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# Solid Waste Management using *q*-rung Orthopair Fuzzy Decision Making based on Sugeno-Weber Prioritized Operator and EDAS Technique

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**Abstract**

The rapidly growing population, urbanization, and advancements in technology have led to a continuous increase in both the quantity and diversity of solid waste (SW). The management of SW stands out as an urgent concern, as the growing volume of garbage places enormous strain on the environment. Addressing this issue necessitates the use of solid waste management methods (SWMMs). Therefore, in this paper, we propose a new Multiattribute group decision-making (MGDM) method under the *q*-rung orthopair fuzzy numbers (*q*-ROFNs) environment to select the optimal sustainable SWMM for effective management of SW. Firstly, we present new operational laws of *q*-ROFNs based on Sugeno-Weber's norm, which overcomes the shortcomings of existing operational laws of *q*-ROFNs. After that, based on proposed operational laws of *q*-ROFNs, we propose the q-rung orthopair fuzzy Sugeno-Weber prioritized weighted arithmetic (*q*-ROFSWPWA) aggregation operator (AO) for aggregating the *q*-ROFNs, which considers the priority relationship among aggregating *q*-ROFNs. Moreover, we propose a new MGDM method for the *q*-ROFNs environment based on the proposed *q*-ROFSWPWA AO and EDAS technique. Furthermore, we consider a case study of selecting the optimal SWMM to demonstrate the proposed MGDM method. We also present a comparative analysis of the proposed MGDM method with existing MGDM methods.

**Keywords-** Solid waste, *q*-ROFNs, Sugeno-weber norms, Decision making, Prioritized operator.

## Abbreviations

| | |
|---|---|
| AO | Aggregation Operator |
| AOL | Addition Operation Law |
| CDMx | Collective Decision Matrix |
| DMExs | Decision Making Experts |
| DMx | Decision Matrix |
| EDAS | Evaluation Based on Distance from Average Solution |
| EPR | Extended Producer Responsibility |
| FS | Fuzzy Set |
| IFS | Intuitionistic Fuzzy Set |
| MG | Membership Grade |
| MABAC | Multi-Attributive Border Approximation Area Comparison |
| MGDM | Multiattribute Group Decision-Making |
| NDMx | Normalized Decision Matrix |
| NMG | Non-Membership Grade |
| PA | Prioritized Average |
| PFS | Pythagorean Fuzzy Set |
| PO | Preference Order |
| $q$-ROFNs | $q$-Rung Orthopair Fuzzy Numbers |
| $q$-ROFSs | $q$-Rung Orthopair Fuzzy Sets |
| $q$-ROFSWPWA | $q$-Rung Orthopair Fuzzy Sugeno-Weber Prioritized Weighted Arithmetic |
| SMOL | Scalar Multiplication Operation Law |
| SPOL | Scalar Power Operation Law |
| SWM | Solid Waste Management |
| SWMMs | Solid Waste Management Methods |

## 1. Introduction

Solid waste is defined as abandoned materials in solid or semi-solid form produced by households, industries, agriculture, and other sectors. With increased urbanization, industrialization, and population growth, efficient waste management has become critical in reducing environmental pollution and health concerns (Paul et al., 2023). Solid waste is classified as municipal, industrial, hazardous, agricultural, biomedical, building and demolition, and mining waste, with each posing unique issues due to its composition and environmental impact. The growing volume of solid waste and its environmental impact necessitate the adoption of advanced methods for efficient solid waste management (SWM) (Sadessa and Balo, 2025). SWM includes systematic garbage collection, transportation, disposal, recycling, and monitoring procedures that aim to reduce the negative effects on ecosystems and human health. Sustainable practices, which emphasize waste reduction, material recycling, and safe residual disposal, help to conserve resources and improve environmental, economic, and social well-being. Therefore, the main important issue is to select the optimal sustainable SWM method (SWMM) influenced by sustainability factors and desirable with respect to multiple conflicting attributes. This issue can be classified as one that falls under the multiattribute decision making category. In addition, it is not possible for a single individual to evaluate all the criteria and the weights that they carry in the decision-making process. As a result, this matter immediately transforms into a problem that requires multiattribute group decision making (MGDM), and in order to collect the necessary information, it is necessary to recruit the assistance of a group of specialists from a variety of fields.

Multiattribute group decision-making (MGDM) is a systematic approach used to deal with complex decision making problems involving several attributes and a group of decision making experts (DMExs). It is beneficial when a judgement must be made considering multiple perspectives and conflicting criteria. The decision-making process in MGDM contains multiple steps, including the problem identification, the attribute and alternative definition, the gathering of evaluations from DMExs, and the employment of fuzzy aggregation techniques to integrate individual judgments. The major issue of MGDM is to aggregate the opinions of many DMExs with varying perspectives and attitudes towards the significance of attributes and the alternatives under assessments. Zadeh (1965) proposed the fuzzy set (FS) theory, which has been integrated with MGDM perfectly by allowing DMExs to express their preferences in a manner that accommodates uncertainty, vagueness, and imprecision. After that, various extensions "intuitionistic FS (IFS)" (Atanassov, 1986) and "Pythagorean FS (PFS)" (Yager, 2014) have been developed to manage the uncertainty, vagueness, and imprecision occurred during the evaluation of alternatives. Under these environments, various applications have been developed. For instance, Jana and Hezam (2024) proposed a classical evaluation based on distance from average solution (EDAS) approach for the MGDM under the multi polar fuzzy environment for sponge iron factory location selection. Joshi et al. (2023) defined the TOPSIS approach in the context of moderator IFSs for renewable energy source selection. Kacprzak (2024) defined the MGDM approach based on the EDAS technique under the fuzzy environment. Similarly, various MGDM techniques (Arora and Garg, 2019; Dhankhar and Kumar, 2023; Kim and Van, 2021; Kumar and Garg, 2018; Rohit et al., 2025) have been proposed under the fuzzy theory and its extensions by various authors.

In Yager (2017), Yager introduced the concept of $q$-rung orthopair fuzzy sets ($q$-ROFSs), where a $q$-rung orthopair fuzzy number ($q$-ROFN) $\varphi = \langle \xi, \upsilon \rangle$ is defined by the membership grade (MG) $\xi$ and the non-MG (NMG) $\upsilon$, satisfying to the constraints $0 \leq \xi \leq 1$, $0 \leq \upsilon \leq 1$ and $0 \leq \xi^q + \upsilon^q \leq 1$, where $q \geq 1$. The $q$-ROFSs are a generalization of IFSs and PFSs. If $q = 1$, then $q$-ROFSs become IFSs; if $q = 2$, then $q$-ROFSs become PFSs. If the value of $q$ increases, then the lawful domain of a $q$-ROFN's MG and NMG enlarges. Consequently, $q$-ROFNs offer greater flexibility for experts in evaluating the features of alternatives compared to IFSs and PFSs. Recently, several applications (Kaur et al., 2023; Liu et al., 2021; Rawat et al., 2024; Wei et al., 2018) have been studied within the framework of $q$-ROFNs. For instance, Liu et al. (2021) defined the MGDM method based on the projection model and entropy measure under the $q$-ROFNs environment. Rawat et al. (2024) proposed the partitioned Hamy mean AOs for aggregating the $q$-ROFNs and MGDM method based on the proposed AOs. Wang et al. (2024) defined the operation laws of $q$-ROFNs based on the Sugeno-Weber norm and decision-making method for the selection of solar panel. A detailed literature is given in Section 1.1.

## 1.1 Literature Review

Ali (2022) introduced the decision-making approach based on the MARCOS approach and score function in the context of $q$-ROFNs with its application in SWM. Bhat et al. (2024) defined the MGDM method under the $q$-ROFNs environment and its application in supply chain management. Chatterjee and Seikh (2024) proposed the decision-making method using the confidence level for the municipal SWM using q-rung orthopair picture fuzzy numbers. Darko and Liang (2020) proposed the MGDM method based on the proposed Hamacher AO and EDAS technique under the $q$-ROFNs environment. Dhankhar and Kumar (2023) proposed the power geometric AO based on Yager norm and decision making method based on the proposed AO under the $q$-ROFNs environment. Ejegwa (2023) defined the distance-smilarity based operators in the context of $q$-ROFNs and their applications in various sectors. Ejegwa and Sarkar (2023) proposed the correlation measure for generalized orthopair fuzzy set and its applications. Khan et al. (2023) proposed the power AOs using the Aczel-Alsina norm for the aggregation of $q$-ROFNs and developed a MGDM approach on the basis of their proposed AOs in the context of $q$-ROFNs. Kumar and Chen (2022) proposed the weighted averaging AO and developed a MGDM approach utilizing their proposed AO under the environment of $q$-ROFNs. Liu et al. (2018) presented the power Maclaurin symmetric mean AO of $q$-ROFNs and developed a MGDM approach based on their proposed AO. Mishra et al. (2023) proposed a novel decision-making method based on MULTIMOORA technique, entropy and discrimination measures in the context of $q$-ROFNs to select the solid waste disposal method. Seikh and Chatterjee (2024) proposed a decision-making method under the Fermatean fuzzy environment for identifying the sustainable method for electronic waste management. Banu et al. (2024) proposed the decision-making method under the complex $q$-ROFNs environment based on the Frank AOs for solid waste management. Thilagasree et al. (2024) introduced a fuzzy decision-making method for management of municipal waste that can be used to generate electricity. Wang et al. (2020) developed a multi-attributive border approximation area comparison (MABAC) technique for MGDM in the environment of $q$-ROFNs. Wu et al. (2024) proposed a MGDM approach using the evident reasoning methodology and attribute reduction techniques. Xing et al. (2020) proposed the Hamy mean AOs of $q$-ROFNs and developed a MGDM approach on the basis of their proposed AOs. Zahid and Akram (2023) focused on municipal waste management in the Azerbaijan region of Iran and explored several waste-to-energy technologies using spherical fuzzy ELECTRE III method.

### 1.2 Motivation of this Study

In this paper, we find that the addition operation law (AOL), the multiplication operation law (MOL), the scalar multiplication operation law (SMOL) and the scalar power operation law (SPOL) of $q$-ROFNs proposed by Wang et al. (2024) have the shortcomings that they do not consistently obtain reasonable outcomes in some cases. Therefore, to overcome the shortcomings of Wang et al.'s operation laws (Wang et al., 2024) of $q$-ROFNs, new basic operation laws for $q$-ROFNs must be develop. Additionally, we also find that Khan et al.'s MGDM method (Khan et al., 2023), Liu et al.'s MGDM method (Liu et al., 2018) and Xing et al.'s MGDM method (Xing et al., 2020) cannot distinguish the preference order (PO) of alternatives in some situations. Therefore, it is also required to develop a new MGDM method to overcome the drawbacks of Khan et al.'s MGDM method (Khan et al., 2023), Liu et al.'s MGDM method (Liu et al., 2018) and Xing et al.'s MGDM method (Xing et al., 2020).

### 1.3 Contribution and Novelty of this Study

In this paper, we propose the new AOL, MOL, SMOL and SPOL of $q$-ROFNs based on Sugeno-Weber's norm (Wang et al., 2024; Weber, 1983) of $q$-ROFNs. The proposed operation laws of $q$-ROFNs can overcome the drawbacks of Wang et al.'s operation laws (Wang et al., 2024) of $q$-ROFNs. Based on the proposed novel AOL and the proposed novel SMOL of $q$-ROFNs, we propose the $q$-rung orthopair fuzzy Sugeno-Weber prioritized weighted arithmetic ($q$-ROFSWPWA) AO for the aggregation of $q$-ROFNs. Additionally, we present some properties of the proposed $q$-ROFSWPWA AO of $q$-ROFNs. The EDAS approach plays a crucial role in decision-making scenarios, particularly MGDM situations characterized by numerous competing attributes. The EDAS approach was first proposed by Keshavarz-Ghorabaee et al. (2015). It is the latest decision-making method and recently used by researchers to handle the decision-making problems under the fuzzy and its extensions environment. Therefore, based on the proposed $q$-ROFSWPWA AO of $q$-ROFNs and the classical EDAS approach (Keshavarz-Ghorabaee et al., 2015), we propose a novel MGDM method for the environmentof $q$-ROFNs. Furthermore, to illustrate the applicability of the proposed MGDM method, we examine a case study of the selection of SWMMS for efficient management and disposal of solid waste. In this case study, we consider five SWMMs "Sanitary Landfilling" ($\chi_1$), "Recycling and Reuse" ($\chi_2$), "Composting" ($\chi_3$), "Waste-to-Resource Innovations" ($\chi_4$) and "Energy Recovery" ($\chi_5$) as alternatives. We also present a comparison study and compare the obtained results with the results obtained from existing MGDM method. The proposed MGDM method can conquer the drawbacks of Khan et al.'s MGDM method (Khan et al., 2023), Liu et al.'s MGDM method

(Liu et al., 2018) and Xing et al.'s MGDM method (Xing et al., 2020) in the context of $q$-ROFNs. Therefore main contributions of this paper are summarized as follows:

- We propose the new operation laws for $q$-ROFNs based on Sugeno-Weber's norm of $q$-ROFNs.
- Based on the proposed operation laws of $q$-ROFNs, we propose $q$-ROFSWPWA AO for aggregating the $q$-ROFNs.
- Based on the proposed $q$-ROFSWPWA AO of $q$-ROFNs and the classical EDAS approach, we propose a novel MGDM method for the environment of $q$-ROFNs.
- We solve a case study of a selection of sustainable SWMM by using the proposed MGDM method.
- We present a Comparative study of the proposed MGDM method with existing MGDM methods given in (Khan et al., 2023; Liu et al., 2018; Xing et al., 2020).

The main novelties of this paper are summarized as follows:

- The proposed operation laws of $q$-ROFNs can overcome the drawbacks of Wang et al.'s operation laws (Wang et al., 2024) of $q$-ROFNs.
- The proposed $q$-ROFSWPWA AO of $q$-ROFNs considers the priority relationship among aggregating $q$-ROFNs.
- The proposed MGDM method is a hybrid approach which is a combination of aggregation operator and classical EDAS technique.
- The proposed MGDM method can conquer the drawbacks of Khan et al.'s MGDM method (Khan et al., 2023), Liu et al.'s MGDM method (Liu et al., 2018) and Xing et al.'s MGDM method (Xing et al., 2020) in the context of $q$-ROFNs.

### 1.4 Structure of this Paper

The remainder of this paper is structured as follows. Section 2 provides the foundational concepts of this paper. In Section 3, we propose some basic operation laws using Sugeno-Weber's $T_n$ and $T_{cn}$ of $q$-ROFNs. In Section 4, we propose the $q$-ROFSWPWA AO of $q$-ROFNs. In Section 5, we propose a novel MGDM method using the proposed $q$-ROFSWPWA AO of $q$-ROFNs and EDAS method. In Section 6, we provide a case study of the selection of best sustainable SWMMs to illustrate the proposed MGDM method and comparative study with the existing MGDM methods. Section 7 presents the advantages and superiority of the proposed MGDM technique over the existing MGDM techniques. The conclusions are presented in Section 8. In the following, **Figure 1** shows a graphical abstract of this study.

