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A Comparative Study using Scale-2 and Scale-3 Haar Wavelet for the Solution of Higher Order Differential Equation

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Abstract
A comparative study of scale-2 and scale-3 Haar wavelet has been presented to illustrate the level of accuracy attained by both the wavelets by applying on higher order differential equations known as Emden fowler equation, which has great importance in the field of astrophysics. Approximation of space variable is done by scale-2 and scale-3 Haar wavelet method by choosing different scales. The method is tested upon several test problems. The results are computed and compared in the form of absolute errors. The numerical tests confirm the accuracy, applicability and efficiency of the proposed method with different levels using both the wavelets. By the help of MATLAB algorithm simplification of the computational process is done.

Keywords- Higher order differential equations, Haar scale-2 wavelet (HS2WM), Haar scale-3 wavelet (HS3WM), Quasi linearisation technique, Emden fowler equation.

1. Introduction

In most of the fields of science, including chemistry, biology, and technology, using differential equations the majority of phenomena are modelled. Meanwhile, nonlinear and singular differential equations are used to model a variety of real-world situations. Due to their numerous applications in most scientific fields, differential equations have recently brought the interest of many researchers in the field of science and engineering (Verma et al., 2022, Kumar et al., 2022). A differential equation that appears in both mathematical physics and astrophysics is the Emden fowler equation. It explains the phenomena of stellar structure and behavior of isothermal gases. It is difficult to solve the Emden-fowler problem numerically, because of the singularity behaviour at the point.

There are other linear and nonlinear singular initial value problems in mathematical physics, astrophysics, biochemistry and quantum mechanics in which singularly perturbed equations are faced computational difficulties in numerical processing (Lin, 2021). For these kinds of problem convergence is directly affected by the small parameters. So, Different numerical solutions have been investigated by the researchers for the solution of this particular kind of problems, homotopy analysis method (Singh, 2018), Genocchi operational matrix for solving Emden-fowler equations (Isah and Phang, 2020) it includes quartic polynomial spline method, fourth order B-spline method (Caglar et al., 1999; Akram, 2011), modifies Adomian decomposition method, differential transformation method, variational iteration method (Khuri, 2001; Hasan and Zhu, 2009; Aruna and Kanth, 2013; Wazwaz, 2015a; Wazwaz, 2015b), In the research article Haar scale wavelet method is discussed for obtaining the approximate solution of linear Emden-fowler equations (Alkan, 2017), Emden fowler equation is solved by using HSWM combined with Newton

Raphson method and for solving nonlinearity quasi-linearisation technique is used and discussed some special cases of Emden-fowler equation (Verma and Kumar, 2019), Fourth order Emden- fowler equation is discussed using Haar scale collocation method and by converting the differential equation in to set of algebraic equations and through various examples discuss the applicability of the proposed technique (Khan et al., 2017),Third order differential equations are solved using Haar scale wavelet method, through different examples discussed the effectiveness of the method for solving higher order differential equations (Singh and Kaur, 2021), in the study the author developed the solution of higher order boundary value problem using Haar scale wavelet method (Heydari et al., 2022). Fourth order Lane-Emden fowler equation also described by two different methods adomain decomposition and quintic B-spline method (Ali et al., 2022).

Wavelet analysis is an area of mathematics that is extensively used in signal analysis, picture processing, and numerical analysis. For tackling calculus-related numerical difficulties, the wavelet approaches have proven to be a very powerful and effective tool. As wavelet-based method gives qualitative improved results in comparison with other methods. Researchers working in different fields have become interested in these techniques, and several articles have been published. The Haar wavelets are one of the members of wavelet families, which are compact, dyadic and orthonormal and they are described by an analytical expression. The idea of Haar wavelets was first described by Haar (1910). His original idea has lately been expanded into a wide range of applications, but basically it used for the representation of different functions using a combination of step functions over intervals of specified widths.

A useful feature of the Haar wavelets is the ability to integrate analytically at any given time. As intermediate boundary conditions, the Haar wavelets are very effective at treating singularities. This technique has a key feature is that, it transforms the differential equation used to describe the numerical problem into a set of algebraic equations.

