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A New Perspective on the Stochastic Fractional Order Materialized by the Exact Solutions of Allen-Cahn Equation

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Abstract
Stochastic fractional differential equations are among the most significant and recent equations in physical mathematics. Consequently, several scholars have recently been interested in these equations to develop analytical approximations. In this study, we highlight the stochastic fractional space Allen-Cahn equation (SFACE) as a major application of this class. In addition, we utilize the simplest equation method (SEM) with a dual sense of Brownian motion to convert the presented equation into an ordinary differential equation (ODE) and apply an effective computational technique to obtain exact solutions. By carefully comparing the derived solutions with solutions from other articles, we prove the distinction of these solutions for their diversity and the discovery of new solutions for SFACE that appear in many scientific fields, such as mathematical biology, quantum mechanics, and plasma physics. The results introduced in this article were obtained by plotting several graphs and examining how noise affects exact solutions using Mathematica and MATLAB software packages.

Keywords- Allen–Cahn equation, Exact solution, Brownian motion, Simplest equation method.

1. Introduction

In recent decades, fractional derivatives have generated substantial interest because of their possible use in several domains, including telegraph transmission (Cascaval et al., 2002), atmospheric science (Korn, 2019), chaotic oscillations (Tavazoei et al., 2008), optical fibers (Yokus and Baskonus, 2022), two-scale thermal science (He, 2021), ecological and economic systems (Saadeh et al., 2023), mechanics (Zhang and Bilige, 2019), chemistry (Yuste et al., 2004), and hydrology (Benson et al., 2000). physics (Abdoon and Hasan, 2022; Prakasha et al., 2023), biology (Amourah et al., 2023; Saadeh et al., 2022), and finance (Wyss, 2000; Raberto et al., 2002). These fractional-order equations represent the memory and heirship of different substances using fractional-order derivatives (Podlubny, 1999) and are preferred over integer-order equations.

However, random disturbances are introduced into physical systems by a wide variety of naturally occurring causes. It is impossible to deny these, and the existence of noise may contribute to the development of certain statistical features and significant events. As a direct consequence, stochastic differential equations came into existence and quickly assumed an increasingly important position in the process of modeling phenomena across many scientific disciplines. Recently, some related studies have been published on the numerical solutions of stochastic fractional equations (Kamrani, 2015; Liu and Yan, 2016; Mohammed, 2021; Ahmad et al., 2021a; Ahmad et al., 2021b; Zou, 2018a). The SFACE that is created by multiplicative noise via Itoˆ sense is considered in this investigation as follows,

$$\frac{\partial u(x,t)}{\partial t} = D_x^{2\alpha} u(x,t) + u(x,t) - u^3(x,t) + \rho u(x,t) r'(t) \tag{1}$$

for $0 < \alpha \leq 1$, where $r(t)$ is the standard Brownian motion and σ is the fractional order of the derivative that it symbolizes, D_x^{α} denotes the Jumarie's -modified Riemann-Liouville derivative (JRLD), $u(x,t)$ is the unknown function of two variables, and ρ is the noise intensity. SFACE Eq. (1) holds when $\rho = 0$ and $\alpha = 1$. It has several uses in the scientific community, ranging from mathematical biology and quantum mechanics to plasma physics. To date, a large number of effective approaches, such as the first integral method, have been presented for this topic (Tascan and Bekir 2009), the tanh-coth method (Wazwaz, 2004), the Haar wavelet method (Hariharan and Kannan, 2009), the double exp-function method (Bekir, 2012), the modified simple equation method (Younis, 2014), and the Riccati-Bernoulli sub-ODE (Mohammed et al. 2021).

In this study, we investigate the solution of SFACE (1) derived from one-dimensional multiplicative noise using SEM (Kudryashov, 2005, Zhao et al., 2013). Additionally, we developed and enhanced previous findings. The exciting physical phenomena could be better understood with the help of the obtained solutions. Therefore, in this study, we focused on finding new solutions for SFACE (1). Moreover, we analyze how the inclusion of a stochastic factor modifies the exact solutions obtained using Mathematica software and illustrate these solutions with a MATLAB program by plotting graphs.

The strength of this study is the importance of the proposed equation, the stochastic Allen-Cahn fractional differential equation, in the sense of the modified Riemann- Liouville derivative, which has been shown in many applications such as mathematical biology, quantum mechanics, and plasma physics. This study investigates the analytical and exact solutions of SFACS, which play a vital role in describing the structure of the dynamics for phase separation in $Fe - Cr - X(X = Mo, Cu)$ ternary alloys. These solutions describe the dynamics of phase separation in iron alloys and are used in solidification and nucleation problems. The dynamics of the phase separation in iron alloys are described by the Cahn-Allen equation, which is regarded as an important model in plasma physics, quantum mechanics, mathematical biology, and fluid dynamics. Additionally, it is used for solidification and nucleation issues (Bulut et al., 2016, Khater et al., 2020). In this research, we discuss the algorithm for establishing a solution based on two approaches: Ricatti and Bernoulli equations. We used an efficient computational technique to acquire precise solutions to the presented equation by converting it into an ODE with the notion of Brownian motion. Moreover, as an advantage of this study, we obtained 18 solutions, all of which are exact solutions; three of which are identical to some solutions obtained in the literature, and the other two are new exact solutions.

