al. (2022)	Price and advertisement dependent	Yes	Spot	No	No	No	Price, Advertisement Effort, Order Cycle Length
This paper	Price and advertisement dependent	Yes	Advance and Spot	Yes	Yes	Yes	Price, Advertisement Expenditure, Order Cycle Length

3. Research Gap

Table 1 provides a comprehensive comparison of this study with past studies. The identified research gap in this study lies in the lack of consideration for the combined impact of ABD, order cancellation, and advertising in previous studies on inventory systems. This study aims to address these gaps by exploring the combined impact of these marketing strategies (ABD, order cancellation, and advertisements) and utilizing the discounted cash flow approach to appropriately recognize different cash flows at different time points due to advance payment in the inventory cycle. The study assumes that order cancellation depends exponentially on the waiting period, with full refunds offered to customers. Moreover, the increasing impact of advertising expenditure on demand is considered, along with offering a discounted price for advance order booking to stimulate demand.

Another identified research gap pertains to the lack of literature on effective revenue management strategies employed by firms to handle advance sales payments. This study examines how firms can enhance profitability by earning interest on funds received from customers in advance and repaying loans taken to purchase stock. The cash received from advance bookings can be deposited into interest-bearing accounts, and payments can be settled when the ordered quantity is received by utilizing proceeds from advance bookings, along with short-term loans during the spot sales period to finance any unsold stock. This approach allows firms to generate additional revenue through interest earned on advance sales. The study utilizes the DCF approach to accurately account for the financial implications of opportunity costs and out-of-pocket costs in inventory analysis.

4. Notations and Assumptions

4.1 Notations

Decision variables

T	Total length of inventory cycle (decision variable)
p	Selling price during spot sales period in ($/unit) (decision variable)
m	Advertisement expenditure per unit in ($/unit) (decision variable)

Fixed variables

$I_S(t)$	Inventory Level at time t during spot sales period
I_{MAX}	Maximum inventory level
I_{CA}	Cumulated advance booking orders received at time T_A
$D(p,m)$	Demand rate
M	Total advertisement budget in ($)
c	Purchase cost per unit in ($/unit)
I	out-of-pocket inventory carrying charge per unit per unit time ($/unit/time)
A	Fixed ordering cost in ($/order)
z	Discount rate representing time value of money per unit time (%)
$\eta(x)$	Fraction of customers at any time t who are willing to wait for the product till the end of advance sales period
Φ	Cost of tracking advance order bookings per unit per unit time ($/unit/time)
Φ'	Cost of tracking order cancellations per unit per unit time ($/unit/time)
I_e	the interest that can be earned per $ per unit time (%)
I_p	the interest charges payable per $ per unit time ($I_p > I_e$) (%)

Note: Time unit can be considered as day, week or year.

4.2 Assumptions

(i) The replenishment rate of the product is instantaneous and the lead time is zero.

(ii) No Shortages are allowed.

(iii) Infinite time horizon is considered with each inventory cycle $[0,T]$ divided into two periods: first advance order booking is made during $[0,T_A]$ followed by usual *spot* sales during $[T_A,T]$.

(iv) Replenishment is received at time T_A and orders received during advance sales are immediately satisfied.

(v) A discounted selling price is offered for booking the product in advance during $[0,T_A]$ given as $p_A = \gamma p;\ \ 0 < \gamma < 1$, where $(1-\gamma)$ is the proportion of discount and p is selling price offered during the spot sales period $[T_A,T]$.

(vi) Demand is assumed to be selling price and advertisement expenditure dependent, and is modeled as $D(p,m) = d(p).g(m)$. Price dependent demand component is assumed to be $d(p) = a - bp; a >$

$0, b \geq 0$. Advertisement expenditure dependent demand function is assumed to be linearly increasing in advertising expenditure given by $g(m) = \alpha + \beta m$; $\alpha, \beta > 0$ (see Figure 1). Such demand functions are frequently used in literature (Yue et al., 2006; Szmerekovsky and Zhang, 2009; Xie and Neyret, 2009; Xie and Wei, 2009; SeyedEsfahani et al., 2011; Li et al., 2013).

(vii) Customers with reservations may cancel their orders during the advance booking period. The fraction of customers at any time t who are willing to wait for the product till the end of the advance booking period depends on the length of the waiting period $x(= T_A - t)$, is given by $\eta(x) = e^{-\lambda x}$, ($\lambda > 0$), where λ is a positive constant and $0 \leq \eta(x) \leq 1$, $\eta(0) = 1$. $x = 0$ being the case of zero waiting time and no sales are cancelled, and while $x = \infty$ corresponds to the case of an infinite waiting time, then all sales are lost (see Figure 2).

(viii) Full refund is offered to customers on order cancellation.

(ix) The DCF approach is used to consider the various costs at various times.

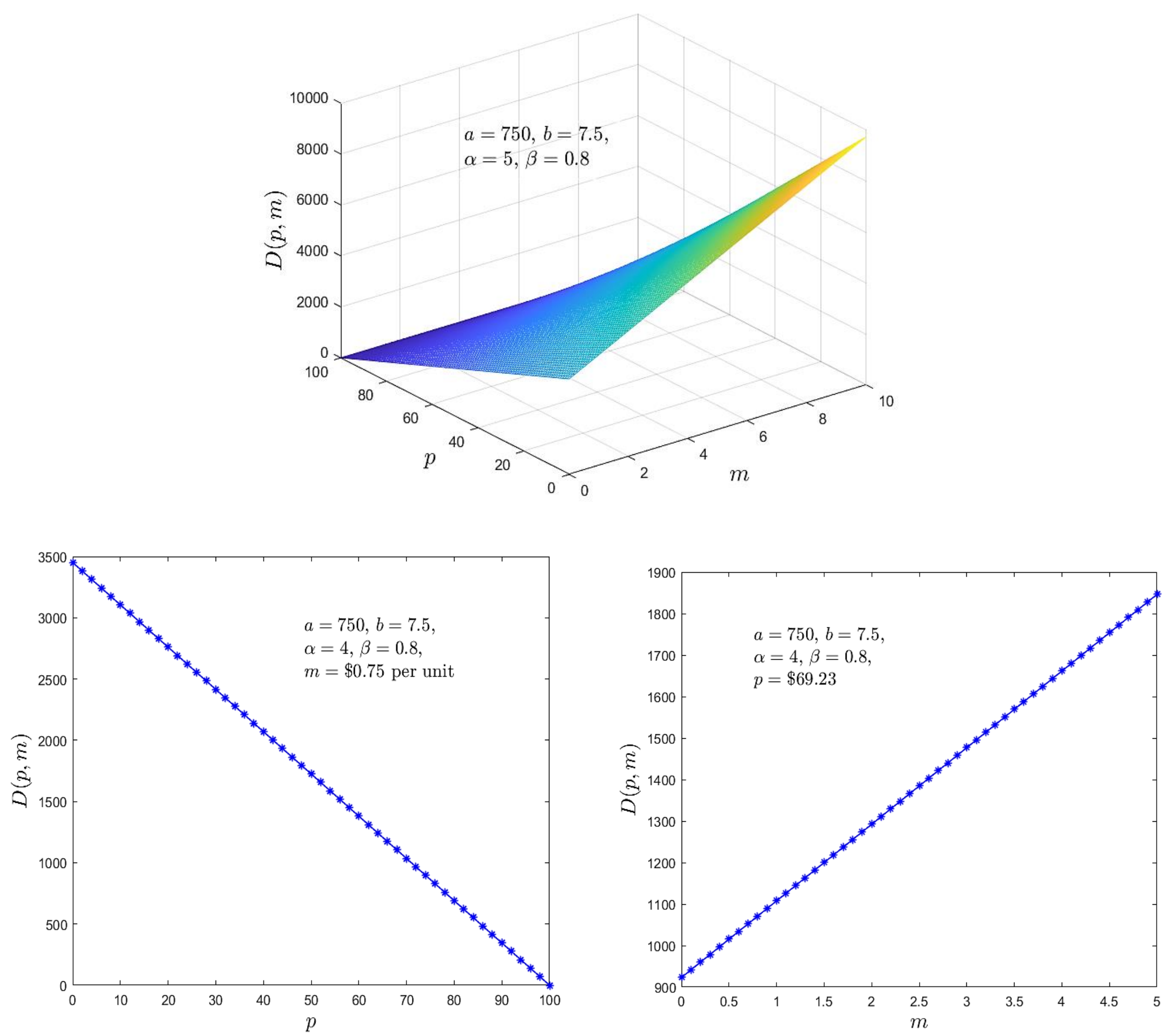


Figure 1. Demand function with respect to price and advetisement expenditure.