In the given study, we discuss the numerical examples of a differential equation having higher order, singularity and non-linearity with the help of Scale-3 Haar wavelet also known as non-dyadic wavelet. In 1995 non-dyadic wavelet is developed by Chui and Lian (1995) to discuss multiresolution analysis. After that, Mittal and Pandit (2018) discussed non-dyadic wavelet-based methods and deal with various kinds of differential equations to examine the results and found it better than other techniques. After that Kumar and Gupta (2022) discussed various equations by using non-dyadic wavelet (Arora et al., 2018, Kumar and Gupta, 2022). But Emden Fowler equation is not yet discussed by non-dyadic wavelet method, which motivates us to work on the solution of this equation.

The purpose of this develop research is to solve singular Emden fowler equation:

$$\chi''(l) + k\chi'(l) + \chi(l) = 0 \tag{1}$$

With initial conditions: $\chi(0) = \alpha, \chi'(0) = 0.$

A Higher order Emden fowler equation is expressed as:

$$\chi^{IV}(l) + \frac{\kappa_1}{l}\chi'''(l) + \frac{\kappa_2}{l^2}\chi''(l) + \frac{\kappa_3}{l^3}\chi'(l) + \kappa_4\chi(l) = F \tag{2}$$

With initial conditions: $\chi(0) = \alpha, \chi'(0) = \beta, \chi''(0) = \eta,$ and $\chi'''(0) = \gamma$ α, β, η *and* γ are constants in the above equation. $\kappa_1, \kappa_2, \kappa_3$ and κ_4 and F are constants and value of a function respectively (Wazwaz, 2015b).

2. Basic Structure of Haar Wavelets

Scale-2 Haar Wavelet

A group of square waves written in the form of set of Haar functions is defined as:

$$h_i(t) = \phi(t) = \begin{cases} 1 & , \frac{\kappa}{m} \leq t < \frac{\kappa+0.5}{m} \\ -1 & , \frac{\kappa+0.5}{m} \leq t < \frac{\kappa+1}{m} \\ 0 & , otherwise. \end{cases}$$

Here the value of $'m'$ represents level of wavelets in terms of 2^j, $'\kappa'$ is a translation parameter and has$'j'$ as its maximum no. of revolution, here, $'i'$ represents the wavelet number which can be expressed as i-1= $\kappa + m$. If we take *m=1*, $\kappa = 0$ then the value of i=2 and further varying the value of k and m different values of i can be obtained.

At *i*= 2^{j+1}, attains its maximum value,

$$h_1(t) = \begin{cases} 1, & t \in [0,1) \\ 0, & elsewhere. \end{cases}$$

Haar function can be obtained by discretising the points in the form of collocation points

$$t = \frac{i-0.5}{2M}.$$

Haar wavelet integrals in generalised form can be written as:

$$\phi_{1,s}(t) = \int_0^x h_{1,s}(t)dt.$$

$$\phi_{1,s+1}(t) = \int_0^x \phi_{1,s}(t)dt.$$

It can be written as:

$$\phi_{i,s+1}(x) = \begin{cases} \frac{t^s}{\Gamma(s+1)} & , \quad \alpha_1 \leq t < \alpha_2 \\ 0 & otherwise. \end{cases}$$

$$\varphi_{i,s+1}(t) = \begin{cases} \frac{[t-\mu_1(i)]^s}{\Gamma(s+1)} & ; \mu(i) \leq t < \mu_2(i) \\ \frac{[t-\mu_1(i)]^s - 2[t-\mu_2(i)]^s}{\Gamma(s+1)} & ; \mu_2(i) \leq t < \mu_3(i) \\ \frac{[t-\mu_1(i)]^s - 2[t-\mu_2(i)]^s + [t-\mu_3(i)]^s}{\Gamma(s+1)}; & ; \mu_3(i) \leq t < \mu_4(i) \\ 0 & ; otherwise. \end{cases}$$