The remainder of this paper is organized as follows. Section 2 presents the definitions of fractional derivatives. Section 3 explains the SEM procedure. In Section 4, we implement the SEM to obtain different exact solutions for SFACE (1). In Section 5, we demonstrate the effect of noise terms on the exact solutions of SFACE (1). Finally, the conclusion of this study is presented.

2. Basic Definitions

The order α for D_x^α that denotes the JRLD of the continuous function $f(x)$ is defined in (Jumarie, 2006), by

$$D^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\xi)^{-\alpha}\big(f(\xi)-f(0)\big)\,d\xi, 0<\alpha<1, \\ \left(f^{(n)}(x)\right)^{\alpha-n}, n\le\alpha<n+1, \qquad n\ge 1. \end{cases}$$

Additionally, we introduce the basic features of the modified Riemann-Liouville fractional operator (see He et al., 2012, Aksoy et al., 2016),

- $D^\alpha c = 0$.
- $D^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}\, x^{\gamma-\alpha}, \gamma > \alpha - 1$.
- $D^{2\alpha} = D^\alpha D^\alpha$.
- $D^\alpha\big(af(x)\big) = aD^\alpha\big(f(x)\big)$.
- $D^\alpha(af(x)+bg(x)) = aD^\alpha\big(f(x)\big) + bD^\alpha\big(g(x)\big)$.
- $D^\alpha\big(f(x)g(x)\big) = f(x)D^\alpha g(x) + g(x)D^\alpha f(x)$.
- $D^\alpha\big(u(f(x))\big) = \sigma_x u'(f(x))D^\alpha(f(x))$, where, σ_x is defined by (He et al., 2012, Aksoy et al., 2016).

3. Algorithm of the SEM

In this section, we present a simple algorithm that illustrates the basic steps of SEM and considers the nonlinear partial differential equation:

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots .) = 0 \tag{2}$$

where, F is a non-linear polynomial function. The steps of the method are as follows:

Step 1. By converting the partial differential equation in (2), into an ordinary differential using substitution: $u(x,t) = \varphi(\zeta)$, and $\zeta = x - ct$. Thus, Eq. (2) becomes,

$$F(u, -cu', u', c^2u'', -cu'', u'', \dots .) = 0 \tag{3}$$

Step 2. Assume that $\varphi(\zeta)$ can be expressed by the following infinite series:

$$\varphi(\zeta) = \textstyle\sum_{i=0}^{n} a_i\chi^i(\zeta) \tag{4}$$

where, $\chi(\zeta)$ fulfills the Riccat or Bernoulli equation, n is a natural number that may be calculated by the balance process, and $a_i, i = 0,1,2, \dots\dots, n$ are constants depending on the target problem. Here, we focus on the Bernoulli equation, which is

$$\chi'(\zeta) = \mu\chi^2(\zeta) + \beta\chi(\zeta), \tag{5}$$

where, μ and β are constants.

The solutions of Eq. (5) can be written as (Yun-Mei et al., 2013),

$$\chi(\zeta) = \frac{-\beta C_1}{\mu\left(C_1+\cosh\left(\beta(\zeta+\zeta_0)\right)-\sinh\left(\beta(\zeta+\zeta_0)\right)\right)},$$
$$\chi(\zeta) = \frac{-\beta\left(\cosh\left(\beta(\zeta+\zeta_0)\right)+\sinh\left(\beta(\zeta+\zeta_0)\right)\right)}{\mu\left(C_2+\cosh\left(\beta(\zeta+\zeta_0)\right)+\sinh\left(\beta(\zeta+\zeta_0)\right)\right)} \tag{6}$$

where, C_1, C_2 and δ_0 are constants.

Riccati equation assume that,

$$\chi'(\zeta) = \mu\chi^2(\zeta) + \beta\chi(\zeta) + \delta \tag{7}$$

where, $\mu, \beta,$ and δ are constants.

The solutions of Eq. (7) can be written as (Jumarie, 2016):

$$\chi(\zeta) = -\frac{\beta}{2\mu} - \frac{\theta}{2\mu}\tanh\left(\frac{\theta}{2}\zeta\right) + \frac{\operatorname{sech}\left(\frac{\theta}{2}\zeta\right)}{C\cosh\left(\frac{\theta}{2}\zeta\right) - \frac{2\beta}{\theta}\sinh\left(\frac{\theta}{2}\zeta\right)} \tag{8}$$

where, $\theta^2 = \beta^2 - 4\mu\delta$.

Step 3. Substitute Eq. (4) into Eq. (3), and use formula in Eq. (5) or Eq. (7), this will convert the LHS of Eq. (3) into a polynomial in $H(\zeta)$. Then, equating each coefficient to zero, produces some equations for $a_i, i = 0,1,2,\ldots\ldots, n,$ and μ, β are nonzero constants. After simple calculations, we solved a set of equations to determine these parameters.