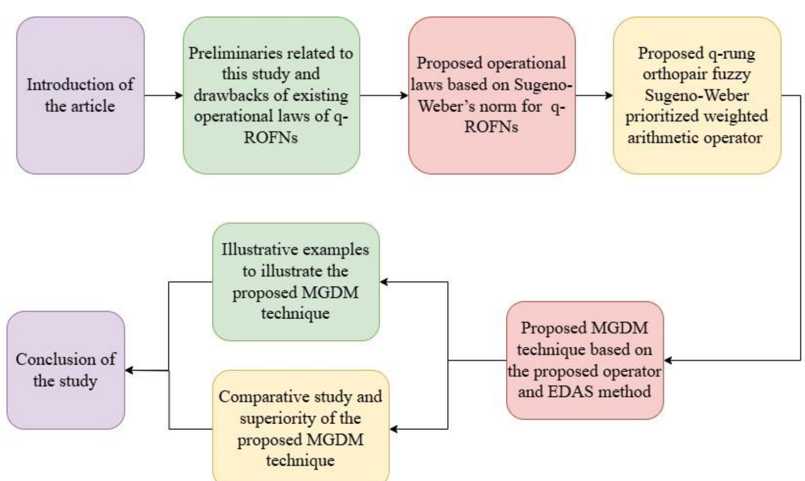


**Figure 1.** Graphical structure of the article's outline.

## 2. Preliminaries

***Definition* 2.1** (Yager, 2017). A $q$-ROFS $\Phi$ in a finite universe of discourse $Y$ is represented as follows:

$$\Phi = \{\langle y_i, \xi_\Phi(y_i), \upsilon_\Phi(y_i)\rangle |\, y_i \in Y\} \tag{1}$$

where, $\xi_\Phi(y_i)$ and $\upsilon_\Phi(y_i)$ denote the MG and the NMG of element $y_i$ belonging to the $q$-ROFS $\Phi$, respectively, $y_i \in Y$ , $0 \le \xi_\Phi(y_i) \le 1$, $0 \le \upsilon_\Phi(y_i) \le 1$, $0 \le \xi_\Phi^q(y_i) + \upsilon_\Phi^q(y_i) \le 1$ and $q \ge 1$. The indeterminacy degree $\pi_\Phi(y_i)$ of element $y_i$ belonging to the $q$-ROFS $\Phi$ is represented as $\pi_\Phi(y_i) = \left(1 - \xi_\Phi^q(y_i) - \upsilon_\Phi^q(y_i)\right)^{\frac{1}{q}}$, where $0 \le \pi_\Phi(y_i) \le 1$, $y_i \in Y$ and $q \ge 1$.

Wang et al. (2020) called the pair $\langle \xi_\Phi(y_i), \upsilon_\Phi(y_i)\rangle$ in the $q$-ROFS $\Phi$ as a $q$-ROFN, where $0 \le \xi_\Phi(y_i) \le 1$, $0 \le \upsilon_\Phi(y_i) \le 1, 0 \le \xi_\Phi^q(y_i) + \upsilon_\Phi^q(y_i) \le 1$ and $q \ge 1$.

***Definition* 2.2** (Wu et al., 2024). The score value $S(\varphi)$ of a $q$-ROFN $\varphi = \langle \xi, \upsilon\rangle$ is defined as follows:

$$S(\varphi) = \frac{1 + \xi^q - \upsilon^q}{2} \tag{2}$$

where, $S(\varphi) \in [0, 1]$ and $q \ge 1$.

***Definition* 2.3** (Yager, 2017). The accuracy value $A(\varphi)$ of a $q$-ROFN $\varphi = \langle \xi, \upsilon\rangle$ is defined as follows:

$$A(\varphi) = \xi^q + \upsilon^q \tag{3}$$

where, $A(\varphi) \in [0, 1]$ and $q \ge 1$.

***Definition* 2.4** (Wang et al., 2020). Consider two $q$-ROFNs $\varphi_1 = \langle \xi_1, \upsilon_1\rangle$ and $\varphi_2 = \langle \xi_2, \upsilon_2\rangle$. Then,

(i) If $S(\varphi_1) > S(\varphi_2)$, then $\varphi_1 \succ \varphi_2$.
(ii) If $S(\varphi_1) < S(\varphi_2)$, then $\varphi_1 \prec \varphi_2$.
(iii) If $S(\varphi_1) = S(\varphi_2)$, then
   a. If $A(\varphi_1) > A(\varphi_2)$, then $\varphi_1 \succ \varphi_2$.
   b. If $A(\varphi_1) < A(\varphi_2)$, then $\varphi_1 \prec \varphi_2$.
   c. If $A(\varphi_1) = A(\varphi_2)$, then $\varphi_1 = \varphi_2$.

***Definition* 2.5** (Kauers et al., 2011; Sarkar et al., 2023; Weber, 1983). The Sugeno-Weber's $T_{cn}$ $W_C(\alpha, \beta)$ and the Sugeno-Weber's $T_n$ $W_N(\alpha, \beta)$ of the real numbers $\alpha$ and $\beta$ are shown as follows:

$$\text{(i) } W_C(\alpha, \beta) = \begin{cases} W_C^D(\alpha, \beta), & \text{if } \tau = -1 \\ \min\left(1, \alpha + \beta - \frac{\tau}{1+\tau}\alpha\beta\right), & \text{if } -1 < \tau < +\infty \\ W_C^P(\alpha, \beta), & \text{if } \tau = +\infty \end{cases} \tag{4}$$

$$\text{(ii) } W_N(\alpha, \beta) = \begin{cases} W_N^D(\alpha, \beta), & \text{if } \tau = -1 \\ \max\left(0, \frac{\alpha + \beta - 1 + \tau\alpha\beta}{1+\tau}\right), & \text{if } -1 < \tau < +\infty \\ W_N^P(\alpha, \beta), & \text{if } \tau = +\infty \end{cases} \tag{5}$$

where, $\tau$ is a parameter of Sugeno-Weber's norm, $W_C^D(\alpha, \beta)$ denotes the drastic $T_{cn}$, $W_N^D(\alpha, \beta)$ denotes the drastic $T_n$, $W_C^P(\alpha, \beta)$ denotes the probabilistic $T_{cn}$ and $W_N^P(\alpha, \beta)$ denotes the probabilistic $T_n$.

***Definition* 2.6** (Yager, 2008). Let the attributes $G_1$, $G_2$,…, and $G_n$ are arranged in a linear priority order, where, $G_1 \succ G_2 \succ \cdots \succ G_n$. If attribute $G_t$ has a higher priority than that of attribute $G_k$, then $t < k$, where $k = 1, 2, \ldots, n$, $t = 1, 2, \ldots, n$ and $k \neq t$. Let $G_t(\chi)$ represent the performance of alternative $\chi$ with respect to attribute $G_t$. The prioritized average (PA) AO of $G_1(\chi)$, $G_2(\chi)$, …, and $G_n(\chi)$ is defined as follows:

$$PA(G_1(\chi), G_2(\chi), \ldots, G_n(\chi)) = \prod_{t=1}^{n} \frac{\zeta_t}{\sum_{j=1}^{n} \zeta_j} G_t(\chi) \tag{6}$$

where, $G_t(\chi) \in [0,1]$, $\zeta_1 = 1$, $\zeta_t = \prod_{s=1}^{t-1} G_s(\chi)$ and $t = 2, 3, \ldots, n$.

***Definition* 2.7** (Wang et al., 2024). Consider two $q$-ROFNs $\varphi_1 = \langle \xi_1, \upsilon_1 \rangle$ and $\varphi_2 = \langle \xi_2, \upsilon_2 \rangle$ with $\delta > 0$, $-1 < \tau < +\infty$ and $q \geq 1$. The operational rules of the $q$-ROFNs $\varphi_1$ and $\varphi_2$ using Sugeno-Weber's $T_{cn}$ and $T_n$ are shown as follows:

(i) $$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[q]{\xi_1^q + \xi_2^q - \frac{\tau}{1+\tau} \xi_1^q . \xi_2^q}, \sqrt[q]{\frac{\upsilon_1^q + \upsilon_2^q - 1 + \tau \upsilon_1^q . \upsilon_2^q}{1+\tau}} \right\rangle \tag{7}$$

(ii) $$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[q]{\frac{\xi_1^q + \xi_2^q - 1 + \tau \xi_1^q . \xi_2^q}{1+\tau}}, \sqrt[q]{\upsilon_1^q + \upsilon_2^q - \frac{\tau}{1+\tau} \upsilon_1^q . \upsilon_2^q} \right\rangle \tag{8}$$

(iii) $$\delta \varphi_1 = \left\langle \sqrt[q]{\frac{1+\tau}{\tau}\left(1 - \left(1 - \xi_1^q \left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)}, \sqrt[q]{\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau \upsilon_1^q + 1}{1+\tau}\right)^{\delta} - 1\right)} \right\rangle \tag{9}$$

(iv) $$\varphi_1^{\delta} = \left\langle \sqrt[q]{\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau \xi_1^q + 1}{1+\tau}\right)^{\delta} - 1\right)}, \sqrt[q]{\frac{1+\tau}{\tau}\left(1 - \left(1 - \upsilon_1^q \left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)} \right\rangle \tag{10}$$

## 2.1 Drawbacks of the Existing Operational Rules of the *q*-ROFNs

***Example* 2.1** Let $\varphi_1 = \langle 0.3, 0.5 \rangle$ and $\varphi_2 = \langle 0.6, 0.4 \rangle$ be two $q$-ROFNs with $\delta = 2$, $\tau = 1$ and $q = 2$. Then, we have

(i) By using Equation (7), we obtain

$$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[q]{\xi_1^q + \xi_2^q - \frac{\tau}{1+\tau} \xi_1^q . \xi_2^q}, \sqrt[q]{\frac{\upsilon_1^q + \upsilon_2^q - 1 + \tau \upsilon_1^q . \upsilon_2^q}{1+\tau}} \right\rangle$$

$$= \left\langle \sqrt[2]{(0.3)^2 + (0.6)^2 - \frac{1}{1+1}(0.3)^2 . (0.6)^2}, \sqrt[2]{\frac{(0.5)^2 + (0.4)^2 - 1 + 1.(0.5)^2 . (0.4)^2}{1+1}} \right\rangle$$

$$= \langle 0.6586, 0.5244i \rangle.$$

From the obtained result $\varphi_1 \oplus \varphi_2 = \langle 0.6586, 0.5244i \rangle$, it is observed that the NMG of the obtained $q$-ROFN $\langle 0.6586, 0.5244i \rangle$ is $0.5244i$, which is not reasonable because $0.5244i$ is an imaginary number. Therefore, Wang et al.'s AOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (7) has the above drawback in this situation.

(ii) By using Equation (8), we obtain

$$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[q]{\frac{\xi_1^q+\xi_2^q-1+\tau\xi_1^q.\xi_2^q}{1+\tau}}, \sqrt[q]{\upsilon_1^q+\upsilon_2^q-\frac{\tau}{1+\tau}\upsilon_1^q.\upsilon_2^q} \right\rangle$$

$$= \left\langle \sqrt[2]{\frac{(0.3)^2+(0.6)^2-1+1.(0.3)^2.(0.6)^2}{1+1}}, \sqrt[2]{(0.5)^2+(0.4)^2-\frac{1}{1+1}(0.5)^2.(0.4)^2} \right\rangle$$

$$= \langle 0.5087i, 0.6245 \rangle.$$

From the obtained result $\varphi_1 \otimes \varphi_2 = \langle 0.5087i, 0.6245 \rangle$, it is observed that the MG of the obtained $q$-ROFN $\langle 0.5087i, 0.6245 \rangle$ is $0.5087i$, which is not reasonable because $0.5087i$ is an imaginary number. Therefore, Wang et al.'s MOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (8) has the above drawback in this situation.

(iii) By using Equation (9), we obtain

$$2\varphi_1 = \left\langle \sqrt[q]{\frac{1+\tau}{\tau}\left(1-\left(1-\xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)}, \sqrt[q]{\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^{\delta}-1\right)} \right\rangle$$

$$= \left\langle \sqrt[2]{\frac{1+1}{1}\left(1-\left(1-(0.3)^2\left(\frac{1}{1+1}\right)\right)^{2}\right)}, \sqrt[2]{\frac{1}{1}\left((1+1)\left(\frac{1(0.5)^2+1}{1+1}\right)^{2}-1\right)} \right\rangle$$

$$= \langle 0.4195,\ 0.4677i \rangle.$$

From the obtained result $2\varphi_1 = \langle 0.4195,\ 0.4677i \rangle$, it is observed that the NMG of the obtained $q$-ROFN $\langle 0.4195,\ 0.4677i \rangle$ is $0.4677i$, which is not reasonable because $0.4677i$ is an imaginary number. Therefore, Wang et al.'s SMOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (9) has the above drawback in this situation.

(iv) By using Equation (10), we obtain

$$(\varphi_1)^2 = \left\langle \sqrt[q]{\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta}-1\right)}, \sqrt[q]{\frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)} \right\rangle$$

$$= \left\langle \sqrt[2]{\frac{1}{1}\left((1+1)\left(\frac{1.(0.3)^2+1}{1+1}\right)^{2}-1\right)}, \sqrt[2]{\frac{1+1}{1}\left(1-\left(1-(0.5)^2\left(\frac{1}{1+1}\right)\right)^{2}\right)} \right\rangle$$

$$= \langle 0.6371i, 0.6847 \rangle.$$

From the obtained result $(\varphi_1)^2 = \langle 0.6371i, 0.6847 \rangle$, it is observed that the MG of the obtained $q$-ROFN $\langle 0.6371i, 0.6847 \rangle$ is $0.6371i$, which is not reasonable because $0.6371i$ is an imaginary number. Therefore, Wang et al.'s SPOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (10) has the above drawback in this situation.

***Example* 2.2** Let $\varphi_1 = \langle 0.4, 0.5\rangle$ and $\varphi_2 = \langle 0.3, 0.7\rangle$ be two $q$-ROFNs with $\delta = 3$, $\tau = 2$ and $q = 2$. Then, we have

(i) By using Equation (7), we obtain

$$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[q]{\xi_1^q + \xi_2^q - \frac{\tau}{1+\tau}\xi_1^q.\xi_2^q}, \sqrt[q]{\frac{\upsilon_1^q + \upsilon_2^q - 1 + \tau\upsilon_1^q.\upsilon_2^q}{1+\tau}} \right\rangle$$

$$= \left\langle \sqrt[2]{(0.4)^2 + (0.3)^2 - \frac{2}{1+2}(0.4)^2.(0.3)^2}, \sqrt[2]{\frac{(0.5)^2 + (0.7)^2 - 1 + 2.(0.5)^2.(0.7)^2}{1+2}} \right\rangle$$

$$= \langle 0.4903, 0.0707i \rangle.$$

From the obtained result $\varphi_1 \oplus \varphi_2 = \langle 0.4903, 0.0707i\rangle$, it is observed that the NMG of the obtained $q$-ROFN $\langle 0.4903, 0.0707i\rangle$ is $0.0707i$, which is not reasonable because $0.0707i$ is an imaginary number. Therefore, Wang et al.'s AOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (7) has the above drawback in this situation.