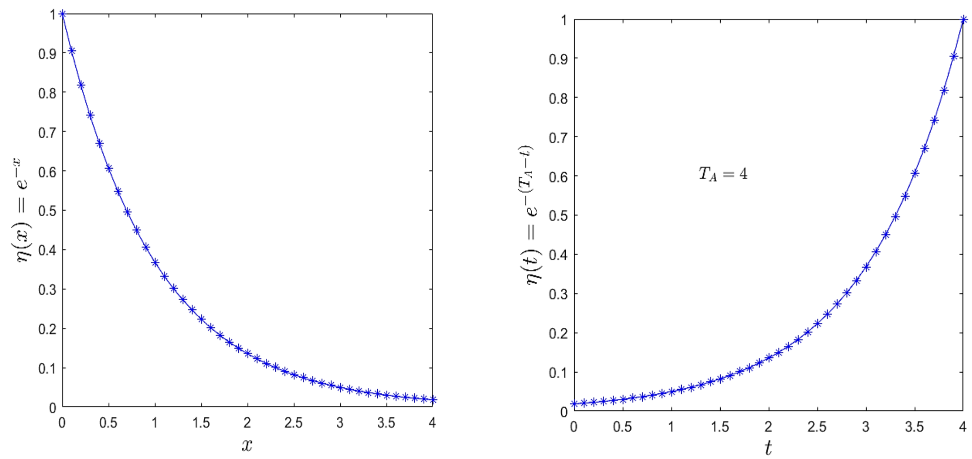


Figure 2. Graphical representation of $\eta(x)$.

5. Mathematical Model

Figure 3 depicts the inventory cycle. At time $t = 0$ the inventory level is zero and till the time T_A, advance order booking is taken at the rate $D(p_A, m)$, some of advance booking orders are cancelled due to waiting period at the rate $e^{-\lambda(T_A - t)}$. Thus, resulting rate change in the inventory level during $[0, T_A]$ is $D(p_A, m). e^{-\lambda(T_A - t)}$. At time T_A, total orders cancelled are N_C and the demand due to advance orders gets cumulate to the level I_{CA}. Immediately after time T_A, replenishment order quantity $Q = I_{MAX} + I_{CA}$ arrives and the advance booking orders are satisfied, and the resulting inventory level is I_{MAX}. Then during $[T_A, T]$, the inventory level decreases due to spot sales at a rate $D(p, m)$ and reaches to zero at time T.

The firm deposits the accumulated revenue from advance booking sales during the period $(0, T_A)$ into an account that earns an interest rate of I_e. At T_A the accounts have to be settled, it is assumed that accounts will be settled by proceeds of sells generated up to T_A and by taking a short term loan at an interest rate of I_p for the duration of $(T - T_A)$ for financing the unsold stock.

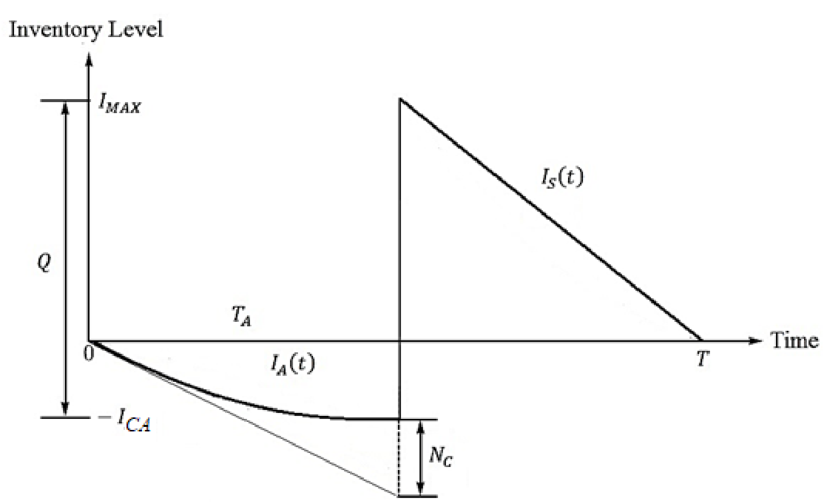


Figure 3. Inventory cycle graphical representation.

During the time interval $[0, T_A]$, due to advance booking and order cancellation, the inventory level changes at the rate $D(p_A, m). e^{-\lambda(T_A - t)}$, thus, the following differential equation governs the inventory level,

$$\frac{dI_A(t)}{dt} = -D(p_A, m). e^{-\lambda(T_A - t)}; \qquad 0 \le t \le T_A \tag{1}$$

Substituting $D(p_A, m) = (a - bp_A)(\alpha + \beta m) = (a - b\gamma p)(\alpha + \beta m)$ in above equation, the resulting equation is,

$$\frac{dI_A(t)}{dt} = -(a - b\gamma p)(\alpha + \beta m). e^{-\lambda(T_A - t)}; \qquad 0 \le t \le T_A \tag{2}$$

Upon applying the boundary condition $\mathrm{I_A}(0) = 0$, the solution is obtained as follows:

$$I_A(t) = -\frac{1}{\lambda}(a - b\gamma p)(\alpha + \beta m)\left(e^{-\lambda(T_A - t)} - e^{-\lambda T_A}\right); \qquad 0 \le t \le T_A \tag{3}$$

During the time interval $[T_A, T]$, inventory level depletes due to spot sales at the rate $D(p, m)$, thus, the following differential equation governs the inventory level,

$$\frac{dI_S(t)}{dt} = -D(p, m); \qquad T_A \le t \le T \tag{4}$$

Substituting $D(p_A, m) = (a - bp)(\alpha + \beta m)$ in the above equation, the resulting equation is,

$$\frac{dI_S(t)}{dt} = -(a - bp)(\alpha + \beta m); \qquad T_A \le t \le T \tag{5}$$

Upon applying the boundary condition $I_S(T) = 0$, the solution is obtained as follows:

$$I_S(t) = (a - bp)(\alpha + \beta m)(T - t); \qquad T_A \le t \le T \tag{6}$$

Now, the order quantity is given as,

$$Q = I_{MAX} + I_{CA} \tag{7}$$

where, I_{MAX} is the maximum level of inventory left after satisfying advance sales demand given by,

$$I_{MAX} = I_S(T_A).$$

and I_{CA} is the amount of cumulated advance orders at time T_A given by $I_{CA} = -I_A(T_A)$.

Thus,

$$Q = (a - bp)(\alpha + \beta m)(T - T_A) + \frac{1}{\lambda}(a - b\gamma p)(\alpha + \beta m)\left(1 - e^{-\lambda T_A}\right) \tag{8}$$

$$= (\alpha + \beta m)\left[(a - bp)(T - T_A) + \frac{1}{\lambda}(a - \gamma bp)\left(1 - e^{-\lambda T_A}\right)\right] \tag{9}$$

Number of total orders during advance booking period $[0, T_A]$, N_A is given as,

$$N_A = (a - b\gamma p)(\alpha + \beta m)T_A \tag{10}$$

Total demand during spot sales period $[T_A, T]$, N_S is given as,

$$N_S = (a - bp)(\alpha + \beta m)(T - T_A) \tag{11}$$

Number of orders cancelled, N_C is given by,

$$N_C = N_A - I_{CA} = N_A + I_A(T_A) = (a - b\gamma p)(\alpha + \beta m)\left[T_A - \frac{1}{\lambda}\left(1 - e^{-\lambda T_A}\right)\right] \tag{12}$$

Total Profit is given by

Total profit= Total Revenue from advance and spot sales−Refund for order cancellation− Ordering cost − Purchase cost−Advertisement cost−Holding cost +Internest earned−Interest payable− Cost of maintaining advance booking system.

The present worth of total profit per unit time is calculated as follows:

(i) Present worth of Interest earned per unit time: The firm deposits the accumulated revenue from cash sales during the period $(0, T_A)$ into an account that earns an interest rate of I_e.

$$IE = \frac{I_e p_A}{T}\left\{\int_0^{T_A} D(p_A, m)\, e^{-zt} dt - \int_0^{T_A} D(p_A, m)\left[1 - e^{-\lambda(T_A - t)}\right] e^{-zt} dt\right\},$$

$$= \frac{I_e p_A}{T}(a - b\gamma p)(\alpha + \beta m)\left\{\frac{1 - e^{-zT_A}}{z} - \left\{\frac{1 - e^{-zT_A}}{z} + e^{-\lambda T_A}\,\frac{(1 - e^{(\lambda - z)T_A})}{\lambda - z}\right\}\right\},$$

$$= \frac{I_e p_A}{T}(a - b\gamma p)(\alpha + \beta m) e^{-\lambda T_A}\,\frac{(1 - e^{(\lambda - z)T_A})}{z - \lambda}.$$

(ii) Present worth of interest payable per unit time: The firm takes short term loan at an interest rate of I_p for the duration of $(T - T_A)$ for financing the unsold stock.

$$IP = \frac{I_p}{T}\left\{cQ - p_A\int_0^{T_A} D(p_A, m)\, e^{-zt} dt - I_e p_A\left[\int_0^{T_A} D(p_A, m)\, e^{-zt} dt - \int_0^{T_A} D(p_A, m)\left[1 - e^{-\lambda(T_A - t)}\right]\right]\right\},$$

$$= \frac{I_p}{T}(\alpha + \beta m)\left\{c\left[(a - bp)(T - T_A) + \frac{1}{\lambda}(a - \gamma bp)\left(1 - e^{-\lambda T_A}\right)\right] - \gamma p(a - b\gamma p)\frac{1 - e^{-zT_A}}{z}\right.$$
$$\left. - I_e \gamma p(a - b\gamma p) e^{-\lambda T_A}\,\frac{\left(1 - e^{(\lambda - z)T_A}\right)}{\lambda - z}\right\}.$$