Step 4. From Step 3, substitute $a_i, i = 0,1,2,\ldots\ldots, n$, into Eq. (4), lead to the precise travel wave solutions for Eq. (2).

4. Solutions of the Allen–Cahn Equation

In this section, we apply the SEM to solve SFACE (1). Using this technique, we create many new exact solutions for the desired equation. To get that, we utilize a wave transformation for a stochastic fractional, such as,

$$u(x,t) = \varphi(\zeta)e^{\rho r(t)-\frac{1}{2}\rho^2 t}, \qquad \zeta = c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha - \lambda t\right) \tag{9}$$

where, φ is a deterministic function, c and λ are nonzero constants, and ρ is the noise intensity. By differentiating u with respect to t, , and twice of fractional order α with respect x, to get the system:

$$\begin{cases} \frac{\partial u}{\partial t} = \varphi(\zeta)e^{\left[\rho r(t)-\frac{1}{2}\rho^2 t\right]}\left(\rho\, r'(t) - \frac{1}{2}\rho^2\right) - c\lambda\,\varphi'(\zeta)e^{\rho r(t)-\frac{1}{2}\rho^2 t}, \\ D_x^\alpha(u(x,t)) = c\sigma_x\varphi'(\zeta)e^{\rho r(t)-\frac{1}{2}\rho^2 t}\ . \\ D_x^{2\alpha}(u(x,t)) = c^2{\sigma_x}^2\varphi''(\zeta)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \end{cases} \tag{10}$$

Additional details on this definition can be seen [35-36].
substituting Eq. (10) in Eq. (1), we obtain the next ODE:

$$\left(\varphi(\zeta)\left(\rho r'(t) - \frac{1}{2}\rho^2\right) - c\lambda\varphi'(\zeta)\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} = c^2{\sigma_x}^2\varphi''(\zeta)e^{\rho r(t)-\frac{1}{2}\rho^2 t} + e^{\left[\rho r(t)-\frac{1}{2}\rho^2 t\right]}(\varphi(\zeta) - \varphi^3(\zeta) + \rho\,\varphi(\zeta)r'(t)) \tag{11}$$

In view of the fact that $E(e^{\rho z}) = e^{\frac{1}{2}\rho^2 t}$ for the real number ρ and z is a standard Gaussian process, the equality $E\left(e^{\rho r(t)}\right) = e^{\frac{1}{2}\rho^2 t}$ as a result of $\rho r(t)$ is distributed like $\rho\sqrt{t}\, z$. Then Eq. (11) can be arranged as follows:

$$c^2 l^2 \varphi''(\zeta) + c\lambda\varphi'(\zeta) - \varphi^3(\zeta) + \left(1 + \tfrac{1}{2}\rho^2\right)\varphi(\zeta) = 0 \tag{12}$$

where, $l = \sigma_x$. We define the solution of $\varphi(\zeta)$ in a finite series as Eq. (4) where $n = 1$ by calculated the balancing process:

$$\varphi(\zeta) = a_0 + a_1\chi(\zeta) \tag{13}$$

According to the simplest equation method, by choosing the Bernoulli equation, we substitute Eq.(13) with Eq.(5) in Eq.(12). After that, by setting the coefficients of $\chi^i(\zeta)$ to zero, we get system of equations in terms a_0, a_1, μ and β. When determining the solution of the system using Mathematica software, we obtained many sets of values for the constants. We chose five sets:

$$B_1 = \left\{\begin{matrix} a_0 = -\dfrac{\sqrt{2+\rho^2}}{\sqrt{2}}, & a_1 = \dfrac{2\sqrt{2}cl\mu + \sqrt{2}cl\mu\rho^2}{2+\rho^2}, \\ \beta = -\dfrac{\sqrt{2+\rho^2}}{cl}, & \lambda = 0 \end{matrix}\right\},$$

$$B_2 = \left\{\begin{matrix} a_0 = \dfrac{\sqrt{2+\rho^2}}{\sqrt{2}}, & a_1 = \dfrac{-2\sqrt{2}cl\mu - \sqrt{2}cl\mu\rho^2}{2+\rho^2}, \\ \beta = -\dfrac{\sqrt{2+\rho^2}}{cl}, & \lambda = 0 \end{matrix}\right\},$$

$$B_3 = \left\{\begin{matrix} a_0 = 0, & a_1 = -\sqrt{2}cl\mu, \\ \beta = -\dfrac{\sqrt{2+\rho^2}}{2cl}, & \lambda = \dfrac{3}{2}l\sqrt{2+\rho^2} \end{matrix}\right\},$$