Scale-3 Haar Wavelet

For the scale-3 wavelet family, detailed equations for Scale-3 wavelet function, a father wavelet, two symmetric and antisymmetric mother wavelets are provided below:

$$h_i(t) = \phi(t) = \begin{cases} 1 & , \quad \alpha_1 \leq t < \alpha_2 \\ 0 & , \quad otherwise. \end{cases} \quad for\ i = 1.$$

$$h_i(t) = \varphi^1(3^m - k) = \frac{1}{\sqrt{2}}\begin{cases} -1 & \mu_1(i) \le t < \mu_2(i) \\ 2 & \mu_2(i) \le t < \mu_3(i) \\ -1 & \mu_3(i) \le t < \mu_4(i) \\ 0 & otherwise. \end{cases} \; for\; i = 2,4, \dots .3p-1$$

$$h_i(t) = \varphi^2(3^m - k) = \sqrt{\frac{3}{2}}\begin{cases} 1 & \mu_1(i) \le t < \mu_2(i) \\ 0 & \mu_2(i) \le t < \mu_3(i) \\ -1 & \mu_3(i) \le t < \mu_4(i) \\ 0 & otherwise. \end{cases} \; for\; i = 3,5, \dots .3p$$

Here, $\mu_1(i) = \frac{\kappa}{p}, \mu_2(i) = \frac{3\kappa+1}{3p},\ \mu_3(i) = \frac{3k+2}{3p},\ \mu_4(i) = \frac{k+1}{p}$, Here the value of $'p'$ represents level of wavelets in terms of 3^j, $'\kappa'$ is a translation parameter and has $'j'$ as its maximum no. of revolution, here,$'i'$ represents the wavelet number which can be expressed as $i - 1 = \kappa + m$. If *we* take *m=1*, $\kappa = 0$ then the value of $i = 2$ and further varying the value of k and m different values of i can be obtained.

At *i*= 3^{j+1}, attained its maximum value,

$$h_1(t) = \begin{cases} 1, & t \in [0,1) \\ 0, & elsewhere. \end{cases}$$

Haar function can be obtained by discretising the points in the form of collocation points,

$$t = \frac{i-0.5}{3M}.$$

Haar wavelet integrals in generalised form can be written as:

$$\phi_{1,s+1}(t) = \int_0^x \phi_{1,s}(t)dt.$$

$$\varphi_{1,s+1}{}^1(t) = \int_0^x \varphi_{1,s}{}^1(t)dt.$$

$$\varphi_{1,s+1}{}^2(t) = \int_0^x \varphi_{1,s}{}^2(t)dt.$$

It can be written as:

$$\phi_{i,s+1}(x) = \begin{cases} \dfrac{t^s}{\Gamma(s+1)} \quad , & \propto_1 \le t < \propto_2 \\ 0 & otherwise. \end{cases}$$

$$\varphi_{i,s+1}{}^1(t) = \frac{1}{\sqrt{2}}\begin{cases} 0 & ;\ 0 \le t < \mu_1(i) \\ \frac{-[t-\mu_1(i)]^s}{\Gamma(s+1)} & ; \mu_1(i) \le t < \mu_2(i) \\ \frac{3[t-\mu_2(i)]^s - [t-\mu_1(i)]^s}{\Gamma(s+1)} & ;\ \mu_2(i) \le t < \mu_3(i) \\ \frac{3[t-\mu_2(i)]^s - 3[t-\mu_3(i)]^s - [t-\mu_1(i)]^s}{\Gamma(s+1)} & ; \mu_3(i) \le t < \mu_4(i) \\ \frac{3[t-\mu_2(i)]^s - 3[t-\mu_3(i)]^s - [t-\mu_1(i)]^s + [t-\mu_4(i)]^s}{\Gamma(s+1)} & ;\ \mu_4(i) \le t < 1. \end{cases}$$