(ii) By using Equation (8), we obtain

$$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[q]{\frac{\xi_1^q + \xi_2^q - 1 + \tau\xi_1^q.\xi_2^q}{1+\tau}}, \sqrt[q]{\upsilon_1^q + \upsilon_2^q - \frac{\tau}{1+\tau}\upsilon_1^q.\upsilon_2^q} \right\rangle$$

$$= \left\langle \sqrt[2]{\frac{(0.4)^2 + (0.3)^2 - 1 + 2.(0.4)^2.(0.3)^2}{1+2}}, \sqrt[2]{(0.5)^2 + (0.7)^2 - \frac{2}{1+2}(0.5)^2.(0.7)^2} \right\rangle$$

$$= \langle 0.4903i, 0.8114 \rangle.$$

From the obtained result $\varphi_1 \otimes \varphi_2 = \langle 0.4903i, 0.8114 \rangle$, it is observed that the MG of the obtained $q$-ROFN $\langle 0.4903i, 0.8114\rangle$ is $0.4903i$, which is not reasonable because $0.4903i$ is an imaginary number. Therefore, Wang et al.'s MOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (8) has the above drawback in this situation.

(iii) By using Equation (9), we obtain

$$3\varphi_1 = \left\langle \sqrt[q]{\frac{1+\tau}{\tau}\left(1 - \left(1 - \xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^\delta\right)}, \sqrt[q]{\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q + 1}{1+\tau}\right)^\delta - 1\right)} \right\rangle$$

$$= \left\langle \sqrt[2]{\frac{1+2}{2}\left(1 - \left(1 - (0.4)^2\left(\frac{2}{1+2}\right)\right)^3\right)}, \sqrt[2]{\frac{1}{2}\left((1+2)\left(\frac{2(0.5)^2 + 1}{1+2}\right)^3 - 1\right)} \right\rangle$$

$$= \langle 0.6562,\ 0.3953i \rangle.$$

From the obtained result $3\varphi_1 = \langle 0.6562,\ 0.3953i \rangle$, it is observed that the NMG of the obtained $q$-ROFN $\langle 0.6562,\ 0.3953i \rangle$ is $0.3953i$ , which is not reasonable because $0.3953i$ is an imaginary number. Therefore, Wang et al.'s SMOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (9) has the above drawback in this situation.

(iv) By using Equation (10), we obtain

$$(\varphi_1)^3 = \left\langle \sqrt[q]{\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta}-1\right)}, \sqrt[q]{\frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)} \right\rangle$$

$$= \left\langle \sqrt[2]{\frac{1}{2}\left((1+2)\left(\frac{2.(0.4)^2+1}{1+2}\right)^{3}-1\right)}, \sqrt[2]{\frac{1+2}{2}\left(1-\left(1-(0.5)^2\left(\frac{2}{1+2}\right)\right)^{3}\right)} \right\rangle$$

$$= \langle 0.4314i\,, 0.7949\,\rangle.$$

From the obtained result $(\varphi_1)^3 = \langle 0.4314i\,, 0.7949\,\rangle$, it is observed that the MG of the obtained $q$-ROFN $\langle 0.4314i\,, 0.7949\,\rangle$ is $0.4314i$, which is not reasonable because $0.4314i$ is an imaginary number. Therefore, Wang et al.'s SPOL (Wang et al., 2024) of $q$-ROFNs shown in Equation (10) has the above drawback in this situation.

From Example 2.1 and Example 2.2, it can be seen that Wang et al.'s AOL, MOL, SMOL and SPOL (Wang et al., 2024) have the shortcomings that they do not consistently obtain reasonable outcomes in some scenarios. Therefore, we need to develop new operational laws of $q$-ROFNs to overcome the drawback of Wang et al.'s AOL, MOL, SMOL and SPOL (Wang et al., 2024) of $q$-ROFNs.

## 3. The Proposed Novel Sugeno-Weber's Operational Rules using Sugeno-Weber's Norm for *q*-ROFNs

This section proposes the new AOL, MOL, SMOL and SPOL for $q$-ROFNs on the basis of Sugeno-Weber norms presented in *Definition* 2.5.

***Definition* 3.1** Let $\varphi_1 = \langle \xi_1, \upsilon_1\rangle$ and $\varphi_2 = \langle \xi_2, \upsilon_2\rangle$ be two $q$-ROFNs with $\delta > 0$, $-1 < \tau < +\infty$ and $q \geq 1$. The proposed AOL, the proposed MOL, the proposed SMOL and the proposed SPOL of the $q$-ROFNs $\varphi_1$ and $\varphi_2$ using Sugeno-Weber's $T_n$ and $T_{cn}$ are shown as follows:

(i) The proposed AOL:

$$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[q]{\min\left\{1, \xi_1^q + \xi_2^q - \frac{\tau}{1+\tau}\xi_1^q.\xi_2^q\right\}}, \sqrt[q]{\max\left\{0, \frac{\upsilon_1^q+\upsilon_2^q-1+\tau\upsilon_1^q.\upsilon_2^q}{1+\tau}\right\}} \right\rangle \tag{11}$$

(ii) The proposed MOL:

$$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[q]{\max\left\{0, \frac{\xi_1^q+\xi_2^q-1+\tau\xi_1^q.\xi_2^q}{1+\tau}\right\}}, \sqrt[q]{\min\left\{1, \upsilon_1^q + \upsilon_2^q - \frac{\tau}{1+\tau}\upsilon_1^q.\upsilon_2^q\right\}} \right\rangle \tag{12}$$

(iii) The proposed SMOL:

$$\delta\varphi_1 = \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1-\left(1-\xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^{\delta}-1\right)\right\}} \right\rangle \tag{13}$$

(iv) The proposed SPOL:

$$\varphi_1^{\delta}=\left\langle\sqrt[q]{\max\left\{0,\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta}-1\right)\right\}},\sqrt[q]{\min\left\{1,\frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)\right\}}\right\rangle \tag{14}$$

***Example* 3.1** Let $\varphi_1=\langle 0.8,0.6\rangle$ and $\varphi_2=\langle 0.5,0.7\rangle$ be two $q$-ROFNs with $\delta=2$, $\tau=2$ and $q=2$. Then, we have

(i) By using the proposed AOL presented in Equation (11), we obtain

$$\varphi_1\oplus\varphi_2=\left\langle\sqrt[q]{\min\left\{1,\xi_1^q+\xi_2^q-\frac{\tau}{1+\tau}\xi_1^q.\xi_2^q\right\}},\sqrt[q]{\max\left\{0,\frac{\upsilon_1^q+\upsilon_2^q-1+\tau\upsilon_1^q.\upsilon_2^q}{1+\tau}\right\}}\right\rangle$$

$$=\left\langle\sqrt[2]{\min\left\{1,(0.8)^2+(0.5)^2-\frac{2}{1+2}(0.8)^2.(0.5)^2\right\}},\sqrt[2]{\max\left\{0,\frac{(0.6)^2+(0.7)^2-1+2.(0.6)^2.(0.7)^2}{1+2}\right\}}\right\rangle$$

$$=\langle 0.8851,0.2600\rangle.$$

(ii) By using the proposed MOL presented in Equation (12), we obtain

$$\varphi_1\otimes\varphi_2=\left\langle\sqrt[q]{\max\left\{0,\frac{\xi_1^q+\xi_2^q-1+\tau\xi_1^q.\xi_2^q}{1+\tau}\right\}},\sqrt[q]{\min\left\{1,\upsilon_1^q+\upsilon_2^q-\frac{\tau}{1+\tau}\upsilon_1^q.\upsilon_2^q\right\}}\right\rangle$$

$$=\left\langle\sqrt[2]{\max\left\{0,\frac{(0.8)^2+(0.5)^2-1+2.(0.8)^2.(0.5)^2}{1+2}\right\}},\sqrt[2]{\min\left\{1,(0.6)^2+(0.7)^2-\frac{2}{1+2}(0.6)^2.(0.7)^2\right\}}\right\rangle$$

$$=\langle 0.2646,0.8558\rangle.$$

(iii) By using the proposed SMOL presented in Equation (13), we obtain

$$2\varphi_1=\left\langle\sqrt[q]{\min\left\{1,\frac{1+\tau}{\tau}\left(1-\left(1-\xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)\right\}},\sqrt[q]{\max\left\{0,\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^{\delta}-1\right)\right\}}\right\rangle$$

$$=\left\langle\sqrt[2]{\min\left\{1,\frac{1+2}{2}\left(1-\left(1-(0.8)^2\left(\frac{2}{1+2}\right)\right)^2\right)\right\}},\sqrt[2]{\max\left\{0,\frac{1}{2}\left((1+2)\left(\frac{2(0.6)^2+1}{1+2}\right)^2-1\right)\right\}}\right\rangle$$

$$=\langle 1,0\rangle.$$

(iv) By using the proposed SPOL presented in Equation (14), we obtain

$$(\varphi_1)^2=\left\langle\sqrt[q]{\max\left\{0,\frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta}-1\right)\right\}},\sqrt[q]{\min\left\{1,\frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)\right\}}\right\rangle$$

$$= \left\langle \sqrt[2]{\max\left\{0, \frac{1}{2}\left((1+2)\left(\frac{2.(0.8)^2+1}{1+2}\right)^2 - 1\right)\right\}}, \sqrt[2]{\min\left\{1, \frac{1+2}{2}\left(1-\left(1-(0.6)^2\left(\frac{2}{1+2}\right)\right)^2\right)\right\}} \right\rangle$$

$$= \langle 0.6053\,, 0.7960\,\rangle.$$

***Example* 3.2** Consider the same $q$-ROFNs $\varphi_1 = \langle 0.3,0.5\rangle$ and $\varphi_2 = \langle 0.6,0.4\rangle$ with $\delta = 2,\ \tau = 1$ and $q = 2$ as given in *Example* 2.1. Then, we have

(i) By using the proposed AOL presented in Equation (11), we obtain

$$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[2]{\min\left\{1, (0.3)^2 + (0.6)^2 - \frac{1}{1+1}(0.3)^2.(0.6)^2\right\}}, \sqrt[2]{\max\left\{0, \frac{(0.5)^2+(0.4)^2-1+1.(0.5)^2.(0.4)^2}{1+1}\right\}} \right\rangle$$

$$= \langle 0.6586, 0\,\rangle.$$

(ii) By using the proposed MOL presented in Equation (12), we obtain

$$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[2]{\max\left\{0, \frac{(0.3)^2+(0.6)^2-1+1.(0.3)^2.(0.6)^2}{1+1}\right\}}, \sqrt[2]{\min\left\{1, (0.5)^2 + (0.4)^2 - \frac{1}{1+1}(0.5)^2.(0.4)^2\right\}} \right\rangle$$

$$= \langle 0, 0.6245\,\rangle.$$

(iii) By using the proposed SMOL presented in Equation (13), we obtain

$$2\varphi_1 = \left\langle \sqrt[2]{\min\left\{1, \frac{1+1}{1}\left(1-\left(1-(0.3)^2\left(\frac{1}{1+1}\right)\right)^2\right)\right\}}, \sqrt[2]{\max\left\{0, \frac{1}{1}\left((1+1)\left(\frac{1(0.5)^2+1}{1+1}\right)^2 - 1\right)\right\}} \right\rangle$$

$$= \langle 0.4195,\ 0\,\rangle.$$

(iv) By using proposed SPOL presented in Equation (14), we obtain

$$(\varphi_1)^2 = \left\langle \sqrt[2]{\max\left\{0, \frac{1}{1}\left((1+1)\left(\frac{1.(0.3)^2+1}{1+1}\right)^2 - 1\right)\right\}}, \sqrt[2]{\min\left\{1, \frac{1+1}{1}\left(1-\left(1-(0.5)^2\left(\frac{1}{1+1}\right)\right)^2\right)\right\}} \right\rangle$$

$$= \langle 0\,, 0.6847\,\rangle.$$

***Example* 3.3** Consider the same $q$-ROFNs $\varphi_1 = \langle 0.4,0.5\rangle$ and $\varphi_2 = \langle 0.3,0.7\rangle$ with $\delta = 3, \tau = 2$ and $q = 2$ as given in *Example* 2.2. Then, we have

(i) By using the proposed AOL presented in Equation (11), we obtain

$$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[2]{\min\left\{1, (0.4)^2 + (0.3)^2 - \frac{2}{1+2}(0.4)^2.(0.3)^2\right\}}, \sqrt[2]{\max\left\{0, \frac{(0.5)^2+(0.7)^2-1+2.(0.5)^2.(0.7)^2}{1+2}\right\}} \right\rangle$$

$$= \langle 0.4903, 0\,\rangle.$$

(ii) By using the proposed MOL presented in Equation (12), we obtain

$$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[2]{\max\left\{0, \frac{(0.4)^2+(0.3)^2-1+2.(0.4)^2.(0.3)^2}{1+2}\right\}}, \sqrt[2]{\min\left\{1, (0.5)^2 + (0.7)^2 - \frac{2}{1+2}(0.5)^2.(0.7)^2\right\}} \right\rangle$$

$$= \langle 0, 0.8114 \rangle.$$

(iii) By using the proposed SMOL presented in Equation (13), we obtain

$$3\varphi_1 = \left\langle \sqrt[2]{\min\left\{1, \frac{1+2}{2}\left(1-\left(1-(0.4)^2\left(\frac{2}{1+2}\right)\right)^3\right)\right\}}, \sqrt[2]{\max\left\{0, \frac{1}{2}\left((1+2)\left(\frac{2(0.5)^2+1}{1+2}\right)^3 - 1\right)\right\}} \right\rangle$$

$$= \langle 0.6562, 0 \rangle.$$

(iv) By using the proposed SPOL presented in Equation (14), we obtain

$$(\varphi_1)^3 = \left\langle \sqrt[2]{\max\left\{0, \frac{1}{2}\left((1+2)\left(\frac{2.(0.4)^2+1}{1+2}\right)^3 - 1\right)\right\}}, \sqrt[2]{\min\left\{1, \frac{1+2}{2}\left(1-\left(1-(0.5)^2\left(\frac{2}{1+2}\right)\right)^3\right)\right\}} \right\rangle$$

$$= \langle 0, 0.7949 \rangle.$$

From *Example* 3.2 and *Example* 3.3, it can be seen that the proposed AOL, MOL, SMOL and SPOL of $q$-ROFNs presented in *Definition* 3.1 can overcome the shortcomings of Wang et al.'s AOL, MOL, SMOL and SPOL (Wang et al., 2024) of $q$-ROFNs.