(iii) Present worth of sales revenue per unit time from advance sales,

$$= \frac{p_A}{T}\int_0^{T_A} D(p_A, m)\, e^{-zt} dt = \frac{\gamma p(a - b\gamma p)(\alpha + \beta m)}{T}\frac{(1 - e^{-zT_A})}{z}.$$

(iv) Present worth of sales revenue per unit time from spot sales,

$$= \frac{pe^{-zT_A}}{T}\int_{T_A}^{T} D(p, m)\, e^{-zt} dt = \frac{p(a - bp)(\alpha + \beta m) e^{-zT_A}}{T}\frac{(e^{-zT_A} - e^{-zT})}{z}.$$

(v) Present worth of refund per unit time due to order cancellation (customers are issued full refund on order cancellation),

$$= \frac{p_A}{T}\int_0^{T_A} D(p_A, m)\left[1 - e^{-\lambda(T_A - t)}\right] e^{-zt} dt.$$

$$= \frac{\gamma p(a - b\gamma p)(\alpha + \beta m)}{T}\left\{\frac{1 - e^{-zT_A}}{z} + e^{-\lambda T_A}\,\frac{(1 - e^{(\lambda - z)T_A})}{\lambda - z}\right\}.$$

(vi) Present worth of cost of out-of-pocket inventory carrying per unit time,

$$= \frac{Ice^{-zT_A}}{T}\int_{T_A}^{T} I_S(t)\, e^{-zt} dt.$$

$$= \frac{Ic(a - bp)(\alpha + \beta m) e^{-zT_A}}{T}\left\{(T - T_A)\frac{e^{-zT_A}}{z} - \frac{(e^{-zT_A} - e^{-zT})}{z^2}\right\}.$$

(vii) Present worth of ordering cost per unit time (since order is placed at time T_A)

$$= \frac{Ae^{-zT_A}}{T}.$$

(viii) Present worth of purchase cost per unit time,

$$= \frac{cQe^{-zT_A}}{T}.$$

(ix) Present worth of advertisement cost per unit time,

$$= \frac{mQ}{T}\int_0^T e^{-zt}\,dt = \frac{mQ}{T}\frac{(1-e^{-zT})}{z}.$$

(x) Present worth of cost of maintaining advance booking system per unit time,

$$\frac{\Phi}{T}\int_0^{T_A} D(p_A,m)\,e^{-zt}dt + \Phi'\int_0^{T_A} D(p_A,m)\left[1-e^{-\lambda(T_A-t)}\right]e^{-zt}dt.$$

$$= \frac{\Phi(a-b\gamma p)(\alpha+\beta m)}{T}\frac{(1-e^{-zT_A})}{z} + \frac{\Phi'(a-b\gamma p)(\alpha+\beta m)}{T}\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\,\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\}.$$

The total profit per unit time is obtained by utilizing the aforementioned components 1-10.

$$\text{Total profit per unit time} = \Pi(p,T,m) = \frac{\gamma p(a-b\gamma p)(\alpha+\beta m)}{T}\frac{(1-e^{-zT_A})}{z} + \frac{p(a-bp)(\alpha+\beta m)e^{-zT_A}}{T}\frac{(e^{-zT_A}-e^{-zT})}{z} - \frac{\gamma p(a-b\gamma p)(\alpha+\beta m)}{T}\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\,\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} - \frac{Ae^{-zT_A}}{T} - \frac{ce^{-zT_A}}{T}(\alpha+\beta m)\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - \frac{m}{T}\frac{(1-e^{-zT})}{z}(\alpha+\beta m)\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - \frac{Ic(a-bp)(\alpha+\beta m)e^{-zT_A}}{T}\left\{(T-T_A)\frac{e^{-zT_A}}{z} - \frac{(e^{-zT_A}-e^{-ZT})}{z^2}\right\} + \frac{I_e\gamma p}{T}(a-b\gamma p)(\alpha+\beta m)e^{-\lambda T_A}\,\frac{(1-e^{(\lambda-z)T_A})}{z-\lambda} - \frac{I_p}{T}(\alpha+\beta m)\left\{c\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - \gamma p(a-b\gamma p)\frac{1-e^{-zT_A}}{z} - I_e\gamma p(a-b\gamma p)e^{-\lambda T_A}\,\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} - \frac{\Phi(a-b\gamma p)(\alpha+\beta m)}{T}\frac{(1-e^{-zT_A})}{z} - \frac{\Phi'(a-b\gamma p)(\alpha+\beta m)}{T}\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\,\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} \tag{13}$$

Now, the profit maximization problem to find the optimal ordering, pricing and advertisement expenditure is given by,

Maximize $\Pi(\mathrm{p},\mathrm{T},\mathrm{m})$
subject to:

$$(\alpha+\beta m)\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m \le M \tag{14}$$

$$T > T_A \ge 0 \tag{15}$$

$$p, m \ge 0 \tag{16}$$

Constraint (14) corresponds to the advertisement budget constraint, where M is the maximum budget of the firm to invest in the advertisement. Constraint (15) ensures that the advance order booking period doesn't exceed the total inventory cycle length, and constraint (16) is the non-negativity constraint.

6. Optimality and Solution Procedure

6.1 Optimal Solution

In this section, the concavity of the objective function (profit) is examined to demonstrate the existence and uniqueness of a global optimum. The proof of concavity requires the demonstration of a negative, definite Hessian matrix. However, due to the high complexity of the profit function (as seen in the Appendix), it is not feasible to establish its concavity through the Hessian matrix. Further, because the profit function is three variable dependent (p, T, and m), it is also not possible to establish concavity graphically. Therefore, a heuristic approach is adopted.

Using the heuristic approach, one variable is kept constant at a time, and the concavity of the profit function with respect to the other two variables is examined. To begin, the concavity of $\Pi(T,p|m)$ in T and p is checked while keeping m fixed. Similarly, for a given T and p, the concavity of $\Pi(m|T,p)$ in m is also examined. As for $\Pi(T,p|m)$, its concavity can be evaluated using the following approach:

$$H = \begin{bmatrix} \frac{\partial \Pi}{\partial T^2} & \frac{\partial \Pi}{\partial T \partial p} \\ \frac{\partial \Pi}{\partial T \partial p} & \frac{\partial \Pi}{\partial p^2} \end{bmatrix}.$$

where, $\frac{\partial \Pi}{\partial T^2}, \frac{\partial \Pi}{\partial T \partial p}$ and $\frac{\partial \Pi}{\partial p^2}$ are given in Appendix.

Because of the complex expressions, it is analytically not possible to establish that Hessian matrix is negative definite. Therefore, its concavity is established graphically through three dimensional plot using MATLAB 2020 (see Figure 4(a)). Next, the concavity of $\Pi(m|T,p)$ is examined.

$$\frac{\partial^2 \Pi(m|T,p)}{\partial m^2} = -2\frac{\beta}{T}\left\{\frac{(1-e^{-zT})}{z}\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]\right\} < 0.$$

Consequently, $\Pi(m|T,p)$ is concave in m , given (T,p). Now, fixing p, the concavity of $\Pi(T,m|p)$ in T and m is established.
For, $\Pi(T,m|p)$,

$$H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial T^2} & \frac{\partial^2 \Pi}{\partial T \partial m} \\ \frac{\partial^2 \Pi}{\partial T \partial m} & \frac{\partial^2 \Pi}{\partial m^2} \end{bmatrix}.$$

where, $\frac{\partial \Pi}{\partial T^2}, \frac{\partial \Pi}{\partial T \partial m}$ and $\frac{\partial \Pi}{\partial m^2}$ are given in Appendix.

Because of the complex expressions, it is analytically not possible to establish that Hessian matrix is negative definite. Therefore, its concavity is established graphically through three dimensional plot using MATLAB 2020 (see Figure 4(b)). Next, the concavity of $\Pi(p|T,m)$ is examined.

$$\frac{\partial^2 \Pi(p|T,m)}{\partial p^2} = -2b\frac{\alpha+\beta m}{T}\left\{\gamma^2\frac{(1-e^{-zT_A})}{z} + \boldsymbol{e^{-zT_A}}\frac{(e^{-zT_A}-e^{-zT})}{z} - \gamma^2\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{\lambda - z}\right\} + I_e\gamma^2 e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{z-\lambda} + I_p\gamma^2\left\{\frac{1-e^{-zT_A}}{z} + I_e e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\}\right\} < 0.$$

Consequently, $\Pi(p|T,m)$ is concave in p, given (T,m). Now, fixing T, the concavity of $\Pi(p,m|T)$ in p and m is examined.
For, $\Pi(p,m|T)$,

$$H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial p^2} & \frac{\partial^2 \Pi}{\partial p \partial m} \\ \frac{\partial^2 \Pi}{\partial p \partial m} & \frac{\partial^2 \Pi}{\partial m^2} \end{bmatrix}.$$

where, $\frac{\partial \Pi}{\partial p^2}, \frac{\partial \Pi}{\partial p \partial m}$ and $\frac{\partial \Pi}{\partial m^2}$ are given in Appendix.