$$\varphi_{i,s+1}{}^{2}(t)=\sqrt{\frac{3}{2}}\begin{cases} 0 & ;\ 0 \le t < \mu_1(i) \\ \dfrac{[t-\mu_1(i)]^s}{\Gamma(s+1)} & ;\ \mu_1(i) \le t < \mu_2(i) \\ \dfrac{[t-\mu_1(i)]^s-[t-\mu_2(i)]^s}{\Gamma(s+1)} & ;\ \mu_2(i) \le t < \mu_3(i) \\ \dfrac{[t-\mu_1(i)]^s-[t-\mu_2(i)]^s-[t-\mu_3(i)]^s}{\Gamma(s+1)} & ;\ \mu_3(i) \le t < \mu_4(i) \\ \dfrac{[t-\mu_1(i)]^s-3[t-\mu_2(i)]^s-[t-\mu_3(i)]^s+[t-\mu_4(i)]^s}{\Gamma(s+1)} & ;\ \mu_4(i) \le t < 1. \end{cases}$$

3. Quasi-Linearisation Process

A generalized form of the Newton– Raphson method is quasi-linearization strategy used for linearizing nonlinear differential equations. Quadratically, it converges to the exact value. As the fact that many nonlinear equations have no analytic solution, but their solution is required as per their application in the physical world is major motivation for employing this strategy stems. If we have a non-linear term, we must employ the recurrence relation shown below:

$$u^2 = f(u^2) = f(x, u^2).$$

$$[u^2]_{l+1} = [u^2]_l + [u_{l+1} - u_l]\left(\frac{\partial}{\partial u}(u^2)\right)_l.$$

$$(u^2)_{l+1} = (u^2)_l + (u_{l+1} - u_l)\, 2u_l.$$

$$(u^2)_{l+1} = (u^2)_l + 2u_{l+1}(u)_l - 2(u^2)_l\,.$$

$$(u^2)_{l+1} = 2(u)_{l+1}u_l - (u^2)_l.$$

4. Approximation of Emden Fowler Differential Equation by Scale-3 Haar Wavelet

Theorem: Let $\chi(z)$ be any square integrable function over the interval $[\sigma_1, \sigma_2)$, whose higher derivative is expressible as a linear combination of Haar wavelet family as $\chi^n(z) = \sum_{i=1}^{3p} a_k h_k(z)$. Then all the derivative of $\chi(z)$ of order less than n are given by (Arora et al; 2018):

$$\chi^k(z) = \sum_{i=1}^{3p} a_i p_{n-k,i}(z) + \sum_{r=0}^{n-k-1} \frac{(z-Q)^r}{r!} \chi^{k+r}\ (Q);\ k = 0,1,2,3,\ldots,n-1 \tag{3}$$

Proof: Let us approximate the higher order derivative as:

$$\chi^n(z) = \sum_{i=1}^{3p} a_k h_k(z). \tag{4}$$

Integrating $\chi^n(z)$ w.r.t 'z' between z to Q we have,

$$\chi^{n-1}(z) = \sum_{i=1}^{3p} a_i p_{1,i}(z) + \chi^{n-1}(Q) \tag{5}$$

As per the rule of mathematical induction on $M = n - k$, Above, theorem is proved.

Take $M = 1 \Rightarrow k = n - 1$, by using $k = n - 1$ we have,

$$\chi^{n-1}(z) = \sum_{i=1}^{3p} a_i p_{1,i}(z) + \chi^{n-1}(Q) \tag{6}$$

It is same as the above calculated results, so for $m = 1$ it is true.

Assume the result is true for $M = n - k = s$.

$$\chi^{n-s}(z) = \sum_{i=1}^{3p} a_i p_{s,i}(z) + \sum_{r=0}^{s-1} \frac{(z-Q)^r}{r!} \chi^{n-s+r}\ (Q) \quad (7)$$

proof of the result at $M = s + 1$, integrating the above equations,

$$\chi^{n-s-1}(z) = \sum_{i=1}^{3p} a_i p_{s+1,i}(z) + \sum_{r=0}^{s-1} \frac{(z-Q)^{r+1}}{(r+1)!} \chi^{n-s+r}\ (Q) + \chi^{n-s-1}\ (Q) \quad (8)$$