$$B_4 = \left\{\begin{matrix} a_0 = -\dfrac{\sqrt{2+\rho^2}}{\sqrt{2}}, & a_1 = \dfrac{2\sqrt{2}cl\mu + \sqrt{2}cl\mu\rho^2}{2+\rho^2}, \\ \beta = -\dfrac{\sqrt{2+\rho^2}}{2cl}, & \lambda = -\dfrac{3}{2}l\sqrt{2+\rho^2} \end{matrix}\right\},$$

$$B_5 = \left\{\begin{matrix} a_0 = \dfrac{\sqrt{2+\rho^2}}{\sqrt{2}}, & a_1 = \dfrac{2\sqrt{2}cl\mu + \sqrt{2}cl\mu\rho^2}{2+\rho^2} \\ ,\beta = \dfrac{\sqrt{2+\rho^2}}{cl}, & \lambda = 0 \end{matrix}\right\},$$

substituting these values for each set in Eq.(13) with Eq.(6), we get solutions equation for $(B_1 - B_5)$ respectively, as

$$\varphi_1(\zeta) = \frac{\sqrt{2+\rho^2}}{\sqrt{2}}\left(-1 + \frac{2C_1}{\text{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right) + \text{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right) + C_1}\right) \tag{14}$$

$$\varphi_2(\zeta)=\frac{\sqrt{2+\rho^2}\left(\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)-\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)-C_2\right)}{\sqrt{2}\left(\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)-\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+C_2\right)} \tag{15}$$

$$\varphi_3(\zeta)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}\left(1-\frac{2C_1}{\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+C_1}\right) \tag{16}$$

$$\varphi_4(\zeta)=\frac{\sqrt{2+\rho^2}\left(-\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+C_2\right)}{\sqrt{2}\left(\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)-\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+C_2\right)} \tag{17}$$

$$\varphi_5(\zeta)=-\frac{\sqrt{2+\rho^2}C_1}{\sqrt{2}\left(\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+C_1\right)} \tag{18}$$

$$\varphi_6(\zeta)=\frac{\sqrt{2+\rho^2}\left(-\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)\right)}{\sqrt{2}\left(\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)-\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+C_2\right)} \tag{19}$$

$$\varphi_7(\zeta)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}\left(-1+\frac{C_1}{\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+C_1}\right) \tag{20}$$

$$\varphi_8(\zeta)=-\frac{\sqrt{2+\rho^2}C_2}{\sqrt{2}\left(\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)-\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{2cl}\right)+C_2\right)} \tag{21}$$

$$\varphi_9(\zeta)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}\left(1-\frac{2C_1}{\operatorname{Cosh}\left(-\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+C_1}\right) \tag{22}$$

$$\varphi_{10}(\zeta)=-\frac{\sqrt{2+\rho^2}\left(\operatorname{Cosh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)-C_2\right)}{\sqrt{2}\left(\operatorname{Cosh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+\operatorname{Sinh}\left(\frac{\sqrt{2+\rho^2}(\zeta+\zeta_0)}{cl}\right)+C_2\right)} \tag{23}$$

where, ζ_0, C_1, C_2 constants, now recently by substituting Eq's (14-23) in Eq.(9), we will show the exact solutions of Eq. (1) respectively,

$$u_1(x,t)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}e^{\rho r(t)-\frac{1}{2}\rho^2 t}\cdot\left(-1+\frac{2C_1}{\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+C_1}\right) \tag{24}$$

$$u_2(x,t)=\frac{\sqrt{2+\rho^2}\cdot e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{\sqrt{2}}\cdot\left(\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)-\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)-C_2\right)\left(\frac{1}{\left(\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)-\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+C_2\right)}\right) \tag{25}$$

$$u_3(x,t)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}\left(1-\frac{2C_1}{\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+C_1}\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \tag{26}$$

$$u_4(x,t)=\sqrt{2+\rho^2}\left(-\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+C_2\right)\cdot\left(\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{\sqrt{2}\left(\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)-\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{cl}\right)+C_2\right)}\right) \tag{27}$$

$$u_5(x,t)=-\frac{-\sqrt{2+\rho^2}C_1 e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{\sqrt{2}\left(\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)+C_1\right)} \tag{28}$$

$$u_6(x,t)=\sqrt{2+\rho^2}\left(-\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)\right)\cdot\left(\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{\sqrt{2}\left(\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)-\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)+C_2\right)}\right) \tag{29}$$

$$u_7(x,t)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}e^{\rho r(t)-\frac{1}{2}\rho^2 t}\left(-1+\frac{C_1}{\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)}{2cl}\right)+C_1}\right) \tag{30}$$

$$u_8(x,t)=\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}\sqrt{2+\rho^2}C_2}{\sqrt{2}\left(\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{2cl}\right)-\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{2cl}\right)C_2\right)} \tag{31}$$

$$u_9(x,t)=\frac{\sqrt{2+\rho^2}}{\sqrt{2}}e^{\rho r(t)-\frac{1}{2}\rho^2 t}\cdot\left(1-\frac{2C_1}{\mathrm{Cosh}\left(-\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{cl}\right)+\mathrm{Sinh}\left(\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{cl}\right)+C_1}\right) \tag{32}$$

$$u_{10}(x,t)=\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}\sqrt{2+\rho^2}}{\sqrt{2}}\left(\mathrm{Cosh}\left[\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{cl}\right]+\mathrm{Sinh}\left[\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{cl}\right]-C_2\right)\cdot$$
$$\left(-\frac{1}{\left(\mathrm{Cosh}\left[\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{cl}\right]+\mathrm{Sinh}\left[\frac{\sqrt{2+\rho^2}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)}{cl}\right]+C_2\right)}\right) \tag{33}$$