Because of the complex expressions, it is analytically not possible to establish that Hessian matrix is negative definite. Therefore, its concavity is established graphically three dimensional plot using MATLAB 2020 (see Figure 4(c)). Next, the concavity of $\Pi(T|p,m)$ is exammined.

Because of the complex expressions, it is not possible to establish that $\frac{\partial^2 \Pi(T|p,m)}{\partial T^2} < 0$. Therefore, its concavity is established graphically.

Combining the above findings, the following solution algorithm is proposed.

6.2 Solution Procedure

The problem involves three variables: m, p and T, which can be obtained using following algorithm. Note that these variables are denoted as $var_j; j = 1,2,3$.

Step 1. Starting at $s = 0$, and $j = 1$ initiate the trial value of $var_j^s = v_j$(say), where v_j is the lower bound on var_j.

Step 2. Obtain var_i^* and var_k^* $i \neq j, k \neq j, i \neq k; i, k = 1,2,3$ for a given v_j by solving corresponding KKT conditions (given in Appendix).

Step 3. Using var_i^* and var_k^* obtained in Step 2, compute the optimal var_j^{s+1} by solving corresponding KKT conditions (given in Appendix).

Step 4. If $\left|var_j^s - var_j^{s+1}\right| < \epsilon$, where ϵ is an arbitrary small number, then $var_j^* = var_j^{s+1}$, and $(var_i^*, var_j^*, var_k^*)$ is the optimal solution, and procedure terminates. Else set $s = s + 1$, and repeat from Step 2.

Step 5. Perform the Steps 1-4 for all $j = 1,2,3$. Compute the total profit Π from Equation (13) in the three cases. Then, optimal total profit Π^* is the maximum and corresponding solution is optimal.

Note: Since the KKT conditions are highly non-linear, MATLAB solver "fsolve" is used to solve them. It uses three different algorithms (Trust-region, Trust-region Dogleg, Levenberg Marquardt), all three were converging to the same solution.

7. Numerical Illustration

In this section, numerical example and sensitivity analysis is presented.

7.1 Numerical Example

Following parameters have been assumed to perform the numerical:
M=\$5,000, $\gamma = 90\%$, $T_A = 6$ weeks, I = \$0.06/unit/week, c = \$10/unit, A = \$100/order, $a = 750, b = 7.5, \alpha = 1.1, \beta = 1.6, \lambda = 0.05$, Φ = \$0.01/unit/week,, Φ' = \$0.03/unit/week,, $I_e = 8\%$, $I_p = 15\%$, $z = 2\%$.

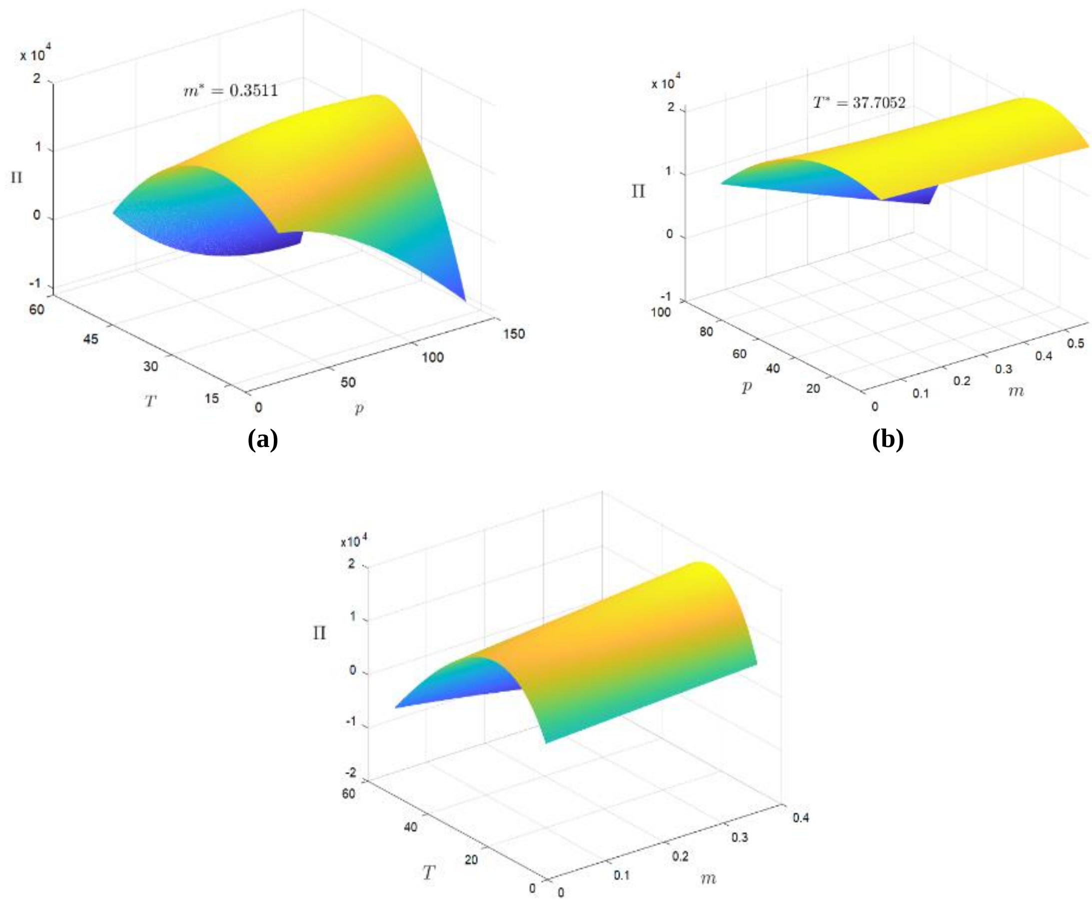
Solution

Solving using the solution algorithm mentioned in previous section, the results are as follows in Table 2.

Table 2. Computational results based on proposed algorithm.

var_j	Optimal Solution			
	m^*	p^*	T^*	Π^*
$\boldsymbol{m}$	**0.3456**	**69.2385**	**37.5856**	**10893.45**
p	0.3431	69.2301	37.5070	10891.49
T	0.3432	69.2321	37.5075	10892.57

Therefore,
$p^* = \$69.2385$ per unit, $T^* = 37.5856$ weeks, $m^* = \$0.3456$ per unit, $Q^* = 14467$ units, $N_C = 381$ units and $\Pi^* = \$\ 10893.45$.



(c)

Figure 4. (a), (b), (c) Concavity of objective function.

7.2 Sensitivity Analysis

Now the effects of changes in the value of parameters γ, h, T_A and c on p^*, T^*, m^*, Q^* and Π^* based on the Example above is studied. First, the impact of discount and compute optimal solutions at the different discount levels is studied. The value of $(1-\gamma)$ is varied as $\{0, 2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20\}$ (see Table 3), $(1-\gamma) = 0$ being the case of no discount given to the customers for booking the product. Computations in Table 3 and Figure 5 show that as discount increases, the total profit also increases, and after attaining its maximum at a certain point, it starts decreasing. This is because offering some discount fetches more sales revenue; however, if the discount offered is too high, the retailer incurs a loss in profit, which is not compensated with the sales revenue generated. Therefore, setting an appropriate discount is very important for the firm to benefit from it.