***Theorem* 3.1** Let $\varphi_1 = \langle \xi_1, \upsilon_1 \rangle$ and $\varphi_2 = \langle \xi_2, \upsilon_2 \rangle$ be two $q$-ROFNs with $\delta$, $\delta_1$ and $\delta_2 > 0$. The proposed operational rules of $q$-ROFNs presented in *Definition* 3.1 fulfills the following properties:

(i) $\varphi_1 \oplus \varphi_2 = \varphi_2 \oplus \varphi_1$,

(ii) $\varphi_1 \otimes \varphi_2 = \varphi_2 \otimes \varphi_1$,

(iii) $\delta(\varphi_1 \oplus \varphi_2) = \delta\varphi_1 \oplus \delta\varphi_2$,

(iv) $\delta_1\varphi_1 \oplus \delta_2\varphi_1 = (\delta_1 + \delta_2)\varphi_1$,

(v) ${\varphi_1}^{\delta_1} \otimes {\varphi_1}^{\delta_2} = {\varphi_1}^{\delta_1+\delta_2}$,

(vi) $(\varphi_1 \otimes \varphi_2)^{\delta} = {\varphi_1}^{\delta} \otimes {\varphi_2}^{\delta}$.

***Proof***: Consider the $q$-ROFNs $\varphi_1 = \langle \xi_1, \upsilon_1 \rangle$ and $\varphi_2 = \langle \xi_2, \upsilon_2 \rangle$.

(i) By using Equation (11), we get

$$\varphi_1 \oplus \varphi_2 = \left\langle \sqrt[q]{\min\left\{1, \xi_1^q + \xi_2^q - \frac{\tau}{1+\tau}\xi_1^q.\xi_2^q\right\}}, \sqrt[q]{\max\left\{0, \frac{\upsilon_1^q+\upsilon_2^q-1+\tau\upsilon_1^q.\upsilon_2^q}{1+\tau}\right\}} \right\rangle$$

$$= \left\langle \sqrt[q]{\min\left\{1, \xi_2^q + \xi_1^q - \frac{\tau}{1+\tau}\xi_2^q.\xi_1^q\right\}}, \sqrt[q]{\max\left\{0, \frac{\upsilon_2^q+\upsilon_1^q-1+\tau\upsilon_2^q.\upsilon_1^q}{1+\tau}\right\}} \right\rangle$$

$$= \varphi_2 \oplus \varphi_1,$$

where, $q \geq 1$ and $-1 < \tau < +\infty$.

(ii) By using Equation (12), we get

$$\varphi_1 \otimes \varphi_2 = \left\langle \sqrt[q]{\max\left\{0, \frac{\xi_1^q + \xi_2^q - 1 + \tau\xi_1^q.\xi_2^q}{1+\tau}\right\}}, \sqrt[q]{\min\left\{1, \upsilon_1^q + \upsilon_2^q - \frac{\tau}{1+\tau}\upsilon_1^q.\upsilon_2^q\right\}} \right\rangle$$

$$= \left\langle \sqrt[q]{\max\left\{0, \frac{\xi_2^q + \xi_1^q - 1 + \tau\xi_2^q.\xi_1^q}{1+\tau}\right\}}, \sqrt[q]{\min\left\{1, \upsilon_2^q + \upsilon_1^q - \frac{\tau}{1+\tau}\upsilon_2^q.\upsilon_1^q\right\}} \right\rangle$$

$$= \varphi_2 \otimes \varphi_1,$$

where, $q \geq 1$ and $-1 < \tau < +\infty$.

(iii) By using Equations (11) and (13), we get

$$\delta(\varphi_1 \oplus \varphi_2) = \delta \left\langle \sqrt[q]{\min\left\{1, \xi_1^q + \xi_2^q - \frac{\tau}{1+\tau}\xi_1^q.\xi_2^q\right\}}, \sqrt[q]{\max\left\{0, \frac{\upsilon_1^q + \upsilon_2^q - 1 + \tau\upsilon_1^q.\upsilon_2^q}{1+\tau}\right\}} \right\rangle$$

$$= \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \left(1 - \xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^\delta \left(1 - \xi_2^q\left(\frac{\tau}{1+\tau}\right)\right)^\delta\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^\delta \left(\frac{\tau\upsilon_2^q+1}{1+\tau}\right)^\delta - 1\right)\right\}} \right\rangle$$

$$= \delta\varphi_1 \oplus \delta\varphi_2,$$

where, $q \geq 1$ and $-1 < \tau < +\infty$.

(iv) By using Equations (11) and (13), we get

$$\delta_1\varphi_1 \oplus \delta_2\varphi_1 = \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \left(1 - \xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta_1}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^{\delta_1} - 1\right)\right\}} \right\rangle$$

$$\oplus \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \left(1 - \xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta_2}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^{\delta_2} - 1\right)\right\}} \right\rangle$$

$$= \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \left(1 - \xi_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta_1+\delta_2}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\upsilon_1^q+1}{1+\tau}\right)^{\delta_1+\delta_2} - 1\right)\right\}} \right\rangle$$

$= (\delta_1 + \delta_2)\varphi_1.$
where, $q \geq 1$ and $-1 < \tau < +\infty$.

(v) By using Equations (12) and (14), we get

$$\varphi_1{}^{\delta_1} \otimes \varphi_1{}^{\delta_2} = \left\langle \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta_1} - 1\right)\right\}}, \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta_1}\right)\right\}}\right\rangle$$

$$\otimes \left\langle \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta_2} - 1\right)\right\}}, \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta_2}\right)\right\}}\right\rangle$$

$$= \left\langle \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta_1+\delta_2} - 1\right)\right\}}, \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta_1+\delta_2}\right)\right\}}\right\rangle$$

$$= \varphi_1{}^{\delta_1+\delta_2}.$$

where, $q \geq 1$ and $-1 < \tau < +\infty$.

(vi) By using Equations (12) and (14), we get

$$(\varphi_1 \otimes \varphi_2)^{\delta} = \left\langle \sqrt[q]{\max\left\{0, \frac{\xi_1^q+\xi_2^q-1+\tau\xi_1^q.\xi_2^q}{1+\tau}\right\}}, \sqrt[q]{\min\left\{1, \upsilon_1^q+\upsilon_2^q-\frac{\tau}{1+\tau}\upsilon_1^q.\upsilon_2^q\right\}}\right\rangle^{\delta}$$

$$= \left\langle \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\left(\frac{\tau\xi_1^q+1}{1+\tau}\right)^{\delta}\left(\frac{\tau\xi_2^q+1}{1+\tau}\right)^{\delta} - 1\right)\right\}}, \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1-\left(1-\upsilon_1^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\left(1-\upsilon_2^q\left(\frac{\tau}{1+\tau}\right)\right)^{\delta}\right)\right\}}\right\rangle$$

$$= \varphi_1{}^{\delta} \otimes \varphi_2{}^{\delta},$$

where, $q \geq 1$ and $-1 < \tau < +\infty$.

## 4. The Proposed $q$-Rung Orthopair Fuzzy Sugeno-Weber Prioritized Weighted Arithmetic AO of $q$-ROFNs

In this section, we propose the $q$-rung orthopair fuzzy Sugeno-Weber prioritized weighted arithmetic ($q$-ROFSWPWA) AO using the proposed AOL and SMOL presented in *Definition* 3.1 and the PA AO shown in *Definition* 2.6.

***Definition* 4.1** Let $\varphi_1 = \langle \xi_1, \upsilon_1 \rangle$, $\varphi_2 = \langle \xi_2, \upsilon_2 \rangle, \ldots,$ and $\varphi_n = \langle \xi_n, \upsilon_n \rangle$ be $q$-ROFNs. The proposed $q$-ROFSWPWA AO of the $q$-ROFNs $\varphi_1 = \langle \xi_1, \upsilon_1 \rangle$, $\varphi_2 = \langle \xi_2, \upsilon_2 \rangle, \ldots,$and $\varphi_n = \langle \xi_n, \upsilon_n \rangle$ is defined as follows:

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t$$

$$= \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \prod_{t=1}^{n}\left(1 - \xi_t^q\left(\frac{\tau}{1+\tau}\right)\right)^{\frac{\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\prod_{t=1}^{n}\left(\frac{\tau\upsilon_k^q+1}{1+\tau}\right)^{\frac{\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}} - 1\right)\right\}} \right\rangle \quad (15)$$

where, $q \geq 1$, $1 < \tau < +\infty$, $\varpi_t$ denotes the weight of $q$-ROFN $\varphi_t$, $\varpi_t \geq 0$, $\sum_{t=1}^{n}\varpi_t = 1$, $\zeta_1 = 1$, $\zeta_r = \prod_{s=1}^{r-1} S(\varphi_s)$, $r = 2,3,\ldots,h$, $S(\varphi_s) = \frac{1 + \xi_s{}^q - \upsilon_s{}^q}{2}$, and $s = 1,2,3,\ldots,r-1$.

***Example* 4.1** Consider three $q$-ROFNs $\varphi_1 = \langle 0.2, 0.4\rangle$, $\varphi_2 = \langle 0.4, 0.5\rangle$ and $\varphi_3 = \langle 0.5, 0.3\rangle$ with the weights $\varpi_1 = 0.4$, $\varpi_2 = 0.2$, and $\varpi_3 = 0.4$, respectively. Firstly, we compute the values of $\zeta_1, \zeta_2$, and $\zeta_3$, respectively, where $\zeta_1 = 1, \zeta_2 = S(\varphi_1) = \frac{1 + \xi_1{}^q - \upsilon_1{}^q}{2} = \frac{1 + (0.2)^2 - (0.4)^2}{2} = 0.44$ and $\zeta_3 = S(\varphi_1) \times S(\varphi_2) = 0.44 \times 0.455 = 0.2002$. Then, we aggregate the $q$-ROFNs $\varphi_1$, $\varphi_2$ and $\varphi_3$ by utilizing the proposed $q$-ROFSWPWA AO of $q$-ROFNs shown in Equation (15), where we take $q = 2$ and $\tau = 2$, shown as follows:

$q$-ROFSWPWA$(\varphi_1, \varphi_2, \varphi_3)$

$$= \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \prod_{t=1}^{n}\left(1 - \xi_t^q\left(\frac{\tau}{1+\tau}\right)\right)^{\frac{\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\prod_{t=1}^{n}\left(\frac{\tau\upsilon_k^q+1}{1+\tau}\right)^{\frac{\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}} - 1\right)\right\}} \right\rangle$$

$= \langle 0.3008, 0.4033\rangle$.

In the following, we present some properties of the proposed $q$-ROFSWPWA AO shown in Equation (15).

***Property* 4.1 (Idempotency).** Let $\varphi_1, \varphi_2, \ldots$, and $\varphi_n$ be $q$-ROFNs with the weights $\varpi_1, \varpi_2, \ldots$, and $\varpi_n$, respectively, where $\varpi_t \geq 0$ and $\sum_{t=1}^{n}\varpi_t = 1$. If $\varphi_t = \varphi$, where $t = 1, 2, \ldots, n$, then

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \varphi.$$

***Proof*:** Because the weights of the $q$-ROFNs $\varphi_1, \varphi_2, \ldots$, and $\varphi_n$ are $\varpi_1, \varpi_2, \ldots$, and, $\varpi_n$, respectively, where $\varpi_t \geq 0$ and $\sum_{t=1}^{n}\varpi_t = 1$, if $\varphi_1 = \varphi_2 = \ldots = \varphi_n = \varphi$, then by utilizing Equation (15), we have,

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \bigoplus_{t=1}^{n} \frac{\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}\varphi_t$$

$$= \bigoplus_{t=1}^{n} \frac{\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}\varphi$$

$$= \frac{\sum_{t=1}^{n}\varpi_t\zeta_t}{\sum_{t=1}^{n}\varpi_t\zeta_t}\varphi$$

$$= \varphi.$$

***Property* 4.2 (Boundedness).** Let $\varphi_1, \varphi_2, \ldots$, and $\varphi_n$ be $q$-ROFNs, let $\varphi^- = \min\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ and let $\varphi^+ = \max\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Then,

$$\varphi^- \leq q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) \leq \varphi^+.$$

***Proof***: Because $\varphi^{-} = \min\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ and $\varphi^{+} = \max\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$, by utilizing Equation (15), we have,

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t \leq \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t^{+} = \frac{\sum_{t=1}^{n} \varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t^{+} = \varphi^{+},$$

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t \geq \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t^{-} = \frac{\sum_{t=1}^{n} \varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t^{-} = \varphi^{-}.$$

Thus, we get $\varphi^{-} \leq q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) \leq \varphi^{+}$.

***Property* 4.3 (Monotonicity).** Let $\varphi_1$, $\varphi_2$, ..., $\varphi_n$ , $\varphi_1^{*}$, $\varphi_2^{*}$, ..., and $\varphi_n^{*}$ be $q$-ROFNs. If $\varphi_t \leq \varphi_t^{*}$, where $t = 1, 2, \ldots, n$, then

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) \leq q\text{-ROFSWPWA}(\varphi_1^{*}, \varphi_2^{*}, \ldots, \varphi_n^{*}).$$

***Proof***: By Equation (15), we have

$$q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t,$$

$$q\text{-ROFSWPWA}(\varphi_1^{*}, \varphi_2^{*}, \ldots, \varphi_n^{*}) = \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t^{*}.$$

Because $\varphi_t \leq \varphi_t^{*}$, $\forall t = 1,2, \ldots, n$, we have

$$\bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t \leq \bigoplus_{t=1}^{n} \frac{\varpi_t \zeta_t}{\sum_{t=1}^{n} \varpi_t \zeta_t} \varphi_t^{*}.$$

Thus, we obtain $q\text{-ROFSWPWA}(\varphi_1, \varphi_2, \ldots, \varphi_n) \leq q\text{-ROFSWPWA}(\varphi_1^{*}, \varphi_2^{*}, \ldots, \varphi_n^{*})$.