$$\chi^{n-s-1}(z) = \sum_{i=1}^{3p} a_i p_{s+1,i}(z) + \chi^{n-s-1}\ (Q) + \left[\frac{(z-Q)^1}{1!}\chi^{n-s} + \frac{(z-Q)^2}{2!}\chi^{(n-s)+1} + \frac{(z-Q)^3}{3!}\chi^{(n-s)+2} + \cdots + \frac{(z-Q)^s}{s!}\chi^{(n-s)+(s-1)}(Q)\right] \quad (9)$$

$$\chi^{n-s-1}(z) = \sum_{i=1}^{3p} a_i p_{s+1,i}(z) + \frac{(z-Q)^0}{0!}\chi^{n-(s+1)} + \left[\frac{(z-Q)^1}{1!}\chi^{(n-(s+1))+1} + \frac{(z-Q)^2}{2!}\chi^{(n-(s+1))+2} + \cdots + \frac{(z-Q)^s}{s!}\chi^{(n-(s+1))+s}(Q)\right] \quad (10)$$

$$\chi^{n-(s+1)}(z) = \sum_{i=1}^{3p} a_i p_{s+1,i}(z) + \sum_{r=0}^{(s+1)-1} \frac{(z-Q)^r}{(r)!} \chi^{(n-(s+1))+r}\ (Q) \quad (11)$$

Hence the above result is verified for $M = (n-k) = s + 1$, therefore given result is valid for all derivative of $\chi(z)$ which satisfies the proof.

Since the members of family of non-dyadic Haar wavelet have a property that they are orthogonal to each other, thus by using properties of wavelet and above theorem, over the interval [0,1); any square integrable function $\chi(z)$ can be expressed as:

$$\chi(z) \approx \alpha_1 h_1(z) + \sum_{even\ index\ i\geq 2}^{\infty} \alpha_i \varphi_i^1(z) + \sum_{odd\ index\ i\geq 3}^{\infty} \alpha_i \varphi_i^2(z) \quad (12)$$

Here α_i's coefficients of Haar wavelet can be calculated as:

$$\alpha_i = \int_0^1 \chi(z) h_i(z) dz,\ i = 1, 2, 3, \ldots.., 3p \quad (13)$$

Considering terms in finite number, for first 3p terms, where $p = 3^j, j = 0,1,2, \ldots.$to approximate the function $\chi(z)$, we get,

$$\chi(z) \approx \chi_{3p} = \sum_{i=1}^{3p} a_i h_i(z) \quad (14)$$

5. Error Analysis with Test Problems

Above scheme is applied on Emden Fowler equation to validate the competency of the scheme and accuracy level gained by the recent technique. $L_2 - error$, $L_\infty - error$ and absolute errors has been found which can written as,

Absolute Error $= |u_{exact}(z_r) - u_{num}(z_r)|$ (15)

$$L_\infty = \max_r |u_{exact}(z_r) - u_{num}(z_r)| \quad (16)$$

$$L_2 = \frac{\sqrt{\sum_{l=1}^{3p} |u_{exact}(z_r) - u_{num}(z_r)|^2}}{\sqrt{\sum_{l=1}^{3p} |u_{exact}(z_r)|^2}} \quad (17)$$

6. Numerical Observations

***Example* 1.** Fourth non-order linear Emden fowler equation:

$$u^{iv}(z) + \frac{12}{z}u'''(z) + \frac{36}{z^2}u''(z) + \frac{24}{z^3}u'(z) + 60(7 - 18z^4 + 3z^8)u^9 = 0 \quad (18)$$

with initial conditions,

$$u(0) = 1, u'(0) = u''(0) = u'''(0) = 0 \quad (19)$$

Exact solution of the given equation:

$$u(z)=\frac{1}{\sqrt{1+z^4}} \quad (20)$$

$$u(z) = \sum_{i=1}^{3p} a_i h_i(z) + \frac{12}{z}P_{1,i}(z) + \frac{36}{z^2}P_{2,i}(z) + \frac{24}{z}P_{2,i}(z) + 60(7 - 18z^4 + 3z^8)(9u_{r+1}u_r^8 - 8u_r^9).$$

Table 1 shows a comparative study of absolute error of proposed methods with other methods and whereas Table 2 represents a comparative study at resolution level *j*=2 and Table 3 signifies value of L_2 and L_∞ error at different level of resolutions. In Figure 1, comparative study is discussed between the approximate solution and analytical solution derived by proposed method and Figure 2 signifies the value of absolute error at different collocation points.