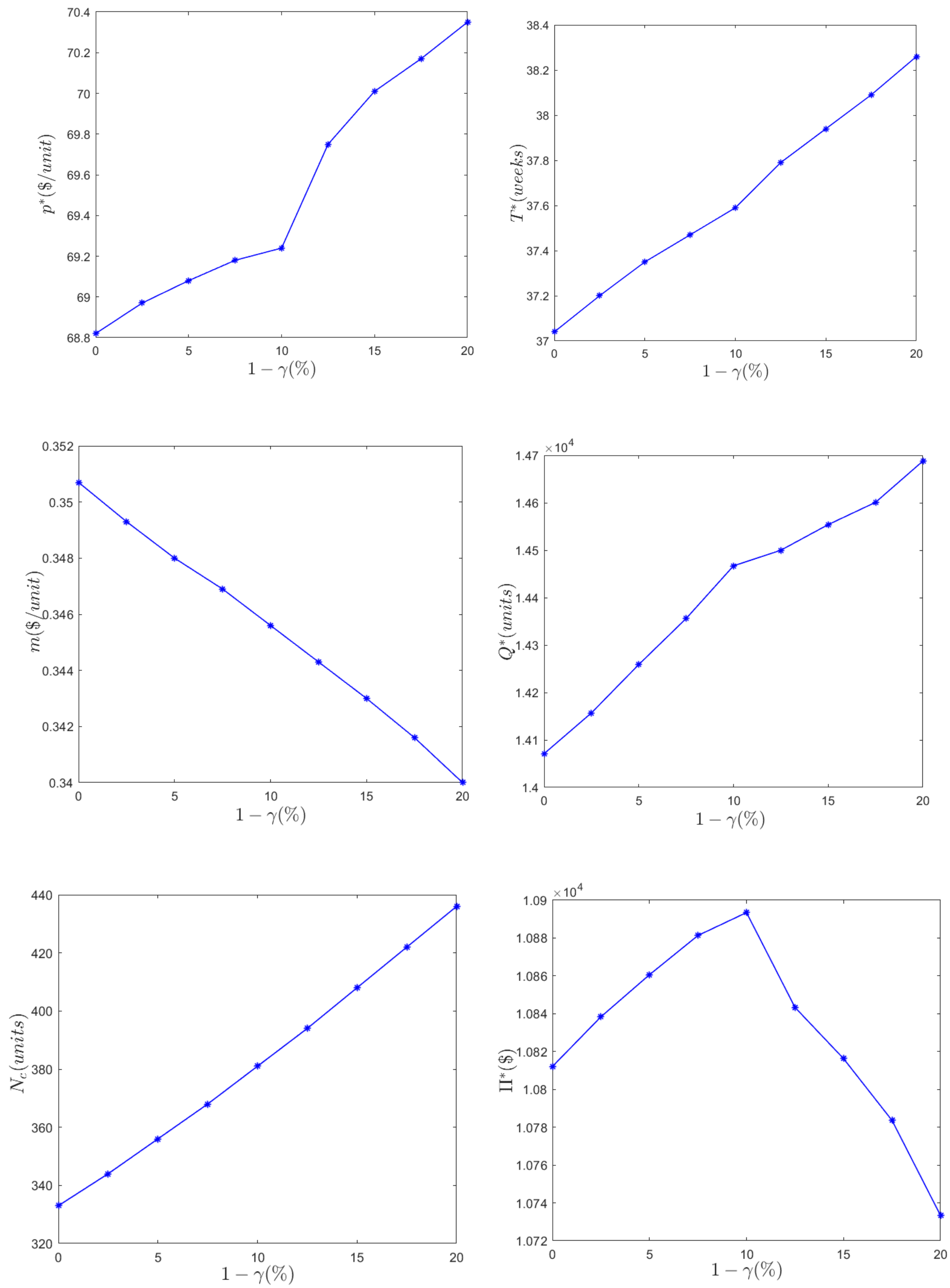


Figure 5. Impact of discount on selling price, cycle length, advertisement expenditure, order cancellations, order quantity and profit.

Now, the effects of various parameters including $h, T_A, c, \Phi, \Phi', I_e, I_p$ are studied. To conduct a sensitivity analysis, each parameter is individually altered by +50%, +25%, -25%, and -50%, while holding the other parameters constant. The resulting computational findings are presented in Table 4.

Table 3. Computational results for sensitivity analysis with respect to discount.

$1-\gamma$ (in %)	p^* ($/unit)	T^* (weeks)	m^*($/unit)	Q^*(units)	N_c(units)	Π^*($)
00.00	68.82	37.04	0.3507	14071	333	10812.08
02.50	68.97	37.20	0.3493	14157	344	10838.35
05.00	69.08	37.35	0.3480	14259	356	10860.58
07.50	69.18	37.47	0.3469	14356	368	10881.35
10.00	69.24	37.59	0.3456	14467	381	10893.45
12.50	69.75	37.79	0.3443	14500	394	10843.31
15.00	70.01	37.94	0.3430	14554	408	10816.25
17.50	70.17	38.09	0.3416	14601	422	10783.56
20.00	70.35	38.26	0.3400	14688	436	10733.28

Table 4. Computational results for Sensitivity analysis with respect to inventory parameters.

Parameter	Percentage change	p_j^*	T^*	m^*	Q^*	N_c	Π^*
		Percentage change					
I	**-50**	-1.81	27.64	-19.27	23.87	-3.55	11.40
	-25	-0.79	11.82	-9.20	10.14	-1.71	5.02
	0	0	0	0	0	0	0
	+25	0.62	-9.25	8.50	-7.83	1.88	-4.06
	+50	1.14	-16.69	16.41	-14.09	3.60	-7.43
T_A	**-50**	0.061	-3.08	2.24	-2.19	-73.55	-5.39
	-25	0.012	-1.48	0.96	-0.94	-42.16	-3.19
	0	0	0	0	0	0	0
	+25	-0.025	1.32	-0.65	0.66	52.49	3.09
	+50	-0.058	2.51	-1.01	1.03	114.37	6.12
c	**-50**	-7.11	54.23	-36.00	56.25	-1.60	36.89
	-25	-3.18	21.39	-17.90	21.81	-0.94	15.63
	0	0	0	0	0	0	0
	+25	2.73	-15.35	18.08	-15.31	1.37	-12.16
	+50	5.15	-27.00	36.52	-26.75	2.76	-21.98
Φ	**-50**	-0.02	-0.05	-0.005	0.003	0.1	0.08
	-25	-0.01	-0.03	-0.003	0.001	0.08	0.07
	0	0	0	0	0	0	0
	+25	0.01	0.02	0.007	-0.003	0.07	-0.07
	+50	0.02	0.03	0.009	-0.005	0.09	-0.09
Φ'	**-50**	-0.07	0.004	-0.006	0.007	-0.10	0.04
	-25	-0.04	0.002	-0.005	0.003	-0.08	0.02
	0	0	0	0	0	0	0
	+25	0.05	0.002	0.005	-0.004	0.09	-0.03
	+50	0.07	0.004	0.007	-0.008	0.12	-0.05
I_e	**-50**	0.07	0.24	-0.06	0.07	-0.22	1.05
	-25	0.04	0.12	-0.03	0.04	-0.12	0.52
	0	0	0	0	0	0	0
	+25	-0.04	-0.14	0.04	-0.03	0.18	-0.52
	+50	-0.08	-0.26	0.07	-0.06	0.26	-1.04
I_p	**-50**	-0.50	2.94	-2.94	3.03	-0.25	0.06
	-25	-0.25	1.46	-1.48	1.51	-0.18	0.02
	0	0	0	0	0	0	0
	+25	0.24	-1.49	1.51	-1.48	0.21	0.02
	+50	0.48	-2.96	3.04	-2.95	0.31	0.07

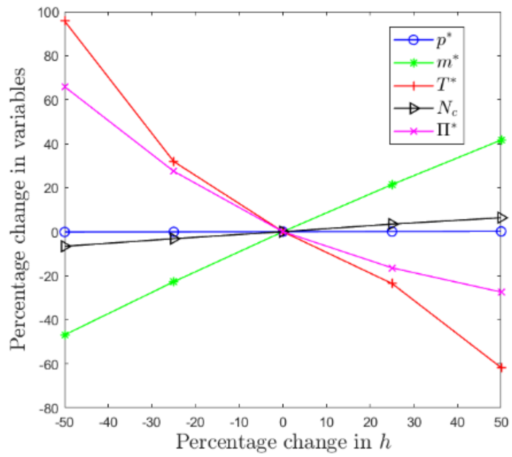


Figure 6. Graphical representation for sensitivity analysis w.r.t. h.

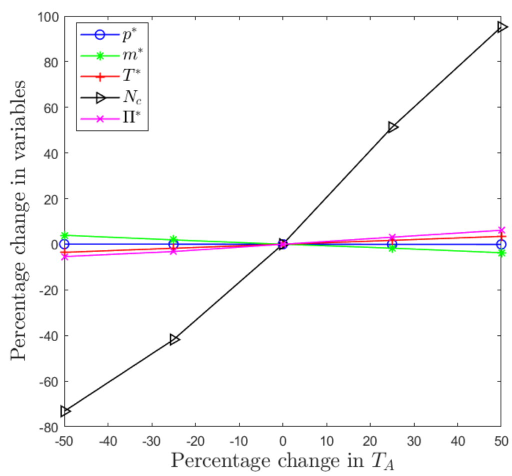


Figure 7. Graphical representation for sensitivity analysis w.r.t. T_A.

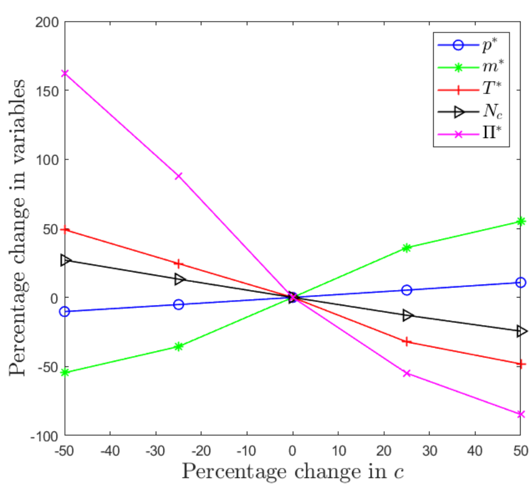


Figure 8. Graphical representation for sensitivity analysis w.r.t. c.