As stated by the simplest equation, the Riccati equation when putting Eq. (13) with Eq. (7) in Eq. (12) also the coefficients of the functions $\chi^i(\zeta)$ equating tobe zero, the system of equations with terms $a_i(i=0,1)$ and $\mu,\ \beta$ is solving, we obtain many sets of values for constants, we choose four sets as the ones that are,

$$R_1=\left\{\begin{matrix} a_0=-\frac{cl\beta}{\sqrt{2}}, & \mathrm{a}_1=-\sqrt{2}cl\mu, \\ \lambda=0, & \delta=\frac{-2+c^2l^2\beta^2-\rho^2}{4c^2l^2\mu} \end{matrix}\right\},$$

$$R_2=\left\{\begin{matrix} a_0=\frac{cl\beta}{\sqrt{2}}, & a_1=\sqrt{2}cl\mu, \\ \lambda=0, & \delta=\frac{-2+c^2l^2\beta^2-\rho^2}{4c^2l^2\mu} \end{matrix}\right\},$$

$$R_3=\left\{\begin{matrix} a_0=-\frac{16cl\beta-8\sqrt{2+\rho^2}}{16\sqrt{2}}, & a_1=-\sqrt{2}cl\mu, \\ \lambda=\frac{3\left(-2l\sqrt{2+\rho^2}-l\rho^2\sqrt{2+\rho^2}\right)}{2(2+\rho^2)}, & \delta=\frac{-2+4c^2l^2\beta^2-\rho^2}{16c^2l^2\mu} \end{matrix}\right\},$$

$$R_4=\left\{\begin{matrix} a_0=\frac{16cl\beta-8\sqrt{2+\rho^2}}{16\sqrt{2}}, & a_1=\sqrt{2}cl\mu, \\ \lambda=\frac{3\left(-2l\sqrt{2+\rho^2}-l\rho^2\sqrt{2+\rho^2}\right)}{2(2+\rho^2)}, & \delta=\frac{-2+4c^2l^2\beta^2-\rho^2}{16c^2l^2\mu} \end{matrix}\right\}.$$

Substituting these values for each set in Eq.(13) with the solutions in Eq.(8), we get the solutions for all value sets R_1-R_4, respectively as,

$$\varphi_1(\zeta)=\frac{cl}{\sqrt{2}}\left(\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Tanh}\left(\frac{1}{2}\sqrt{\frac{2+\rho^2}{c^2l^2}}(\zeta+\zeta_0)\right)\right) \tag{34}$$

$$\varphi_2(\zeta)=\frac{2c^2l^2\mu\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Sinh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(1+\text{Tanh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)}{\sqrt{2}cl\left(2\mu\text{Sinh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)}-\frac{(2+\rho^2)\text{Cosh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\text{Tanh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1}{\sqrt{2}cl\left(2\mu\text{Sinh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)} \quad (35)$$

$$\varphi_3(\zeta)=-\frac{cl}{\sqrt{2}}\left(2\beta+\sqrt{\frac{2+\rho^2}{c^2l^2}}\,\text{Tanh}\left(\frac{1}{2}\sqrt{\frac{2+\rho^2}{c^2l^2}}(\zeta+\zeta_0)\right)\right) \quad (36)$$

$$\varphi_4(\zeta)=\frac{-2c^2l^2\mu\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Sinh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(1+\text{Tanh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)}{\sqrt{2}cl\left(2\mu\text{Sinh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)}+\frac{(2+\rho^2)\text{Cosh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\text{Tanh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1}{\sqrt{2}cl\left(2\mu\text{Sinh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)} \quad (37)$$

$$\varphi_5(\zeta)=\frac{1}{2\sqrt{2}}\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\,\text{Tanh}\left(\frac{1}{4}\sqrt{\frac{2+\rho^2}{c^2l^2}}(\zeta+\zeta_0)\right)\right) \quad (38)$$

$$\varphi_6(\zeta)=\frac{4cl\mu\,\text{Sinh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Tanh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)}{2\sqrt{2}cl\left(4\mu\,\text{Sinh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1\right)}-$$
$$\frac{\text{Cosh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(cl\sqrt{2+\rho^2}\sqrt{\frac{2+\rho^2}{c^2l^2}}+(2+\rho^2)\text{Tanh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)C_1}{2\sqrt{2}cl\left(4\mu\,\text{Sinh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1\right)} \quad (39)$$