## 5. The Proposed MGDM Technique based on the Proposed *q*-ROFSWPWA AO of *q*-ROFNs and EDAS Approach

In this section, we introduce a novel MGDM technique by utilizing the proposed $q$-ROFSWPWA AO of $q$-ROFNs and the EDAS approach (Keshavarz-Ghorabaee et al., 2015). Let $\chi_1$, $\chi_2$, …, and $\chi_m$ be alternatives and let $G_1$, $G_2$,…,and $G_n$ be attributes. Let $\varpi_1$, $\varpi_2$, …., $\varpi_n$ be the weights of the attributes $G_1$, $G_2$,…,and $G_n$ , respectively, where $\varpi_t \geq 0$, $t = 1, 2, \ldots, n$ and $\sum_{t=1}^{n} \varpi_t = 1$. Let $\psi_1$, $\psi_2$, …, and $\psi_p$ be decision making experts (DMExs) and let $\vartheta_1$, $\vartheta_2$, ..., and $\vartheta_p$ be the weights of the DMExs $\psi_1$, $\psi_2$, …, and $\psi_p$, respectively, where $\vartheta_i \geq 0$, $i = 1, 2, \ldots, p$ and $\sum_{i=1}^{p} \vartheta_i = 1$. Each DMEx $\psi_i$ uses a $q$-ROFN $\tilde{\varphi}_{kt}^{i} =$

$\langle \tilde{\xi}_{kt}^{i}, \tilde{\upsilon}_{kt}^{i} \rangle$ to assess alternative $\chi_k$ towards the attribute $G_t$ for composing the decision matrix (DMx) $\tilde{R}^i = (\tilde{\varphi}_{kt}^{i})_{m\times n}$, shown as follows:

$$\tilde{R}^i = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \vdots \\ \chi_m \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & \dots & G_n \end{matrix} \\ \begin{pmatrix} \tilde{\varphi}_{11}^{i} & \tilde{\varphi}_{12}^{i} & \cdots & \tilde{\varphi}_{1n}^{i} \\ \tilde{\varphi}_{21}^{i} & \tilde{\varphi}_{22}^{i} & \dots & \tilde{\varphi}_{2n}^{i} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\varphi}_{m1}^{i} & \tilde{\varphi}_{m2}^{i} & \dots & \tilde{\varphi}_{mn}^{i} \end{pmatrix} \end{matrix},$$

The steps of the proposed MGDM technique are shown as follows:

**Step 1:** Based on (Kumar and Chen, 2022), convert the DMx $\tilde{R}^i = (\tilde{\varphi}_{kt}^{i})_{m\times n} = (\langle \tilde{\xi}_{kt}^{i}, \tilde{\upsilon}_{kt}^{i} \rangle)_{m\times n}$ into the normalized decision matrix (NDMx) $R^i = (\varphi_{kt}^{i})_{m\times n} = (\langle \xi_{kt}^{i}, \upsilon_{kt}^{i} \rangle)_{m\times n}$, where $k = 1, 2, \dots, m$, $t = 1, 2, \dots, n$ and $i = 1,2, \dots, p$, as follows:

$$\varphi_{kt}^{i} = \begin{cases} \langle \tilde{\xi}_{kt}^{i}, \tilde{\upsilon}_{kt}^{i} \rangle, & \text{if } G_t \text{ is a benefit type attribute} \\ \langle \tilde{\upsilon}_{kt}^{i}, \tilde{\xi}_{kt}^{i} \rangle, & \text{if } G_t \text{ is a cost type attribute} \end{cases} \tag{16}$$

**Step 2:** Compute the weight $\varrho_{kt}^{1}, \varrho_{kt}^{2}, \dots, \varrho_{kt}^{p}$ of $q$-ROFN $\varphi_{kt}^{1}, \varphi_{kt}^{2}, \dots, \varphi_{kt}^{p}$ by utilizing weight $\vartheta_1, \vartheta_2, \dots$, and $\vartheta_p$ of the DMExs $\psi_1, \psi_2, \dots$, and $\psi_p$, respectively, as follows:

$$\varrho_{kt}^{i} = \frac{\vartheta_i \zeta_{kt}^{i}}{\sum_{i=1}^{p} \vartheta_i \zeta_{kt}^{i}} \tag{17}$$

where, $\varrho_{kt}^{i} \geq 0$, $\sum_{i=1}^{p} \varrho_{kt}^{i} = 1$, $\zeta_{kt}^{i} = \begin{cases} 1 & \text{if } i = 1, \\ \prod_{h=1}^{i-1} \mathrm{S}(\varphi_{kt}^{h}) & \text{if } i = 2,3, \dots, p \end{cases}$, $\mathrm{S}(\varphi_{kt}^{h}) = \frac{1 + (\xi_{kt}^{h})^q - (\upsilon_{kt}^{h})^q}{2}$, $\mathrm{S}(\varphi_{kt}^{h}) \in [0, 1]$, $q \geq 1$, $k = 1, 2, \dots, m$, $t = 1, 2, 3, \dots, n$, $i = 1, 2, \dots, p$ and $h = 1, 2, \dots, i-1$.

**Step 3:** By using the proposed $q$-ROFSWPWA AO given in Equation (15), aggregate the $q$-ROFNs $\varphi_{kt}^{1}$, $\varphi_{kt}^{2}$, …, and $\varphi_{kt}^{p}$ shown in NDMxs $R^1 = (\varphi_{kt}^{1})_{m\times n} = (\langle \xi_{kt}^{1}, \upsilon_{kt}^{1} \rangle)_{m\times n}$, $R^2 = (\varphi_{kt}^{2})_{m\times n} = (\langle \xi_{kt}^{2}, \upsilon_{kt}^{2} \rangle)_{m\times n}$, …, and $R^p = (\varphi_{kt}^{p})_{m\times n} = (\langle \xi_{kt}^{p}, \upsilon_{kt}^{p} \rangle)_{m\times n}$, respectively, into a $q$-ROFN $\varphi_{kt}$, for constructing the collective DMx (CDMx) $R = (\varphi_{kt})_{m\times n} = (\langle \xi_{kt}, \upsilon_{kt} \rangle)_{m\times n}$, where

$$\varphi_{kt} = q - \mathrm{ROFSWPWA}(\varphi_{kt}^{1}, \varphi_{kt}^{2}, \dots, \varphi_{kt}^{p})$$

$$= \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \prod_{i=1}^{p}\left(1 - (\xi_{kt}^{i})^q \left(\frac{\tau}{1+\tau}\right)\right)^{\varrho_{kt}^{i}}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\prod_{i=1}^{p}\left(\frac{\tau(\upsilon_{kt}^{i})^q + 1}{1+\tau}\right)^{\varrho_{kt}^{i}} - 1\right)\right\}} \right\rangle \tag{18}$$

$k = 1, 2, \dots, m$, $t = 1, 2, \dots, n$, $q \geq 1$ and $1 < \tau < +\infty$.

**Step 4:** Compute the weight $\epsilon_{kt}$ of $q$-ROFN $\varphi_{kt}$ of the obatined CDMx $R = (\varphi_{kt})_{m\times n} = (\langle \xi_{kt}, \upsilon_{kt} \rangle)_{m\times n}$, as follows:

$$\epsilon_{kt} = \frac{(1/m)\zeta_{kt}}{\sum_{s=1}^{m}(1/m)\zeta_{st}} \tag{19}$$

where, $\zeta_{1t} = 1$, $\zeta_{kt} = \prod_{s=1}^{k-1} S(\varphi_{st})$, $S(\varphi_{st}) = \frac{1+(\xi_{st})^q-(\upsilon_{st})^q}{2}$, $S(\varphi_{st}) \in [0,1]$, $q \geq 1$, $k = 1,2,\ldots,m$, and $t = 2,\ 3,\ldots,n$.

**Step 5:** Compute the mean value $MV_t = \langle \xi_t, \upsilon_t \rangle$ for each attribute $G_t$ by using the proposed $q$-ROFSWPWA AO given in Equation (15) and obtained weight $\epsilon_{kt}$ of $q$-ROFN $\varphi_{kt}$, where, $k = 1,2,\ldots,m$ and $t = 1,2,\ldots,n$, shown as follows:

$$MV_t = \langle \xi_t, \upsilon_t \rangle$$
$$= \left\langle \sqrt[q]{\min\left\{1, \frac{1+\tau}{\tau}\left(1 - \prod_{k=1}^{m}\left(1-(\xi_{kt})^q\left(\frac{\tau}{1+\tau}\right)\right)^{\epsilon_{kt}}\right)\right\}}, \sqrt[q]{\max\left\{0, \frac{1}{\tau}\left((1+\tau)\prod_{k=1}^{m}\left(\frac{\tau(\upsilon_{kt})^q+1}{1+\tau}\right)^{\epsilon_{kt}} - 1\right)\right\}} \right\rangle \tag{20}$$

where, $t$ =$1,2,\ldots,n$, $q \geq 1$ and $1 < \tau < +\infty$.

**Step 6:** Compute the positive distance $PDM_{kt}$ from mean and negative distance $NDM_{kt}$ from mean for each $q$-ROFN $\varphi_{kt}$ of the CDMx $R = (\varphi_{kt})_{m\times n} = (\langle \xi_{kt}, \upsilon_{kt} \rangle)_{m\times n}$, as follows:

$$PDM_{kt} = \frac{max(0, S(\varphi_{kt}) - S(MV_t))}{S(MV_t)} \tag{21}$$

$$NDM_{kt} = \frac{max(0, S(MV_t) - S(\varphi_{kt}))}{S(MV_t)} \tag{22}$$

where, $S(\varphi_{kt}) = \frac{1+(\xi_{kt})^q-(\upsilon_{kt})^q}{2}$, $S(MV_t) = \frac{1+(\xi_t)^q-(\upsilon_t)^q}{2}$, $q \geq 1$, $k = 1,2,\ldots,m$, and $t = 1,\ 2,\ldots,n$.

**Step 7:** Compute the weighted sum $SP_k$ of $PDM_{kt}$ and weighted sum $SN_k$ of $NDM_{kt}$ by using weights $\varpi_1$, $\varpi_2$, …., $\varpi_n$ of the attributes $G_1$, $G_2$,…,and $G_n$, respectively, as follows:

$$SP_k = \sum_{t=1}^{n} \varpi_t . PDM_{kt} \tag{23}$$
$$SN_k = \sum_{t=1}^{n} \varpi_t . NDM_{kt} \tag{24}$$

where, $k = 1,2,\ldots,m$, and $t = 2,\ 3,\ldots,n$.

**Step 8:** Compute the normalize values $NSP_k$ and $NSN_k$ of the $SP_k$ and $SN_k$, respectively, for each alternative $\chi_k$, as follows:

$$NSP_k = \frac{SP_k}{\max_k(SP_k)} \tag{25}$$

$$NSN_k = 1 - \frac{SN_k}{\max_k(SN_k)} \tag{26}$$

where, $k = 1,2,\ldots,m$.

**Step 9:** Compute the appraisal score $AS_k$ for each alternative $\chi_k$, as follows:

$$AS_k = \frac{1}{2}(NSP_k + NSN_k) \tag{27}$$

where, $k = 1,2,\ldots,m$.

**Step 10:** Obtain the preference order (PO) of the alternatives by arranging the values of $AS_1$, $AS_2$, …, $AS_m$ in descending order. The alternative $\chi_k$ with highest $AS_k$ is the optimal alternative.

In the following, **Figure 2** represents a comprehensive flow chart of the proposed MGDM technique.

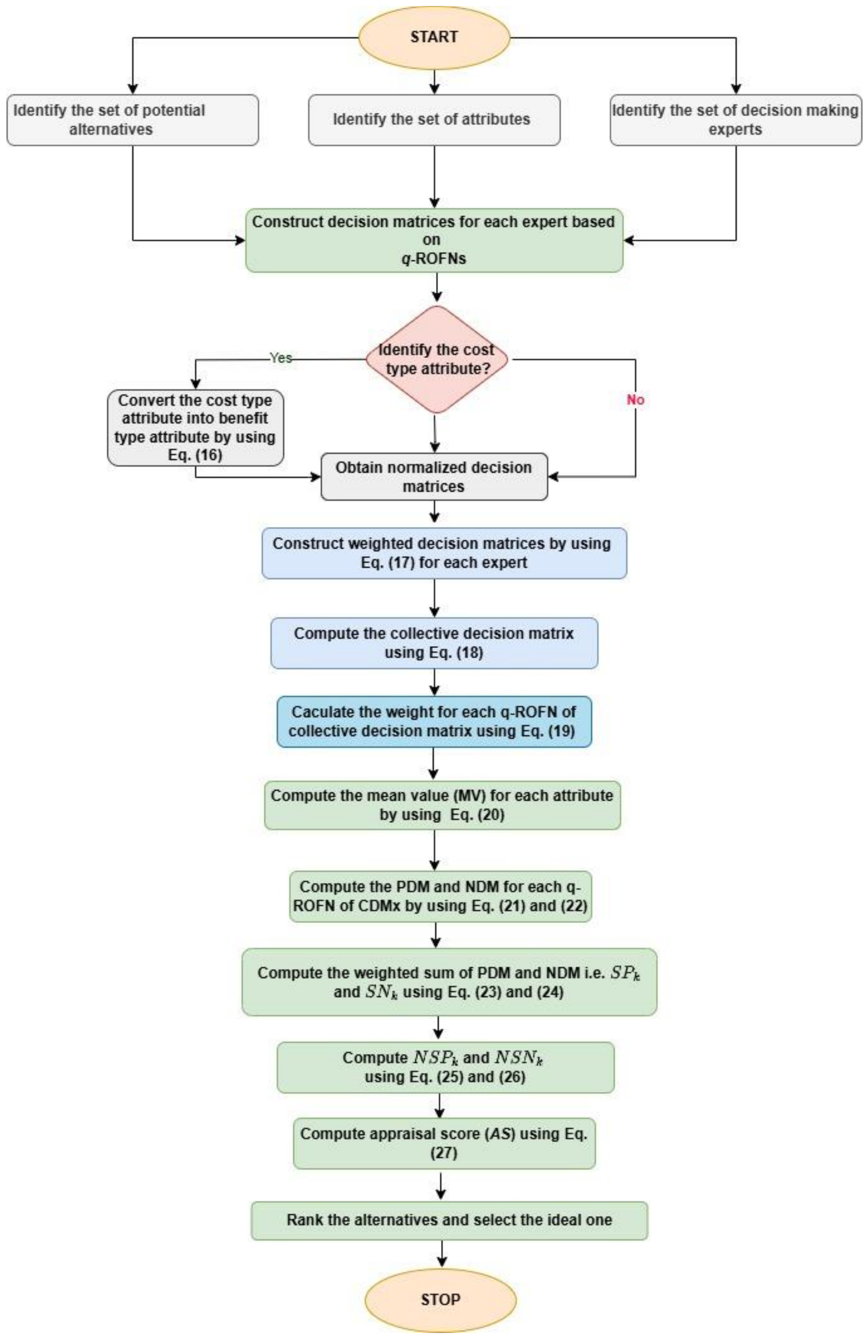


**Figure 2.** Flow chart of the proposed MGDM technique.

## 6. Case study on Selection of Best Sustainable SWMMs by using Proposed MGDM Method

Here, we consider a real-life MGDM example of selection of best sustainable SWMM for sustainable management of solid waste to illustrate the proposed MGDM technique.

***Example* 6.1** Solid waste is defined as discarded materials in solid or semi-solid form produced by households, industries, agriculture, and other sectors. Solid waste management is critical in lowering environmental pollution and health concerns. With rising urbanization, industrialization, and population growth, proper waste management has become critical for addressing emerging concerns. Solid waste is classified into several categories, each with its own distinct features. Solid waste includes home and commercial waste such as food scraps, paper, plastics, metals, and glass. Chemicals, slag, fly ash, and process byproducts are all examples of industrial waste generated during manufacturing processes. Medical waste, pesticides, and electronic garbage (e-waste) are examples of hazardous waste. These compounds are toxic, reactive, or corrosive. Agricultural waste includes biodegradable elements such as agricultural residues and animal dung, whereas biomedical from healthcare facilities includes syringes, bandages, and pathological waste. Construction and demolition waste includes debris like concrete, bricks, wood, and steel, while mining waste consists of overburden, tailings, and waste rock.