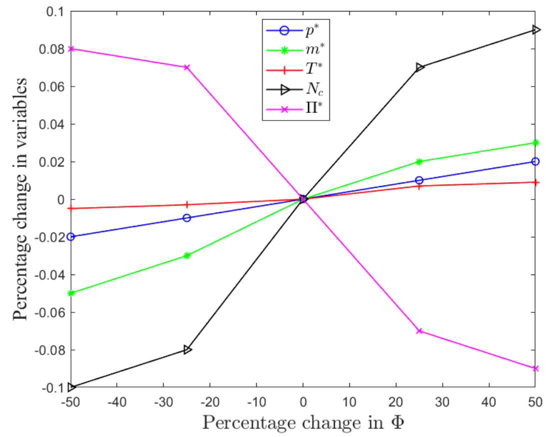


Figure 9. Graphical representation for sensitivity analysis w.r.t. Φ.

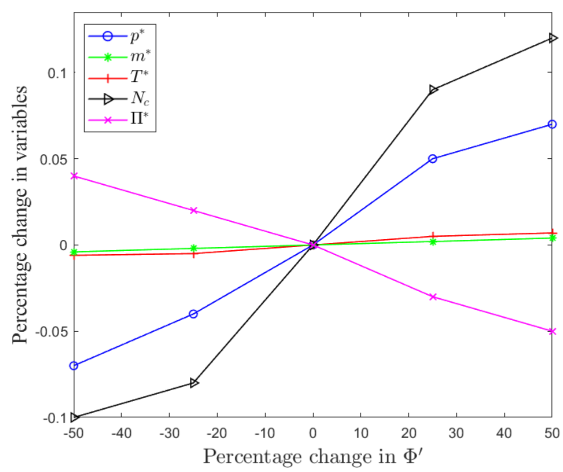


Figure 10. Graphical representation for sensitivity analysis w.r.t. Φ'.

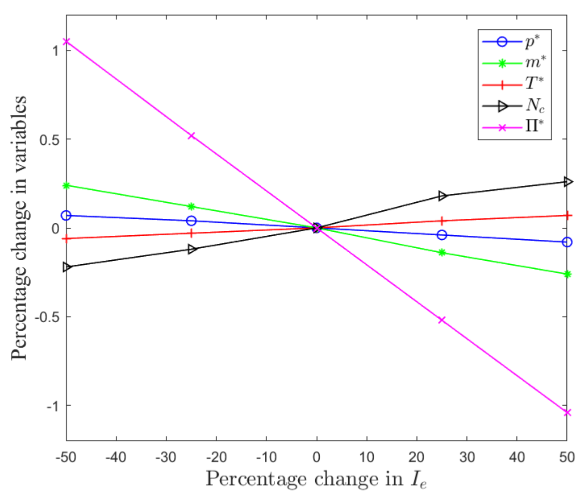


Figure 11. Graphical representation for sensitivity analysis w.r.t. I_e.

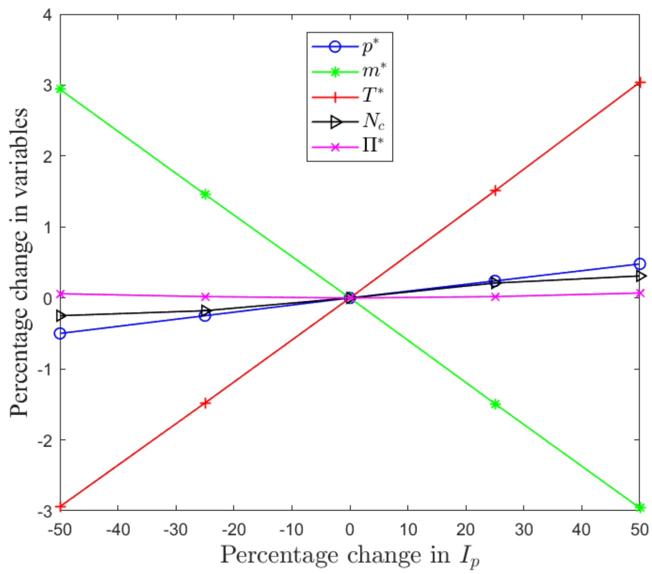


Figure 12. Graphical representation for sensitivity analysis w.r.t. I_p.

The sensitivity analysis shown in Table 4 and Figures 6-12 indicate the following observations:

(i) Table 4 and Figure 6 illustrate that advertisement expenditure, order quantity, and total profit are highly sensitive to changes in unit holding cost, whereas the number of orders cancelled and selling price are less sensitive. The results show that an increase in unit holding cost leads to a decrease in order quantity and total profit, and an increase in advertisement expenditure investment. The findings confirm that a higher holding cost typically reduces the inventory cycle length, which can have a negative impact on the total profit of the firm. Therefore, managing holding costs can help the firm optimize its inventory management process and increase its total profit.

(ii) Table 4 and Figure 7 results indicate that the number of orders cancelled is highly sensitive to changes in advance booking period length, while order quantity, selling price, advertisement expenditure, and total profit are less sensitive. The results suggest that if the advance booking period length is increased, customers tend to cancel more orders as the waiting time for delivery increases. However, it is interesting to note that although the number of orders cancelled has increased, the total profit has also increased with an increase in advance booking period length. This implies that offering an advance sales discount along with an extended advance booking period may benefit firms gain more profit. Overall, the findings highlight the importance of carefully managing the advance booking period length to optimize inventory management and increase profitability.

(iii) Table 4 and Figure 8 results indicate that advertisement expenditure, order quantity, number of orders cancelled, total profit, and unit selling price are significantly affected by changes in the unit cost of the item. As the unit cost of the item increases, the selling price also increases, while the total profit decreases. Moreover, an increase in unit cost leads to a decrease in order quantity, and more orders are cancelled during the advance order booking period. The findings highlight the importance of carefully managing the unit cost of the item to optimize inventory management and profitability. It is essential to balance the selling price with the unit cost of the item to ensure that the firm can generate maximum revenue while maintaining profitability.

(iv) Table 4 and Figures 9-10 suggest that the model is not highly sensitive to changes in the per-unit advance booking cost. When the per-unit advance booking cost increases, the unit selling price and total inventory cycle length increase marginally, while the per-unit advertisement investment, number of order cancellations, and total profit decrease marginally. This indicates that the changes in the per-unit advance booking cost have a relatively small impact on the overall performance of the inventory management model. The marginal changes in unit selling price and total inventory cycle length suggest that the model is flexible enough to absorb changes in the per-unit advance booking cost without significantly affecting its performance. The decrease in per-unit advertisement investment and number of order cancellations suggests that the model may be more cost-effective with higher per- unit advance booking costs.

(v) Table 4 and Figures 11-12 suggest that the expected changes in the variables occur as the rate of interest earned or payable is increased. As the rate of interest earned increases, the unit selling price and per-unit advertisement expenditure decrease, and the profit increases. This can be explained by the fact that a higher rate of interest earned on the advance sales payments enables the firm to finance its inventory requirements at a lower cost. This, in turn, allows the firm to reduce its unit selling price and advertisement expenditure, while still maintaining a reasonable profit margin. On the other hand, an increase in the rate of interest payable has the opposite effect. The unit selling price and per-unit advertisement expenditure increase, while the profit decreases. This is because a higher rate of interest payable on loans taken to finance inventory requirements increases the cost of inventory, and the firm needs to increase the selling price and advertisement expenditure to maintain a reasonable profit margin.

8. Managerial Insights

The managerial insights gleaned from this study are critical for companies operating in the rapidly growing e-commerce industry. First, the optimization model presented in this paper offers a comprehensive approach to decision-making that integrates inventory, pricing, and advertising. By considering these factors simultaneously, firms can make more informed decisions that lead to improved profitability. Secondly, the model's incorporation of interest earned and payable is a critical component that can help managers better understand the overall profit function. By considering the time value of money, companies can make more informed decisions regarding their use of capital, such as investing in advertising or repaying loans. Thirdly, the study highlights the potential benefits of offering advance booking discounts. By incentivizing customers to book in advance, firms can drive demand and increase revenue. However, the analysis also underscores the importance of setting an appropriate discount level to achieve maximum profit. A discount that is too high may generate more revenue but fail to cover total costs, resulting in reduced profitability.

Finally, the study emphasizes the importance of balancing revenue and cost considerations when making pricing and advertising decisions. While increasing prices or advertising budgets may boost revenue, it is critical to consider the associated costs and potential impact on profitability. Particularly, an e-commerce company that offers advanced booking discounts could leverage the insights from this study to optimize its pricing, inventory, and advertising decisions. By using the optimization model presented in this paper, the company could determine the appropriate discount level, pricing strategy, and advertising budget that maximize profitability. The company could also leverage the incorporation of interest earned and payable to better understand the overall impact of its financial decisions on its profitability.