$$\varphi_7(\zeta)=-\frac{1}{2\sqrt{2}}\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\,\text{Tanh}\left(\frac{1}{4}\sqrt{\frac{2+\rho^2}{c^2l^2}}(\zeta+\zeta_0)\right)\right) \quad (40)$$

$$\varphi_8(\zeta)=\frac{-4cl\mu\text{Sinh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Tanh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)}{2\sqrt{2}cl\left(4\mu\text{Sinh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1\right)}+$$
$$\frac{\text{Cosh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(cl\sqrt{2+\rho^2}\sqrt{\frac{2+\rho^2}{c^2l^2}}+(2+\rho^2)\text{Tanh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)C_1}{2\sqrt{2}cl\left(4\mu\text{Sinh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\text{Cosh}\left(\frac{1}{4}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1\right)} \quad (41)$$

where, ζ_0, C_1 constants, now recently by substituting Eq's (34-41) in Eq.(9), we present the exact solutions of Eq.(1), respectively,

$$u_{11}(x,t)=\frac{cl}{\sqrt{2}}\left(\sqrt{\frac{2+\rho^2}{c^2l^2}}\,\text{Tanh}\left(\frac{1}{2}\sqrt{\frac{2+\rho^2}{c^2l^2}}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^{\alpha}-\lambda t\right)+\zeta_0\right)\right)\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \quad (42)$$

$$u_{12}(x,t)=\left(\frac{2c^2l^2\mu\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Sinh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(1+\mathrm{Tanh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)}{\sqrt{2}cl\left(2\mu\mathrm{Sinh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Cosh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)}-\frac{(2+\rho^2)\mathrm{Cosh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\mathrm{Tanh}\left(\frac{1}{2}\zeta\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1}{\sqrt{2}cl\left(2\mu\mathrm{Sinh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Cosh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)}\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \tag{43}$$

$$u_{13}(x,t)=-\frac{cl}{\sqrt{2}}\left(2\beta+\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Tanh}\left(\frac{1}{2}\sqrt{\frac{2+\rho^2}{c^2l^2}}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)\right)\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \tag{44}$$

$$u_{14}(x,t)=\left(-2c^2l^2\mu\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Sinh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(1+\mathrm{Tanh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)+(2+\rho^2)\mathrm{Cosh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\mathrm{Tanh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1\right)\cdot\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{\sqrt{2}cl\left(2\mu\mathrm{Sinh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Cosh}\left(\frac{c}{2}\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)c_1\right)} \tag{45}$$

$$u_{15}(x,t)=\frac{1}{2\sqrt{2}}\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Tanh}\left(\frac{1}{4}\sqrt{\frac{2+\rho^2}{c^2l^2}}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)\right)\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \tag{46}$$

$$u_{16}(x,t)=\left(4cl\mu\,\mathrm{Sinh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Tanh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)-\mathrm{Cosh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(cl\sqrt{2+\rho^2}\sqrt{\frac{2+\rho^2}{c^2l^2}}+(2+\rho^2)\mathrm{Tanh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)C_1\right)\cdot\left(\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{2\sqrt{2}cl\left(4\mu\,\mathrm{Sinh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Cosh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)C_1\right)}\right) \tag{47}$$

$$u_{17}(x,t)=-\frac{1}{2\sqrt{2}}\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Tanh}\left(\frac{1}{4}\sqrt{\frac{2+\rho^2}{c^2l^2}}\left(c\left(\frac{1}{\Gamma(1+\alpha)}x^\alpha-\lambda t\right)+\zeta_0\right)\right)\right)e^{\rho r(t)-\frac{1}{2}\rho^2 t} \tag{48}$$

$$u_{18}(x,t)\left(=-4cl\mu\mathrm{Sinh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(\sqrt{2+\rho^2}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}+cl\sqrt{\frac{2+\rho^2}{c^2l^2}}\mathrm{Tanh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\right)+\mathrm{Cosh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2l^2}}\right)\left(cl\sqrt{2+\rho^2}\sqrt{\frac{2+\rho^2}{c^2l^2}}+(2+\rho^2)\mathrm{Tanh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\right.\right.\right.$$

$$\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2 l^2}}\right)\right)C_1\right)\cdot\left(\frac{e^{\rho r(t)-\frac{1}{2}\rho^2 t}}{2\sqrt{2}cl\left(4\mu \text{Sinh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2 l^2}}\right)-\sqrt{\frac{2+\rho^2}{c^2 l^2}}\text{Cosh}\left(\frac{c}{4}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\lambda t\right)\sqrt{\frac{2+\rho^2}{c^2 l^2}}\right)C_1\right)}\right) \quad (49)$$

***Remark* 1**

We obtain identical solutions from previous studies for these three cases:

- If we put $\rho = 0$, in $u_{15}(x,t)$ in ref. (Aksoy et al., 2016).
- If we put $\rho = 0, \alpha = 1$ in $u_{17}(x,t)$ in ref. (Mohammed et al. 2021).
- The solutions $u_{11}(x,t)$ and $u_{13}(x,t)$ in ref. (Albosaily et al., 2022).