Solid waste management (SWM) refers to the systematic collection, transportation, disposal, recycling, and monitoring of solid waste generated by human activities. It lowers environmental damage caused by unmanaged rubbish accumulation. Poor waste management can cause air, water, and soil contamination, harming human health and ecosystems. Sustainable solid waste management aims to limit garbage's environmental impact while promoting resource recovery and reuse. To achieve long-term environmental, economic, and social benefits, this strategy focuses on waste reduction, material recycling, and safe trash disposal. Effective waste management promotes sustainability, saves resources, and enhances overall quality of life. Sustainable solid waste management focuses on minimizing the environmental impact of trash while optimizing resource recovery and reuse. These strategies strive to limit waste generation, recycle materials, and safely dispose of leftover garbage, assuring long-term environmental, economic, and social advantages. Here are some popular sustainable solid waste management methods (SWMMs).

**Alternative Descriptions**

**(i)** ***Sanitary Landfilling*** **($\chi_1$):** Sanitary landfilling remains an integral part of waste management. Garbage is spread out in thin layers, compressed and further covered with soil or plastic foam. They are designed in such a way that the bottom of the landfill is layered with several layers of plastic and sand to prevent groundwater contamination due to leaching or percolation. Similarly, when complete, it is covered with layers of sand, clay and gravel to prevent seepage of water. Proper landfill management minimizes environmental damage and provides a safe means for disposing of residual garbage.

**(ii)** ***Recycling and Reuse*** **($\chi_2$):** Recycling and reuse are essential components of sustainable waste management. Recycling transforms waste items like paper, glass, metals, and plastics into new goods, lowering the demand for raw materials and energy. Reuse increases the life of things by repurposing materials such as storage containers or renovating furniture. These measures help to save resources, reduce landfill use, and lower greenhouse gas emissions.

**(iii)** ***Composting*** **($\chi_3$):** Composting is another important way for turning organic waste, such as food scraps and yard trash, into nutrient-rich compost. This natural process increases soil fertility while decreasing methane emissions from organic waste in landfills. Advanced composting systems, such as vermicomposting, rely on earthworms to break down organic matter more efficiently.

**(iv)** ***Waste-to-Resource Innovations*** **($\chi_4$):** Innovative waste-to-resource technologies are emerging as viable options. These include recycling plastic trash into fuel and repurposing construction debris to make environmen friendly building materials. These approaches lessen reliance on virgin resources while providing economic incentives for good waste management. Extended Producer Responsibility

(EPR) is another important initiative that holds producers responsible for their products' whole lifecycle, including post-consumer waste. EPR emphasizes eco-friendly product design, take-back programs, and investment in recycling infrastructure, supporting responsible waste management and sustainable practices.

**(v)** ***Energy Recovery*** **($\chi_5$):** Energy recovery involves converting non-recyclable waste into usable energy through various technologies:

- ***Incineration*:** Burns waste to produce heat, which is converted into electricity. Modern incinerators use advanced emission control technologies to minimize pollution.
- ***Anaerobic Digestion*:** Decomposes organic waste in the absence of oxygen, producing biogas (a renewable energy source) and nutrient-rich digestate, which can be used as a fertilizer.
- ***Pyrolysis and Gasification*:** Thermal processes that convert waste into syngas, a clean energy source, and other by-products.

**Attributes Descriptions**

Each SWMM is characterized by its own specific advantages and disadvantages. For example, landfill disposal ranks low on cost and implementation simplicity but requires considerable land areas while doing little in reducing the volume of the waste. On the other hand, pyrolysis produces high-quality fuel, but further processing is required to obtain targeted product. Allied with these trade-offs, DMExs find a great challenge in the choice of any one suitable sustainable SWMM among the five sustainable SWMMs. In order to evaluate the SWMMs, four key criteria were set up:

**(i)** ***Environmental Impact*** **($G_1$):** Environmental impact examines each technology's ecological impacts, which are crucial in assessing its sustainability. This includes assessing the greenhouse gas emissions generated during operation as well as the overall carbon footprint of the device. Landfills, for example, are large sources of methane, a potent greenhouse gas, whereas pyrolysis releases fewer greenhouse gases but requires careful management to reduce by-product emissions. Another consideration is the contamination potential of each option such as leachate contaminating soil and groundwater in landfills as a result of leachate or toxic gas being released during burning or pyrolysis. Further, resource conservation technologies, such as recycling and reuse, are a much better fit with sustainability goals because they reduce raw material extraction and promote circular economic activities. This criterion offsets environmentally damaging practices and ensures compliance with environmentally legislation, enhances ecological balance, and enhances public acceptance of SWMMs.

**(ii)** ***Investment Cost*** **($G_2$):** The investment cost is a measure of the financial resources required for the establishment of initial setup, which is comprised of infrastructure, equipment, and installation. The landfilling technologies usually have lower capital investments, rendering them suitable for sites that operate with relatively small budgets. But other advanced technologies such as pyrolysis demand more initial investment because of their sophisticated gear and infrastructure prerequisites. A cost assessment could assist DMExs in ascertaining economic viability and in resource allocation.

**(iii)** ***Revenue Generation*** **($G_3$):** This is profit generation: Within that subject, the financial returns on each technology are examined to secure long-term commercial viability. The generation of an income would occur through the recovery of metals and plastics in recycling or in pyrolysis via the value-added products of its synthetic gas and bio-oil production. By creating organic fertilizers for use in agriculture, composting provides further sources of income. Indirect benefits like obtaining carbon credits or lowering landfill taxes are also taken into consideration by this criterion.

**(iv)** ***Technical Specifications*** **($G_4$):** Technical specifications encompass functional and operational features of the technology, such as maintenance, energy efficiency, and scalability; however, its applicability very heavily depends on the product quality, such as use of biofuels or average purity of the recycled materials. The decision-makers will pick the best waste management solution providing for environmental, financial, and technical aspects for local needs.

In the following, **Figure 3** illustrates a hierarchical framework for the selection of best sustainable SWMM.

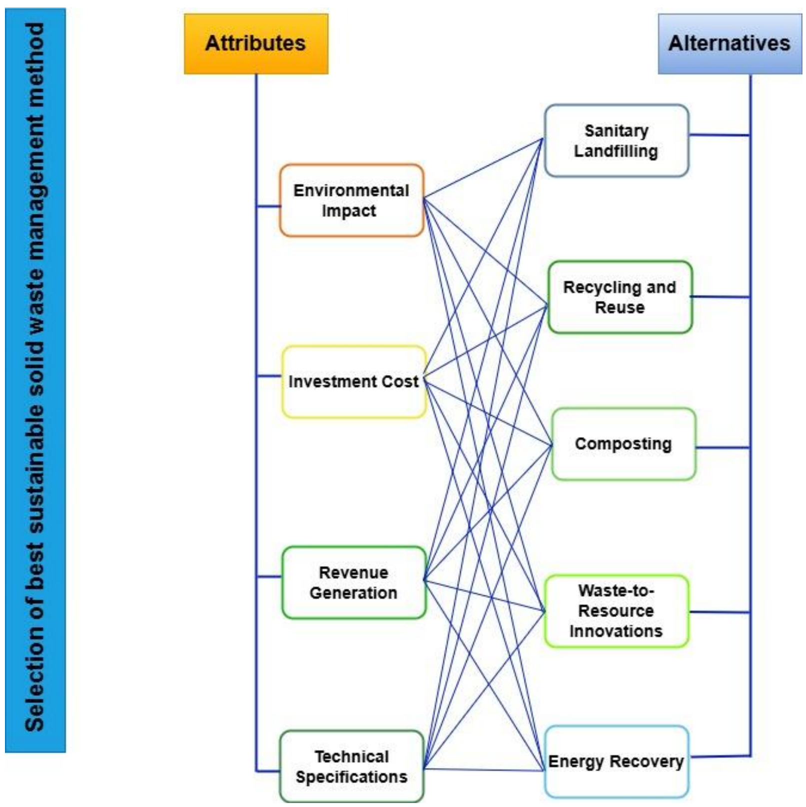


**Figure 3.** Hierarchical framework for the selection of best sustainable SWMM.

The objective of this case study is to evaluate the most sustainable and best SWMM under the above considered attributes. In the evaluation process of this case study, there is a group of three DMExs $\psi_1, \psi_2$ and $\psi_3$. DMExs utilize the $q$-ROFNs to assess these SWMMs towards the given attributes. The $q$-ROFN is an effective tool to reduce the uncertainty of DMExs during the decision-making process and it has a wide acceptability over the other environment.

Furthermore, three DMExs $\psi_1, \psi_2$ and $\psi_3$ assess the SWMMs "Sanitary Landfilling" $(\chi_1)$, "Recycling and Reuse" $(\chi_2)$, "Composting" $(\chi_3)$, "Waste-to-Resource Innovations" $(\chi_4)$ and "Energy Recovery" $(\chi_5)$ under the four attributes "Environmental Impact" $(G_1)$, "Investment cost" $(G_2)$, "Revenue generation" $(G_3)$ and "Technical Specifications" $(G_4)$. Let the weights of the attributes $G_1, G_2, G_3$ and $G_4$ are $\varpi_1 = 0.2$, $\varpi_2 = 0.1$, $\varpi_3 = 0.3$ and $\varpi_4 = 0.4$, respectively. The weights of the DMks $\psi_1, \psi_2$ and $\psi_3$ are $\vartheta_1 = 0.35$, $\vartheta_2 = 0.4$ and $\vartheta_3 = 0.25$, respectively. Each DMEx $\psi_1, \psi_2$ and $\psi_3$ evaluates alternatives $\chi_1, \chi_2, \chi_3, \chi_4$ and $\chi_5$

with respect to attribute $G_1, G_2,\ G_3$ and $G_4$ by using an $q$-ROFN $\tilde{\varphi}_{kt}^{i}$ to construct the DMx $\tilde{R}^i = \left(\tilde{\varphi}_{kt}^{i}\right)_{5\times4}$, where, $i$ =1,2,3, $k$ =1,2,3,4,5, and $t$ =1,2,3,4 as demonstrated below:

$$\tilde{R}^1 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & G_3 & G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.5,0.2\rangle & \langle 0.6,0.4\rangle & \langle 0.4,0.3\rangle & \langle 0.5,0.2\rangle \\ \langle 0.3,0.6\rangle & \langle 0.2,0.4\rangle & \langle 0.5,0.1\rangle & \langle 0.5,0.2\rangle \\ \langle 0.2,0.4\rangle & \langle 0.2,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.3\rangle \\ \langle 0.3,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.2\rangle \\ \langle 0.1,0.6\rangle & \langle 0.4,0.4\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.1\rangle \end{pmatrix} \end{matrix},$$

$$\tilde{R}^2 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & G_3 & G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.6,0.3\rangle & \langle 0.7,0.2\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.3\rangle \\ \langle 0.1,0.5\rangle & \langle 0.3,0.5\rangle & \langle 0.6,0.2\rangle & \langle 0.4,0.3\rangle \\ \langle 0.3,0.5\rangle & \langle 0.2,0.6\rangle & \langle 0.6,0.1\rangle & \langle 0.5,0.4\rangle \\ \langle 0.4,0.5\rangle & \langle 0.6,0.2\rangle & \langle 0.5,0.1\rangle & \langle 0.6,0.3\rangle \\ \langle 0.2,0.7\rangle & \langle 0.3,0.4\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.2\rangle \end{pmatrix} \end{matrix},$$

$$\tilde{R}^3 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & G_3 & G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.7,0.2\rangle & \langle 0.4,0.5\rangle & \langle 0.4,0.5\rangle & \langle 0.2,0.6\rangle \\ \langle 0.2,0.5\rangle & \langle 0.4,0.6\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.1\rangle \\ \langle 0.3,0.6\rangle & \langle 0.2,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0.5,0.4\rangle \\ \langle 0.5,0.4\rangle & \langle 0.4,0.5\rangle & \langle 0.6,0.2\rangle & \langle 0.5,0.2\rangle \\ \langle 0.3,0.6\rangle & \langle 0.1,0.5\rangle & \langle 0.6,0.4\rangle & \langle 0.6,0.1\rangle \end{pmatrix} \end{matrix}.$$

To address this MGDM problem, we employ the newly proposed MGDM technique as demonstrated below:

**Step 1:** Since $G_1$ and $G_2$ are cost type attributes while $G_3$ and $G_4$ are beneficial attributes, therefore by utilizing Equation (16), we determine the NDMx as follows:

$$R^1 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & G_3 & G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.2,0.5\rangle & \langle 0.4,0.6\rangle & \langle 0.4,0.3\rangle & \langle 0.5,0.2\rangle \\ \langle 0.6,0.3\rangle & \langle 0.4,0.2\rangle & \langle 0.5,0.1\rangle & \langle 0.5,0.2\rangle \\ \langle 0.4,0.2\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.3\rangle \\ \langle 0.5,0.3\rangle & \langle 0.3,0.5\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.2\rangle \\ \langle 0.6,0.1\rangle & \langle 0.4,0.4\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.1\rangle \end{pmatrix} \end{matrix},$$

$$R^2 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & G_3 & G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.3,0.6\rangle & \langle 0.2,0.7\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.3\rangle \\ \langle 0.5,0.1\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle & \langle 0.4,0.3\rangle \\ \langle 0.5,0.3\rangle & \langle 0.6,0.2\rangle & \langle 0.6,0.1\rangle & \langle 0.5,0.4\rangle \\ \langle 0.5,0.4\rangle & \langle 0.2,0.6\rangle & \langle 0.5,0.1\rangle & \langle 0.6,0.3\rangle \\ \langle 0.7,0.2\rangle & \langle 0.4,0.3\rangle & \langle 0.6,0.3\rangle & \langle 0.6,0.2\rangle \end{pmatrix} \end{matrix},$$

$$R^3 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{matrix} \begin{matrix} \begin{matrix} G_1 & G_2 & G_3 & G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.2,0.7\rangle & \langle 0.5,0.4\rangle & \langle 0.4,0.5\rangle & \langle 0.2,0.6\rangle \\ \langle 0.5,0.2\rangle & \langle 0.6,0.4\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.1\rangle \\ \langle 0.6,0.3\rangle & \langle 0.5,0.2\rangle & \langle 0.5,0.3\rangle & \langle 0.5,0.4\rangle \\ \langle 0.4,0.5\rangle & \langle 0.5,0.4\rangle & \langle 0.6,0.2\rangle & \langle 0.5,0.2\rangle \\ \langle 0.6,0.3\rangle & \langle 0.5,0.1\rangle & \langle 0.6,0.4\rangle & \langle 0.6,0.1\rangle \end{pmatrix} \end{matrix}.$$