9. Conclusion

In order to thrive in the rapidly expanding e-commerce industry, companies must prioritize providing exceptional customer service and facilitating smooth product purchases. Marketing strategies such as easy order cancellation, full refunds, and advertisements are key components of driving product demand, which in turn influences inventory decisions. This paper presents a mathematical model that simultaneously optimizes inventory, pricing, and advertising decisions in a scenario where customers make advance payments at a discounted price with the option of full refunds for cancelled orders prior to the scheduled delivery date. The demand for the product is affected by the amount invested in advertising, the selling price, and the number of order cancellations. The DCF approach is utilized to accurately calculate various components of the profit function as it recognizes different cash flows occurring at different time points.

Additionally, the model considers the interest earned and payable by firms, as they can increase profits by earning interest on advance payments and repaying loans used to purchase stock. Numerical analysis reveals some noteworthy insights, including the potential benefits of offering advance booking discounts to increase overall demand and generate higher revenue. Results suggest that the rate of interest earned and payable has a significant impact on the performance of the model. Therefore, it is essential for firms to carefully manage their interest expenses to improve their profitability. Another key takeaway is that it is crucial to set the appropriate discount level to maximize profit, as setting a discount too high may generate more revenue but fail to cover total costs. Moreover, the paper's managerial insights stress the importance of joint decision-making concerning marketing strategies and inventory management within the e-commerce industry.

This study has certain limitations that need to be addressed in future research. For instance, the model assumes a constant demand rate with respect to time and does not consider seasonal variations in demand, which may affect the accuracy of the results. The model does not take into account the product return policy commonly used by e-commerce companies. Product returns can significantly impact inventory decisions and should be considered in future research. Additionally, the model does not consider the impact of external factors such as competition and market trends on the inventory system. Hence, future research should focus on developing more comprehensive models that incorporate these external factors and consider dynamic demand patterns to improve the accuracy of the results.

Conflict of Interest

The authors declare no conflict of interest.

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APPENDIX

Hessian Matrix

$$H = \begin{bmatrix} \frac{\partial^2 \Pi}{\partial p^2} & \frac{\partial^2 \Pi}{\partial p \partial m} & \frac{\partial^2 \Pi}{\partial p \partial T} \\ \frac{\partial^2 \Pi}{\partial p \partial m} & \frac{\partial^2 \Pi}{\partial m^2} & \frac{\partial^2 \Pi}{\partial m \partial T} \\ \frac{\partial^2 \Pi}{\partial p \partial T} & \frac{\partial^2 \Pi}{\partial m \partial T} & \frac{\partial^2 \Pi}{\partial T^2} \end{bmatrix}.$$

where,

$$\frac{\partial^2\Pi}{\partial p^2} = -2b\frac{\alpha+\beta m}{T}\left\{\gamma^2\frac{(1-e^{-zT_A})}{z} + \boldsymbol{e^{-zT_A}}\frac{(e^{-zT_A}-e^{-zT})}{z} - \gamma^2\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} + I_e\gamma^2 e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{z-\lambda} + I_p\gamma^2\left\{\frac{1-e^{-zT_A}}{z} + I_e e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\}\right\} < 0.$$

$$\frac{\partial^2\Pi}{\partial T^2} = 2\frac{\alpha+\beta m}{T^3}\left\{\gamma p(a-b\gamma p)\frac{(1-e^{-zT_A})}{z} + p(a-bp)\boldsymbol{e^{-zT_A}}\frac{(e^{-zT_A}-e^{-zT})}{z} - \gamma p(a-b\gamma p)\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} - Ic(a-bp)e^{-zT_A}\left\{(T-T_A)\frac{e^{-zT_A}}{z} - \frac{(e^{-zT_A}-e^{-ZT})}{z^2}\right\} - ce^{-zT_A}\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - m\frac{(1-e^{-zT})}{z}\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] + I_e\gamma p(a-b\gamma p)e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{z-\lambda} - I_p\left\{c\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - p_A(a-b\gamma p)\frac{1-e^{-zT_A}}{z} - I_e\gamma p(a-b\gamma p)e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} - \Phi(a-b\gamma p)\frac{(1-e^{-zT_A})}{z} - \Phi'(a-b\gamma p)\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\}\right\} - 2\frac{\alpha+\beta m}{T^2}\left\{\gamma p(a-bp)\boldsymbol{e^{-zT_A}}e^{-zT} - Ic(a-bp)e^{-zT_A}\left\{\frac{e^{-zT_A}}{z} - \frac{e^{-zT}}{z}\right\} - ce^{-zT_A}(a-bp) - me^{-zT}\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - m\frac{(1-e^{-zT})}{z}(a-bp) - I_p[c(a-bp) - p_A(a-b\gamma p)e^{-zT}]\right\} + \frac{\alpha+\beta m}{T}\left\{-zp(a-bp)\boldsymbol{e^{-zT_A}}e^{-zT} - Ic(a-bp)e^{-zT_A}e^{-zT} - 2m(a-bp)e^{-zT} - I_p z p_A(a-b\gamma p)e^{-zT}\right\} - 2\frac{Ae^{-zT_A}}{T^3}.$$

$$\frac{\partial^2\Pi}{\partial m^2} = -2\frac{\beta}{T}\left\{\frac{(1-e^{-zT})}{z}\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]\right\} < 0.$$

$$\frac{\partial^2\Pi}{\partial T\partial p} = -\frac{\alpha+\beta m}{T^2}\left\{\gamma(a-2b\gamma p)\frac{(1-e^{-zT_A})}{z} + (a-2bp)\boldsymbol{e^{-zT_A}}\frac{(e^{-zT_A}-e^{-zT})}{z} - \gamma(a-2b\gamma p)\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} + bIce^{-zT_A}\left\{(T-T_A)\frac{e^{-zT_A}}{z} - \frac{(e^{-zT_A}-e^{-ZT})}{z^2}\right\} + bce^{-zT_A}\left[(T-T_A) + \frac{1}{\lambda}\gamma\left(1-e^{-\lambda T_A}\right)\right] + bm\frac{(1-e^{-zT})}{z}\left[(T-T_A) + \frac{1}{\lambda}\gamma\left(1-e^{-\lambda T_A}\right)\right] + I_e\gamma(a-2b\gamma p)e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{z-\lambda} - I_p\left\{-bc\left[(T-T_A) + \frac{1}{\lambda}\gamma\left(1-e^{-\lambda T_A}\right)\right] + b\gamma p\frac{1-e^{-zT_A}}{z} - I_e\gamma(a-2b\gamma p)e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} + b\gamma\Phi\frac{(1-e^{-zT_A})}{z} + b\gamma\Phi'\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\}\right\} + \frac{\alpha+\beta m}{T}\left\{(a-2bp)\boldsymbol{e^{-zT_A}}e^{-zT} + bIce^{-zT_A}\left\{\frac{e^{-zT_A}}{z} - \frac{e^{-zT}}{z}\right\} + bce^{-zT_A} \mp bme^{-zT}\left[(T-T_A) + \frac{1}{\lambda}\gamma\left(1-e^{-\lambda T_A}\right)\right] + bm\frac{(1-e^{-zT})}{z} + bI_p[c-\gamma^2 pe^{-zT}]\right\}.$$

$$\frac{\partial^2\Pi}{\partial T\partial m} = -\frac{\beta}{T^2}\left\{\gamma p(a-b\gamma p)\frac{(1-e^{-zT_A})}{z} + p(a-bp)\boldsymbol{e^{-zT_A}}\frac{(e^{-zT_A}-e^{-zT})}{z} - \gamma p(a-b\gamma p)\left\{\frac{1-e^{-zT_A}}{z} + e^{-\lambda T_A}\ \frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\right\} - Ic(a-bp)e^{-zT_A}\left\{(T-T_A)\frac{e^{-zT_A}}{z} - \frac{(e^{-zT_A}-e^{-ZT})}{z^2}\right\} - ce^{-zT_A}\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] - m\frac{(1-e^{-zT})}{z}\left[(a-bp)(T-T_A) + \frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right] + I_e\gamma p(a-$$

$b\gamma p)e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{z-\lambda} - I_p\Big\{c\Big[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big] - p_A(a-$
$b\gamma p)\frac{1-e^{-zT_A}}{z} - I_e\gamma p(a-b\gamma p)e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\Big\} - \Phi(a-b\gamma p)\frac{(1-e^{-zT_A})}{z} - \Phi'(a-b\gamma p)\Big\{\frac{1-e^{-zT_A}}{z}+$
$e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\Big\}\Big\} + \frac{\alpha+\beta m}{T^2}\frac{(1-e^{-zT})}{z}\Big[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big] + \frac{\beta}{T}\Big\{p(a-$
$bp)\boldsymbol{e^{-zT_A}}e^{-zT} - Ic(a-bp)e^{-zT_A}\Big\{\frac{e^{-zT_A}}{z}-\frac{e^{-zT}}{z}\Big\} - ce^{-zT_A}(a-bp) - me^{-zT}\Big[(a-bp)(T-T_A)+$
$\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big] - m\frac{(1-e^{-zT})}{z}(a-bp) - I_p[c(a-bp)\ - p_A(a-b\gamma p)e^{-zT}]\Big\} +$
$\frac{\alpha+\beta m}{T}\Big\{e^{-zT}\Big[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big]\frac{(1-e^{-zT})}{z}(a-bp)\Big\}.$