However, the other obtained solutions in this study are completely new exact solutions.

5. The Impact of Noise on the New Solutions of SFACE (1)

Here, we examine how noise affects the exact solutions of SFACE (1). The study of how noise influences the accuracy of exact solutions to SFACE (1) in a stochastic fractional space represents the bright spot of this study. Therefore, we provide several visual representations to explain the behavior of these solutions.

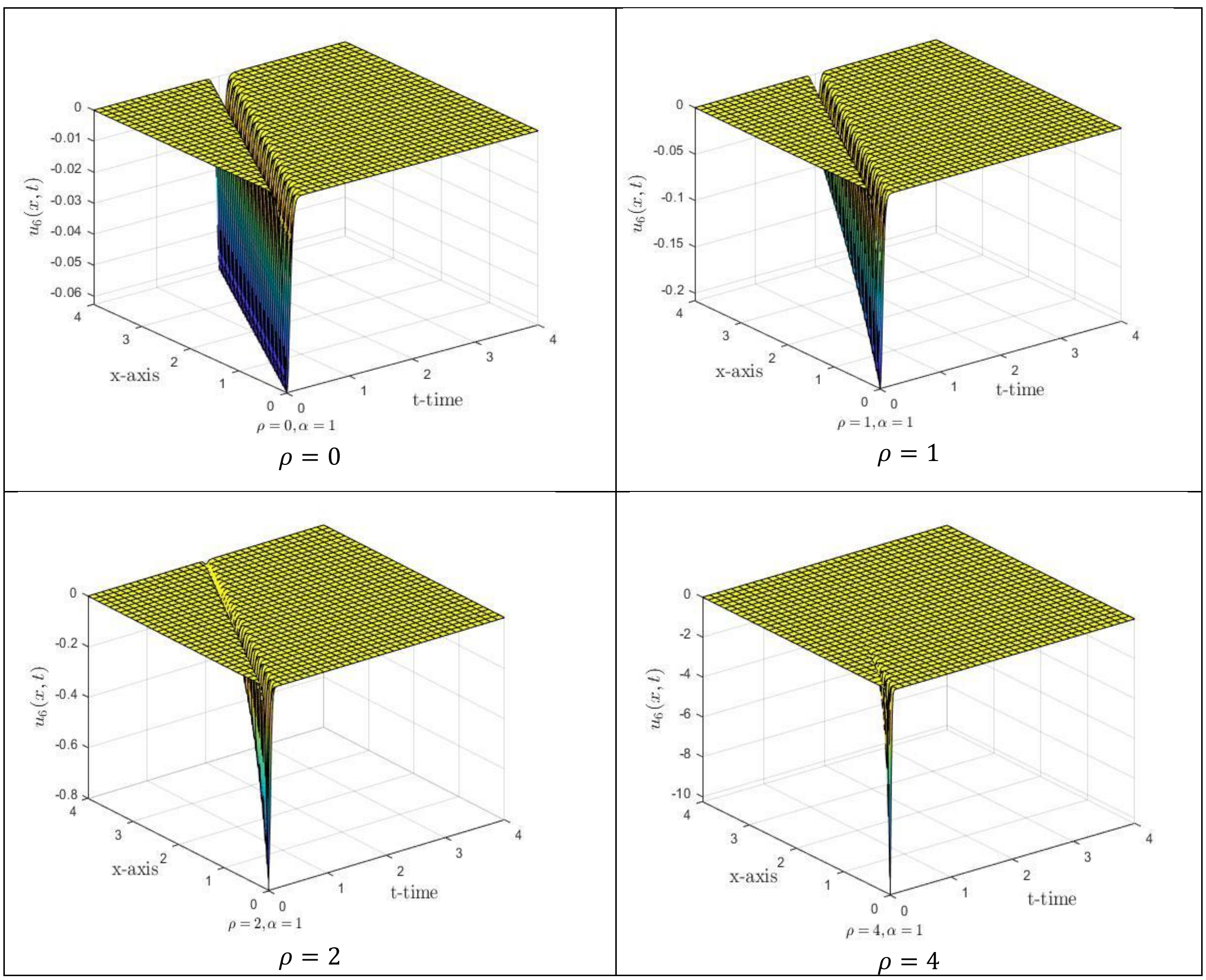

Figure 1. Graph of the solution $u_6(x,t)$ in Eq. (15) with $\alpha = 1$ and different values of ρ.

Several graphs were generated in MATLAB for various noise intensities, as shown in Figures 1 and 2. By simulation of the solution $u_6(x,t)$ in Eq. (29) for $t \in [0,4]$ and $x \in [0,4]$ we can note in the first graphs of Figures 1 and 2, when $\rho = 0$ the surface is less flat compared to other shapes. However, after some minor transit behaviors, the surface becomes more planar when noise is introduced and the intensity of the noise increases ($\rho = 1,2,4$) as shown in the rest graphs in Figure 1 and 2. This demonstrates the stability of the solutions under the influence of noise.