**Step 2:** By utilizing Equation (17), the weight $\vartheta_1 = 0.35$, $\vartheta_2 = 0.4$ and $\vartheta_3 = 0.25$ of the DMExs $\psi_1$, $\psi_2$ and $\psi_3$, respectively, we determine the weight $\varrho_{kt}^i = \frac{\vartheta_i \zeta_{kt}^i}{\sum_{i=1}^p \vartheta_i \zeta_{kt}^i}$ of each $q$-ROFN $\varphi_{kt}^i$, for $q = 2$ where, $\varrho_{11}^1 = 0.6433$, $\varrho_{12}^1 = 0.6512$, $\varrho_{13}^1 = 0.5495$, $\varrho_{14}^1 = 0.5149$, $\varrho_{21}^1 = 0.4983$, $\varrho_{22}^1 = 0.5342$, $\varrho_{23}^1 = 0.4998$, $\varrho_{24}^1 = 0.5201$, $\varrho_{31}^1 = 0.5342$, $\varrho_{32}^1 = 0.5059$, $\varrho_{33}^1 = 0.5148$, $\varrho_{34}^1 = 0.5495$, $\varrho_{41}^1 = 0.5295$, $\varrho_{42}^1 = 0.6321$, $\varrho_{43}^1 = 0.5104$, $\varrho_{44}^1 = 0.5087$, $\varrho_{51}^1 = 0.4715$, $\varrho_{52}^1 = 0.5674$, $\varrho_{53}^1 = 0.4869$, $\varrho_{54}^1 = 0.4786$, $\varrho_{11}^2 = 0.2904$, $\varrho_{12}^2 = 0.2977$, $\varrho_{13}^2 = 0.3360$, $\varrho_{14}^2 = 0.3560$, $\varrho_{21}^2 = 0.3616$, $\varrho_{22}^2 = 0.3419$, $\varrho_{23}^2 = 0.3541$, $\varrho_{24}^2 = 0.3596$, $\varrho_{31}^2 = 0.3419$, $\varrho_{32}^2 = 0.3498$, $\varrho_{33}^2 = 0.3412$, $\varrho_{34}^2 = 0.3360$, $\varrho_{41}^2 = 0.3510$, $\varrho_{42}^2 = 0.3034$, $\varrho_{43}^2 = 0.3529$, $\varrho_{44}^2 = 0.3517$, $\varrho_{51}^2 = 0.3637$, $\varrho_{52}^2 = 0.3242$, $\varrho_{53}^2 = 0.3673$, $\varrho_{54}^2 = 0.3692$, $\varrho_{11}^3 = 0.0663$, $\varrho_{12}^3 = 0.0512$, $\varrho_{13}^3 = 0.1145$, $\varrho_{14}^3 = 0.1291$, $\varrho_{21}^3 = 0.1401$, $\varrho_{22}^3 = 0.1239$, $\varrho_{23}^3 = 0.1461$, $\varrho_{24}^3 = 0.1203$, $\varrho_{31}^3 = 0.1239$, $\varrho_{32}^3 = 0.1443$, $\varrho_{33}^3 = 0.1440$, $\varrho_{34}^3 = 0.1145$, $\varrho_{41}^3 = 0.1195$, $\varrho_{42}^3 = 0.0645$, $\varrho_{43}^3 = 0.1367$, $\varrho_{44}^3 = 0.1396$, $\varrho_{51}^3 = 0.1648$, $\varrho_{52}^3 = 0.1084$, $\varrho_{53}^3 = 0.1458$ and $\varrho_{54}^3 = 0.1523$.

**Step 3:** By applying Equation (18), we get the consolidated $q$-ROFN $\varphi_{kt}$ through the process of aggregating the $q$-ROFNs $\varphi_{kt}^1, \varphi_{kt}^2$ and $\varphi_{kt}^3$ appeared in NDMxs $R^1 = \left(\varphi_{kt}^1\right)_{5\times4} = \left(\langle \xi_{kt}^1, \upsilon_{kt}^1 \rangle\right)_{5\times4}$, $R^2 = \left(\varphi_{kt}^2\right)_{5\times4} = \left(\langle \xi_{kt}^2, \upsilon_{kt}^2 \rangle\right)_{5\times4}$ and $R^3 = \left(\varphi_{kt}^3\right)_{5\times4} = \left(\langle \xi_{kt}^3, \upsilon_{kt}^3 \rangle\right)_{5\times4}$, respectively, for assembling the CDMx $R = (\varphi_{kt})_{5\times4} = (\langle \xi_{kt}, \upsilon_{kt} \rangle)_{5\times4}$, where

$$R = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{array} \begin{array}{c} \begin{array}{cccc} \quad G_1 \quad & \quad G_2 \quad & \quad G_3 \quad & \quad G_4 \quad \end{array} \\ \begin{pmatrix} \langle 0.2338, 0.5440 \rangle & \langle 0.3604, 0.6214 \rangle & \langle 0.4367, 0.3614 \rangle & \langle 0.4735, 0.3079 \rangle \\ \langle 0.5529, 0.2311 \rangle & \langle 0.4657, 0.2669 \rangle & \langle 0.5383, 0.1787 \rangle & \langle 0.4554, 0.2324 \rangle \\ \langle 0.4657, 0.2510 \rangle & \langle 0.5379, 0.2000 \rangle & \langle 0.5370, 0.2490 \rangle & \langle 0.4484, 0.3478 \rangle \\ \langle 0.4894, 0.3640 \rangle & \langle 0.2929, 0.5256 \rangle & \langle 0.5152, 0.1712 \rangle & \langle 0.5381, 0.2394 \rangle \\ \langle 0.6392, 0.1836 \rangle & \langle 0.4123, 0.3462 \rangle & \langle 0.6000, 0.2740 \rangle & \langle 0.6000, 0.1448 \rangle \end{pmatrix} \end{array}.$$

**Step 4:** By using Equation (19), we compute the weight $\epsilon_{kt}$ of each $q$-ROFN $\varphi_{kt}$ of the CDMx $R = (\varphi_{kt})_{5\times4}$, where, $\epsilon_{11} = 0.2500$, $\epsilon_{12} = 0.2500$, $\epsilon_{13} = 0.2500$, $\epsilon_{14} = 0.2500$, $\epsilon_{21} = 0.2055$, $\epsilon_{22} = 0.2014$, $\epsilon_{23} = 0.2871$, $\epsilon_{24} = 0.3059$, $\epsilon_{31} = 0.2141$, $\epsilon_{32} = 0.1920$, $\epsilon_{33} = 0.3004$, $\epsilon_{34} = 0.2935$, $\epsilon_{41} = 0.2107$, $\epsilon_{42} = 0.2046$, $\epsilon_{43} = 0.3143$, $\epsilon_{44} = 0.2704$, $\epsilon_{51} = 0.2081$, $\epsilon_{52} = 0.1478$, $\epsilon_{53} = 0.3467$ and $\epsilon_{54} = 0.2974$.

**Step 5:** By using Equation (20), we compute $MV_t = \langle \xi_t, \upsilon_t \rangle$ for each attribute $G_t$, where $t = 1, 2, 3, 4$, $MV_1 = \langle 0.5112, 0.2496 \rangle$, $MV_2 = \langle 0.4193, 0.4340 \rangle$, $MV_3 = \langle 0.6415, 0 \rangle$ and $MV_4 = \langle 0.5961, 0 \rangle$.

**Step 6:** By using Equations (21) and (22), we compute $PDM_{kt}$ and $NDM_{kt}$ for each $q$-ROFN $\varphi_{kt}$ of the CDMx $R = (\varphi_{kt})_{5\times4} = (\langle \xi_{kt}, \upsilon_{kt} \rangle)_{5\times4}$, respectively, where $PDM_{11} = 0$, $PDM_{12} = 0$, $PDM_{13} = 0$, $PDM_{14} = 0$, $PDM_{21} = 0.0444$, $PDM_{22} = 0.1602$, $PDM_{23} = 0$, $PDM_{24} = 0$, $PDM_{31} = 0$, $PDM_{32} = 0.2652$, $PDM_{33} = 0$, $PDM_{34} = 0$, $PDM_{41} = 0$, $PDM_{42} = 0$, $PDM_{43} = 0$, $PDM_{44} = 0$, $PDM_{51} = 0.1466$, $PDM_{52} = 0.0634$, $PDM_{53} = 0$, $PDM_{54} = 0$ $NDM_{11} = 0.3673$, $NDM_{12} = 0.2468$, $NDM_{13} = 0.2489$, $NDM_{14} = 0.1667$, $NDM_{21} = 0$, $NDM_{22} = 0$, $NDM_{23} = 0.1089$, $NDM_{24} = 0.1490$, $NDM_{31} = 0.0376$, $NDM_{32} = 0$, $NDM_{33} = 0.1312$,

$NDM_{34}$ = 0.2031, $NDM_{41}$ = 0.0767, $NDM_{42}$ = 0.1802, $NDM_{43}$ = 0.1242, $NDM_{44}$ = 0.0909, $NDM_{51}$ = 0, $NDM_{52}$ = 0, $NDM_{53}$ = 0.0896, $NDM_{54}$ = 0.0120.

**Step 7:** By using Equations (23) and (24), we compute the weighted sum $SP_k$ and $SN_k$ of $PDM_{kt}$ and $NDM_{kt}$, respectively, where, $k = 1, 2, 3, 4, 5$, $SP_1 = 0$, $SP_2 = 0.0249$, $SP_3 = 0.0265$, $SP_4 = 0$, $SP_5 = 0.0357$, $SN_1 = 0.2395$, $SN_2 = 0.0923$, $SN_3 = 0.1281$, $SN_4 = 0.1070$, and $SN_5 = 0.0317$.

**Step 8:** By using Equations (25) and (26), we compute the normalize values $NSP_k$ and $NSN_k$ of the $SP_k$ and $SN_k$, respectively, where $k = 1, 2, 3, 4, 5$, $NSP_1 = 0$, $NSP_2 = 0.6983$, $NSP_3 = 0.7435$, $NSP_4 = 0$, $NSP_5 = 1$, $NSN_1 = 0$, $NSN_2 = 0.6148$, $NSN_3 = 0.4651$, $NSN_4 = 0.5533$, and $NSN_5 = 0.8676$.

**Step 9:** By using Equation (27), we compute the appraisal scores $AS_1$, $AS_2$, $AS_3$, $AS_4$ and $AS_5$ for the alternatives $\chi_1$, $\chi_2$, $\chi_3$, $\chi_4$ and $\chi_5$, respectively, where $AS_1 = 0$, $AS_2 = 0.6565$, $AS_3 = 0.6043$, $AS_4 = 0.2767$, and $AS_5 = 0.9338$.

**Step 10:** Since $AS_5 > AS_2 > AS_3 > AS_4 > AS_1$ where $AS_1 = 0$, $AS_2 = 0.6565$, $AS_3 = 0.6043$, $AS_4 = 0.2767$, and $AS_5 = 0.9338$, according to the proposed MGDM technique, the PO of the five possible alternatives is "$\chi_5 \succ \chi_2 \succ \chi_3 \succ \chi_4 \succ \chi_1$". Hence, "Energy Recovery" ($\chi_5$) is the best sustainable SWMM among the "Sanitary Landfilling" ($\chi_1$), "Recycling and Reuse" ($\chi_2$), "Composting" ($\chi_3$), "Waste-to-Resource Innovations" ($\chi_4$) and "Energy Recovery" ($\chi_5$).

## 6.1 Comparison Analysis with the Existing MGDM Techniques

In the following, we compare the POs of the alternatives $\chi_1$, $\chi_2$, $\chi_3$, $\chi_4$ and $\chi_5$ obtained by the proposed MGDM techniques with Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018) and Xing et al.'s MGDM technique (Xing et al., 2020) for *Example* 6.1 to show the practical applicability and validity of the proposed MGDM technique.

**Table 1** and **Figure 4** present a comparison of the POs of the SWMMs "Sanitary Landfilling" ($\chi_1$), "Recycling and Reuse" ($\chi_2$), "Composting" ($\chi_3$), "Waste-to-Resource Innovations" ($\chi_4$) and "Energy Recovery" ($\chi_5$) obtained by the different MGDM techniques for *Example* 6.1.

**Table 1.** Comparative analysis of the alternatives POs derived by various MGDM techniques for *Example* 6.1.

| MGDM techniques | POs |
|---|---|
| Khan et al.'s MGDM technique (Khan et al., 2023) | $\chi_5 \succ \chi_2 \succ \chi_3 \succ \chi_4 \succ \chi_1$ |
| Liu et al.'s MGDM technique (Liu et al., 2018) | $\chi_5 \succ \chi_2 \succ \chi_3 \succ \chi_4 \succ \chi_1$ |
| Xing et al.'s MGDM technique (Xing et al., 2020) | $\chi_5 \succ \chi_2 \succ \chi_3 \succ \chi_4 \succ \chi_1$ |
| The proposed MGDM technique | $\chi_5 \succ \chi_2 \succ \chi_3 \succ \chi_4 \succ \chi_1$ |

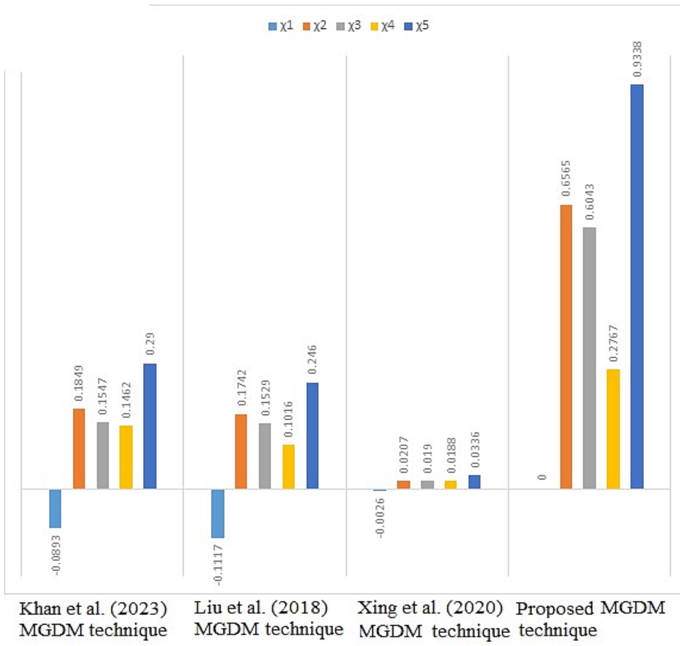


**Figure 4.** Graphical comparison of POs derived by various MGDM techniques for *Example* 6.1.

**Table 1** and **Figure 4** show that Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018), Xing et al.'s MGDM technique (Xing et al., 2020) and proposed MGDM technique obtain the same PO "$\chi_5 \succ \chi_2 \succ \chi_3 \succ \chi_4 \succ \chi_1$" for the alternatives $\chi_1$, $\chi_2$, $\chi_3$, $\chi_4$ and $\chi_5$. Hence, results presented in **Table 1** and **Figure 4** prove the validity and applicability of the proposed MGDM technique.

## 7. Superiority of the Proposed MGDM Technique over the Existing MGDM Techniques

In the following, we consider the two numerical illustrations to show the advantages and superiority of the proposed MGDM technique over Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018), and Xing et al.'s MGDM technique (Xing et al., 2020).