$\frac{\partial^2\Pi}{\partial p\partial m} = \beta\Big\{\gamma(a-2b\gamma p)\frac{(1-e^{-zT_A})}{z} + (a-2bp)e^{-zT_A}\frac{(e^{-zT_A}-e^{-zT})}{z} - \gamma(a-2b\gamma p)\Big\{\frac{1-e^{-zT_A}}{z} -$
$e^{-\lambda T_A}\frac{(1-e^{(\lambda-z)T_A})}{\lambda-z}\Big\} + bhe^{-zT_A}\Big\{(T-T_A)\frac{e^{-ZT_A}}{z} - \frac{(e^{-ZT_A}-e^{-ZT})}{z^2}\Big\} - \Big(ce^{-zT_A}+m\frac{(1-e^{-zT})}{z}\Big)\Big[-b(T-$
$T_A) - \gamma b\frac{1}{\lambda}\big(1-e^{-\lambda T_A}\big)\Big]\Big\} - (\alpha+\beta m)\frac{(1-e^{-zT})}{z}\Big[-b(T-T_A) - \gamma b\frac{1}{\lambda}\big(1-e^{-\lambda T_A}\big)\Big].$

KKT Conditions

Case 1. Here first, m is fixed, then, for the fixed m, total profit $\Pi(T,p|m)$ is maximized, and the model reduces to-

$$\max_{T,p} \quad \Pi(T,p|m).$$

subjected to:

$$(\alpha+\beta m)\Big[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big]m \le M;$$
$$T > T_A,$$
$$p \ge 0.$$

The Kuhn Tucker conditions for the problem (P2) is given as follows:

(P2-i): $\frac{\partial \mathrm{L}(T,p,\mu_1,\mu_2,\mu_3)}{\partial p} = 0.$
(P2-ii): $\frac{\partial \mathrm{L}(T,p,\mu_1,\mu_2,\mu_3)}{\partial T} = 0.$
(P2-iii): $\mu_1\Big((\alpha+\beta m)\Big[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big]m - M\Big) = 0.$
(P2-iv): $\mu_2(T-T_A) = 0.$
(P2-iv): $\mu_3 p = 0.$
(P2-v): $\mu_1,\mu_2,\mu_3,p \ge 0.$

where, $\mathrm{L}(T,p,\mu_1,\mu_2,\mu_3)$ is Lagrangian function given as follows, and $\mu_i\ \forall\ i = 1,2,3$ are Lagrange multipliers.

$\mathrm{L}(T,p,\mu_1,\mu_2,\mu_3)= \Pi(T,p|m) + \mu_1\Big((\alpha+\beta m)\Big[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\big(1-e^{-\lambda T_A}\big)\Big]m -$
$M\Big) - \ \mu_2(T-T_A) - \mu_3 p.$

The problem maximizing $\Pi(m|T,p)$ over m for any given T,p is given as follows:

$\max_{m} \quad \Pi(m|T,p).$
subjected to:
$(\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp^*)\left(1-e^{-\lambda T_A}\right)\right]m \le M,$
$m \ge 0.$

For any T, p, the KKT conditions for the above problem are given as follows:
(P3-i): $\frac{\partial L(m,\mu_1,\mu_2)}{\partial m}=0.$
(P3-ii): $\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)=0.$
(P3-iii): $\mu_2 m=0.$
(P3-iv): $\mu_1,\mu_2,m \ge 0.$
Where $L(m,\mu_1,\mu_2)$ is Lagrangian function given as follows, and $\mu_i\ \forall\ i=1,2$ are Lagrange multipliers.
$L(m,\mu_1,\mu_2,\mu_3)= \Pi(m|T,p)+\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)-\mu_2 m.$

Case 2. Here first, p is fixed, then, for the fixed p, total profit $\Pi(T,m|p)$ is maximized, and the model reduces to,

$$\max_{T,p} \quad \Pi(T,m|p).$$

subjected to:
$(\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m \le M;$
$T > T_A,$
$m \ge 0.$

The Kuhn Tucker conditions for the problem (P2) is given as follows:
(P2-i): $\frac{\partial L(T,m,\mu_1,\mu_2,\mu_3)}{\partial m}=0.$
(P2-ii): $\frac{\partial L(T,m,\mu_1,\mu_2,\mu_3)}{\partial T}=0.$
(P2-iii): $\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)=0.$
(P2-iv): $\mu_2(T-T_A)=0.$
(P2-v): $\mu_3 m=0.$
(P2-vi): $\mu_1,\mu_2,\mu_3 \ge 0.$

where, $L(T,p,\mu_1,\mu_2)$ is Lagrangian function given as follows, and $\mu_i\ \forall\ i=1,2,3$ are Lagrange multipliers.
$L(T,m,\mu_1,\mu_2,\mu_3)= \Pi(T,m|p)+\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)-\ \mu_2(T-T_A)-\mu_3 m.$

The problem maximizing $\Pi(p|T,m)$ over p for any given T, m is given as follows:
$\max_{m} \quad \Pi(p|T,m).$
subjected to:

$(\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m \leq M;$
$p \geq 0,$
For any T, p, the KKT conditions for the above problem are given as follows:
(P3-i): $\frac{\partial \mathrm{L}(p,\mu_1,\mu_2)}{\partial p}=0.$
(P3-ii): $\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)=0.$
(P3-iii): $\mu_2 p=0.$
(P3-iv): $\mu_1,\mu_2,p \geq 0.$

where, $\mathrm{L}(m,\mu_1,\mu_2)$ is Lagrangian function given as follows, and $\mu_i\ \forall\ i=1,2$ are Lagrange multipliers.
$\mathrm{L}(p,\mu_1,\mu_2,\mu_3)=\Pi(\mathrm{p}|\mathrm{T},\mathrm{m})+\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)-\mu_2 p.$

Case 3. Here first, T is fixed, then, for the fixed T, total profit $\Pi(m,p|T)$ is maximized, and the model reduces to-
$\max_{\mathrm{T,p}} \quad \Pi(\mathrm{m},\mathrm{p}|\mathrm{T}).$
subjected to:
$(\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m \leq M;$
$p \geq 0,$
$m \geq 0.$

The Kuhn Tucker conditions for the problem (P2) is given as follows:
(P2-i): $\frac{\partial \mathrm{L}(m,p,\mu_1,\mu_2,\mu_3)}{\partial m}=0.$
(P2-ii): $\frac{\partial \mathrm{L}(m,p,\mu_1,\mu_2,\mu_3)}{\partial p}=0.$
(P2-iii): $\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)=0.$
(P2-iv): $\mu_2 p=0.$
(P2-v): $\mu_3 m=0.$
(P2-vi): $\mu_1,\mu_2,\mu_3 \geq 0.$

where, $\mathrm{L}(T,p,\mu_1,\mu_2)$ is Lagrangian function given as follows, and $\mu_i\ \forall\ i=1,2,3$ are Lagrange multipliers.
$\mathrm{L}(m,p,\mu_1,\mu_2,\mu_3)=\Pi(m,p|T)+\mu_1\left((\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp)\left(1-e^{-\lambda T_A}\right)\right]m-M\right)-\mu_2 p-\mu_3 m.$

The problem maximizing $\Pi(\mathrm{T}|\mathrm{p},\mathrm{m})$ over p for any given T, m is given as follows:
$\max_{\mathrm{m}} \quad \Pi(\mathrm{T}|\mathrm{p},\mathrm{m}).$
subjected to:
$(\alpha+\beta m)\left[(a-bp)(T-T_A)+\frac{1}{\lambda}(a-\gamma bp^*)\left(1-e^{-\lambda T_A}\right)\right]m \leq M.$
$T \geq T_A.$

For any $T, p,$ the KKT conditions for the above problem are given as follows:

(P3-i): $\frac{\partial \mathrm{L}(T,\mu_1,\mu_2)}{\partial T} = 0.$

(P3-ii): $\mu_1 \left((\alpha + \beta m) \left[(a - bp)(T - T_A) + \frac{1}{\lambda}(a - \gamma bp)\left(1 - e^{-\lambda T_A}\right) \right] m - M \right) = 0.$

(P3-iii): $\mu_2 (T - T_A) = 0.$

(P3-iv): $\mu_1, \mu_2, T \geq 0.$

Where $\mathrm{L}(T, \mu_1, \mu_2)$ is Lagrangian function given as follows, and $\mu_i \; \forall \; i = 1{,}2$ are Lagrange multipliers.

$\mathrm{L}(T, \mu_1, \mu_2, \mu_3) = \Pi(T|p, m) + \mu_1 \left((\alpha + \beta m) \left[(a - bp)(T - T_A) + \frac{1}{\lambda}(a - \gamma bp)\left(1 - e^{-\lambda T_A}\right) \right] m - M \right) - \mu_2 (T - T_A).$

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