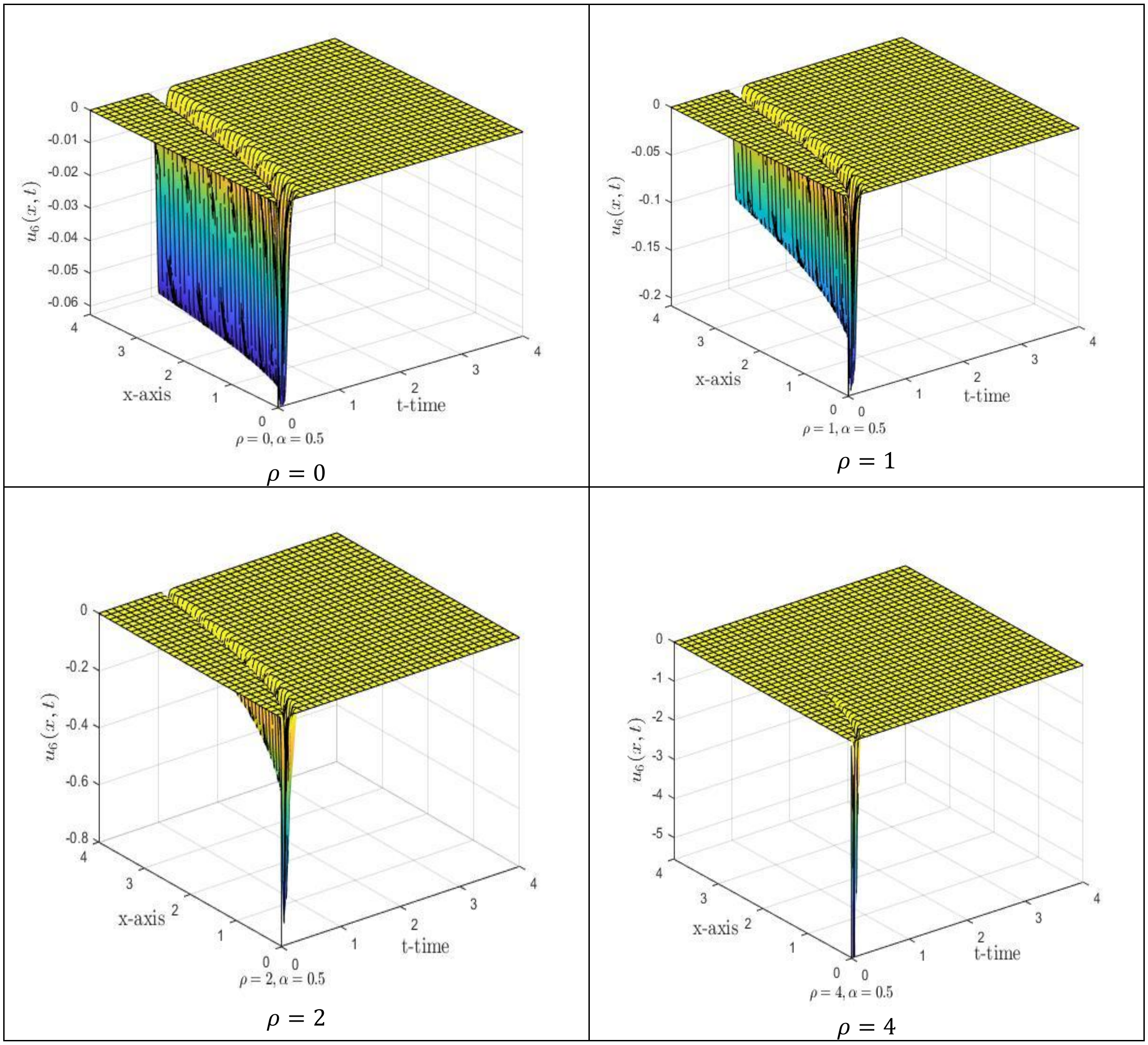

Figure 2. Graph of the solution $u_6(x,t)$ in Eq. (29) with $\alpha = 0.5$ and different values of ρ.

6. Conclusion

The different and new exact solutions are derived by utilizing the simplest equation method, SFACE, which is described in the multiplicative noise by Itô sense. This method has proven its efficiency and fluency in finding many new exact solutions using the Mathematica software. Additionally, we broadened our scope and improved the quality of other findings, including those described in the literature (He et al., 2012; Aksoy et al., 2016). These results are important for gaining knowledge on a variety of physical phenomena. In conclusion, we demonstrated the impact of the stochastic term t on the exact solutions of SFACE, where we noticed that the stability of the solutions is influenced by the noise increase by plotting several graphs in the MATLAB package. This research was carried out in the hope that it will be a useful resource for future applications and explorations of exact solutions using different methods and investigating new methods such as those in (Salah et al., 2023, Qazza et al., 2023a, Saadeh et al., 2023, Qazza et al., 2023b).

Conflict of Interests

The authors confirm that there is no conflict of interest to declare for this publication.

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