***Example* 7.1** Let $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$ be four alternatives and $G_1$, $G_2$, $G_3$ and $G_4$ be four benefit kind attributes. Let the weights $\varpi_1$, $\varpi_2$, $\varpi_3$ and $\varpi_4$ of the attributes $G_1, G_2, G_3$ and $G_4$ are 0.2, 0.1, 0.3 and 0.4, respectively, i.e., $\varpi_1 = 0.2$, $\varpi_2 = 0.1$, $\varpi_3 = 0.3$ and $\varpi_4 = 0.4$. Let $\psi_1$, $\psi_2$ and $\psi_3$ be three DMExs and let weights $\vartheta_1$, $\vartheta_2$ and $\vartheta_3$ of the DMExs $\psi_1$, $\psi_2$ and $\psi_3$ are 0.3, 0.4 and 0.3, respectively, i.e., $\vartheta_1 = 0.3$, $\vartheta_2 = 0.4$ and $\vartheta_3 = 0.3$. Each DMEx $\psi_1$, $\psi_2$ and $\psi_3$ evaluates alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$ with respect to attribute $G_1, G_2, G_3$ and $G_4$ by using a $q$-ROFN $\tilde{\varphi}_{kt}^{i}$ to construct the DMx $\tilde{R}^i = \left(\tilde{\varphi}_{kt}^{i}\right)_{4\times 4}$, where, $i = 1, 2, 3$, $k = 1, 2, 3, 4$ and $t = 1, 2, 3, 4$, as demonstrated below:

$$\tilde{R}^1 = \begin{matrix} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{matrix}\begin{matrix} \begin{matrix} G_1 & \quad G_2 & \quad G_3 & \quad G_4 \end{matrix} \\ \begin{pmatrix} \langle 0.3,0.5\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.5,0.2\rangle \\ \langle 0.5,0.3\rangle & \langle 0.4,0.5\rangle & \langle 0.5,0.3\rangle & \langle 0.4,0.2\rangle \\ \langle 0.6,0.2\rangle & \langle 0.4,0.2\rangle & \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0.5,0.3\rangle & \langle 0.3,0.4\rangle & \langle 0.4,0.5\rangle & \langle 0.5,0.2\rangle \end{pmatrix} \end{matrix},$$

$$\tilde{R}^2 = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{pmatrix} \langle 0.5,0.4\rangle & \langle 0.3,0.4\rangle & \langle 0.5,0.4\rangle & \langle 0.1,0.3\rangle \\ \langle 0.5,0.4\rangle & \langle 0.4,0.5\rangle & \langle 1,0\rangle & \langle 1,0\rangle \\ \langle 0.5,0.4\rangle & \langle 0.5,0.4\rangle & \langle 0.6,0.4\rangle & \langle 0.5,0.4\rangle \\ \langle 0.4,0.5\rangle & \langle 0.3,0.6\rangle & \langle 0.5,0.5\rangle & \langle 0.5,0.4\rangle \end{pmatrix} \end{array},$$

$$\tilde{R}^3 = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{pmatrix} \langle 0.4,0.5\rangle & \langle 1,0\rangle & \langle 1,0\rangle & \langle 0.5,0.2\rangle \\ \langle 0.5,0.4\rangle & \langle 0.4,0.4\rangle & \langle 0.4,0.3\rangle & \langle 0.4,0.3\rangle \\ \langle 0.6,0.3\rangle & \langle 0.2,0.5\rangle & \langle 0.6,0.4\rangle & \langle 0.5,0.4\rangle \\ \langle 0.4,0.6\rangle & \langle 0.5,0.2\rangle & \langle 0.4,0.3\rangle & \langle 0.3,0.6\rangle \end{pmatrix} \end{array}.$$

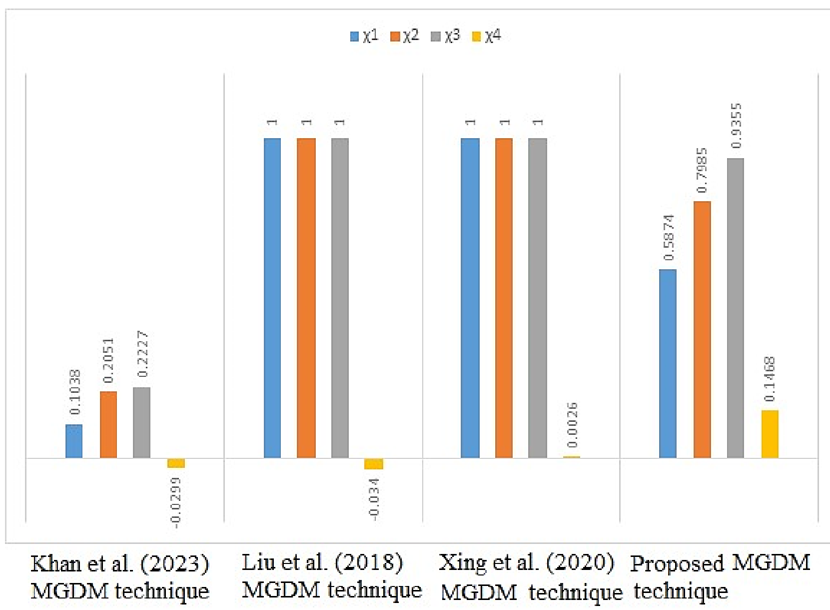


**Figure 5.** Graphical comparison of POs derived by various MGDM techniques for *Example* 7.1.

To address this MGDM problem, we employ Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018), Xing et al.'s MGDM technique (Xing et al., 2020) and proposed MGDM technique. The POs of the alternatives $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$ obtained by Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018), Xing et al.'s MGDM technique (Xing et al., 2020) and proposed MGDM technique are summarized in **Table 2** and **Figure 5** for *Example* 7.1.

**Table 2.** Comparative analysis of the alternatives POs derived by various MGDM techniques for *Example* 7.1.

| MGDM techniques | POs |
|---|---|
| Khan et al.'s MGDM technique (Khan et al., 2023) | $\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$ |
| Liu et al.'s MGDM technique (Liu et al., 2018) | $\chi_1 = \chi_2 = \chi_3 \succ \chi_4$ |
| Xing et al.'s MGDM technique (Xing et al., 2020) | $\chi_1 = \chi_2 = \chi_3 \succ \chi_4$ |
| The proposed MGDM technique | $\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$ |

**Table 2** and **Figure 5** show that Liu et al.'s MGDM technique (Liu et al., 2018) and Xing et al.'s MGDM technique (Xing et al., 2020) yield the same PO "$\chi_1 = \chi_2 = \chi_3 \succ \chi_4$" for the alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$, indicating their inability to differentiate the PO among the alternatives $\chi_1$, $\chi_2$, and $\chi_3$ in this case. However, Khan et al.'s MGDM technique (Khan et al., 2023) and proposed MGDM technique obtain the same PO "$\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$" for the alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$. Hence, the proposed MGDM technique can overcome the limitations of Liu *et al.*'s MGDM technique (Liu et al., 2018) and Xing et al.'s MGDM technique (Xing et al., 2020) in this particular scenario.

***Example* 7.2** Let $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$ be four alternatives and $G_1$, $G_2$, $G_3$ and $G_4$ be four benefit kind attributes. Let the weights $\varpi_1$, $\varpi_2$, $\varpi_3$ and $\varpi_4$ of the attributes $G_1, G_2, G_3$ and $G_4$ are 0.2, 0.2, 0.3 and 0.3, respectively, i.e., $\varpi_1 = 0.2$, $\varpi_2 = 0.2$, $\varpi_3 = 0.3$ and $\varpi_4 = 0.3$. Let $\psi_1$, $\psi_2$ and $\psi_3$ be three DMExs and let weights $\vartheta_1$, $\vartheta_2$ and $\vartheta_3$ of the DMExs $\psi_1$, $\psi_2$ and $\psi_3$ are 0.3, 0.4 and 0.3, respectively, i.e., $\vartheta_1 = 0.3$, $\vartheta_2 = 0.4$ and $\vartheta_3 = 0.3$. Each DMEx $\psi_1$, $\psi_2$ and $\psi_3$ evaluates alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$ with respect to attribute $G_1, G_2, G_3$ and $G_4$ by using a $q$-ROFN $\tilde{\varphi}_{kt}^{i}$ to construct the DMx $\tilde{R}^i = \left(\tilde{\varphi}_{kt}^{i}\right)_{4\times 4}$, where $i = 1, 2, 3$, $k = 1, 2, 3, 4$ and $t = 1, 2, 3, 4$, as demonstrated below:

$$\tilde{R}^1 = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{pmatrix} \langle 0.2,0.8\rangle & \langle 0.2,0.6\rangle & \langle 0.4,0.5\rangle & \langle 0.1,0.9\rangle \\ \langle 0.2,0.3\rangle & \langle 0.4,0.2\rangle & \langle 0.5,0.1\rangle & \langle 0.5,0.2\rangle \\ \langle 0.7,0.2\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.3\rangle & \langle 0.6,0.3\rangle \\ \langle 0.2,0.6\rangle & \langle 0.2,0.5\rangle & \langle 0.5,0.5\rangle & \langle 0,1\rangle \end{pmatrix} \end{array},$$

$$\tilde{R}^2 = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{pmatrix} \langle 0,1\rangle & \langle 0.3,0.7\rangle & \langle 0.2,0.6\rangle & \langle 0.4,0.6\rangle \\ \langle 0.5,0.2\rangle & \langle 0.5,0.3\rangle & \langle 0.7,0.2\rangle & \langle 0.5,0.3\rangle \\ \langle 0.6,0.3\rangle & \langle 0,1\rangle & \langle 0.4,0.2\rangle & \langle 0.6,0.1\rangle \\ \langle 0.2,0.7\rangle & \langle 0.2,0.3\rangle & \langle 0.5,0.4\rangle & \langle 0.5,0.3\rangle \end{pmatrix} \end{array},$$

$$\tilde{R}^3 = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{pmatrix} \langle 0.5,0.3\rangle & \langle 0.8,0.1\rangle & \langle 0.5,0.4\rangle & \langle 0.6,0.3\rangle \\ \langle 0,1\rangle & \langle 0.3,0.4\rangle & \langle 0.5,0.2\rangle & \langle 0.4,0.4\rangle \\ \langle 0.4,0.3\rangle & \langle 0.4,0.3\rangle & \langle 0.5,0.1\rangle & \langle 0.5,0.1\rangle \\ \langle 0.2,0.6\rangle & \langle 0.3,0.5\rangle & \langle 0.3,0.5\rangle & \langle 0.2,0.8\rangle \end{pmatrix} \end{array}.$$

To address this MGDM problem, we employ Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018), Xing et al.'s MGDM technique (Xing et al., 2020) and proposed MGDM technique. The POs of the alternatives $\chi_1$, $\chi_2$, $\chi_3$ and $\chi_4$ obtained by Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018), Xing et al.'s MGDM technique (Xing et al., 2020) and proposed MGDM technique are summarized in **Table 3** and **Figure 6** for *Example* 7.2.

**Table 3.** Comparative analysis of the alternatives POs derived by various MGDM techniques for Example 7.2.

| MGDM techniques | POs |
|---|---|
| Khan et al.'s MGDM technique (Khan et al., 2023) | $\chi_1 = \chi_2 = \chi_3 = \chi_4$ |
| Liu et al.'s MGDM technique (Liu et al., 2018) | $\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$ |
| Xing et al.'s MGDM technique (Xing et al., 2020) | $\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$ |
| The proposed MGDM technique | $\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$ |

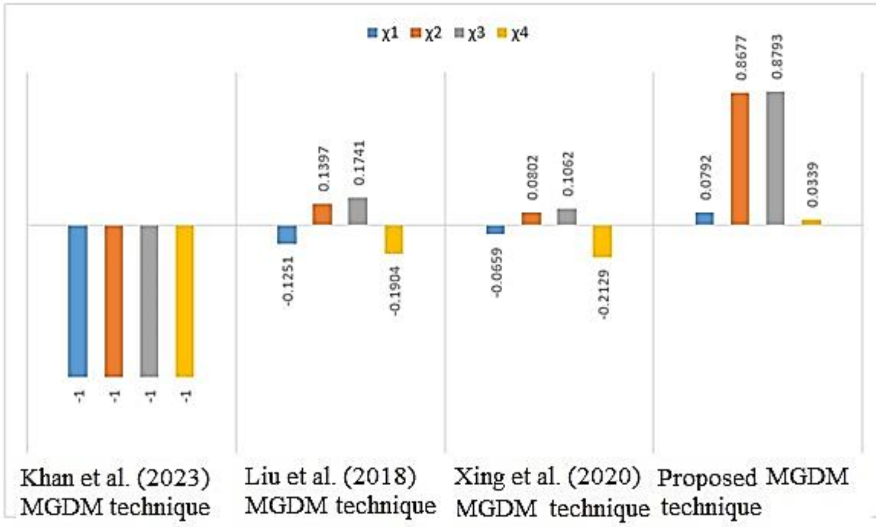


**Figure 6.** Graphical comparison of POs derived by various MGDM techniques for Example 7.2.

**Table 3** and **Figure 6** show that Khan et al.'s MGDM technique (Khan et al., 2023) obtains the PO "$\chi_1 = \chi_2 = \chi_3 = \chi_4$" for the alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$, indicating that it fails to differentiate the PO among the alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$ in this case. However, Liu et al.'s MGDM technique (Liu et al., 2018), Xing et al.'s MGDM technique (Xing et al., 2020) and proposed MGDM technique obtain the same PO "$\chi_3 \succ \chi_2 \succ \chi_1 \succ \chi_4$" for the alternatives $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$. Therefore, in this particular scenario the proposed MGDM technique can overcome the limitations of Khan et al.'s MGDM technique (Khan et al., 2023).

## 8. Conclusion

Solid waste management is a critical component of urban planning and public health, directly influencing environmental sustainability and socio-economic growth. The management of solid waste stands out as an urgent concern, as the growing volume of garbage places enormous strain on the environment and public health. Therefore, in this article, we have proposed the new operational laws of $q$-ROFNs based on Sugeno-Weber's norm, which can overcome the drawbacks of Wang et al.'s operation laws (Wang et al., 2024) of $q$-ROFNs. Moreover, we have proposed $q$-ROFSWPWA AO of $q$-ROFNs based on the proposed operation laws of $q$-ROFNs. However, based on the $q$-ROFSWPWA AO and EDAS approach, we have proposed a new MGDM technique for $q$-ROFNs environment. Furthermore, we have solved a case study of the selection of sustainable by using the proposed MGDM technique for sustainable management of solid waste. According to the proposed MGDM technique, we obtained that "Energy Recovery" ($\chi_5$) is the best sustainable SWMM among the "Sanitary Landfilling" ($\chi_1$), "Recycling and Reuse" ($\chi_2$), "Composting" ($\chi_3$), "Waste-to-Resource Innovations" ($\chi_4$) and "Energy Recovery" ($\chi_5$). To show the advantages we have compared the obtained PO of the proposed MGDM technique with the POs obtained from the existing MGDM technique. From the results of comparative study, it is clear that the proposed MGDM technique can conquer the shortcomings of Khan et al.'s MGDM technique (Khan et al., 2023), Liu et al.'s MGDM technique (Liu et al., 2018) and Xing et al.'s MGDM technique (Xing et al., 2020) of $q$-ROFNs, when sometimes these techniques can not differentiate the POs of the alternatives. The proposed MGDM technique offer us highly practical techniques for handling MGDM problems in the $q$-ROFNs environment.

In future, we can solve the other real-life problems like supplier selection, renewable energy source selection etc., and also we can extend the proposed MGDM method for other environments.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**AI Disclosure**

The authors declare that no assistance is taken from generative AI to write this article.

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