Table 1. A comparison study of Absolute errors of proposed method with other methods.

z	Absolute error by VIM (Wazwaz, 2015a)	Absolute error by Haar scale 2 (Khan et al., 2017)	Absolute error by Haar scale3
0.930	4.5E-02	1.1E-02	1.32E-05
0.943	6.2E-02	1.3E-02	1.35E-05
0.961	8.5E-01	1.5E-02	1.42E-05
0.977	1.2E-01	1.7E-02	1.48E-05
0.992	1.6E-01	2.0E-02	1.53E-05

Table 2. A comparison study of Absolute errors by proposed method at different points for resolution level *j*=2.

z	Exact solution	Approximate solution by HS3WM	Absolute Error by HS3WM	Absolute Error by ADM (Ali et al., 2022)	Absolute Error by QBSM (Ali et al., 2022)
0.1	0.999858843193092	0.999858845358450	2.165E-09	1.1713E−05	2.9749E−06
0.2	0.999140153146428	0.999140184430897	3.128E-08	-	-
0.3	0.995124236352257	0.995124652439060	4.160E-07	-	-
0.4	0.983937555332222	0.983940019395591	2.464E-06	6.5734E−06	1.7577E−06
0.5	0.970136312766435	0.970142500145332	6.187E-06	5.2624E−07	4.7039E−07
0.6	0.936785055989435	0.936803613298913	1.855E-05	7.5805E−06	3.9870E−06
0.7	0.886591328782529	0.886633635115172	4.230E-05	-	-
0.8	0.821293712001356	0.821369965673296	7.625E-05	-	-
0.9	0.771883253099228	0.771983726870444	1.004E-04	3.6837E−05	1.0483E−05

Table 3. Value of the L_2 and L_∞ error at different number of resolutions.

No. of Resolution	L_2 error at Haar scale 3	L_∞ error at Haar scale 3
J=1	4.85313E-04	1.05739E-03
J=2	5.15079E-05	1.21669E-04
J=3	5.69559E-06	1.37822E-05
J=4	6.32507E-07	1.54229E-06
J=5	7.02744E-08	1.71784E-07
J=6	7.80821E-09	1.91028E-08

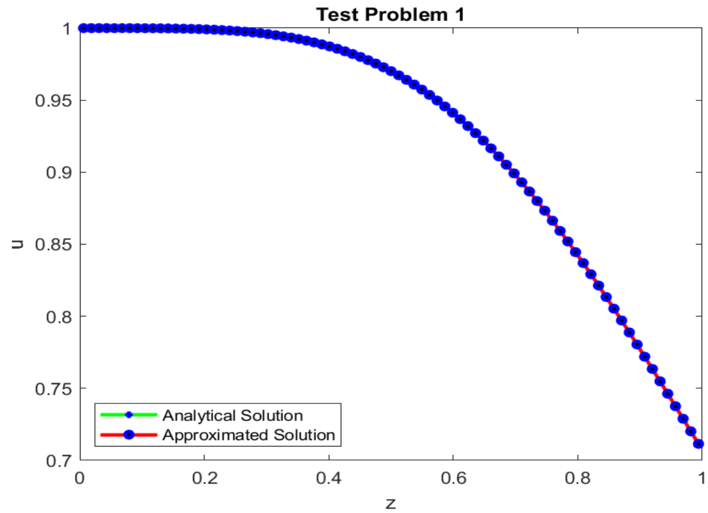

Figure 1. For numerical experiment no.1, graphical presentation of analytical solution and approximate solution of *u (z, t)* at distinct point of *z*.

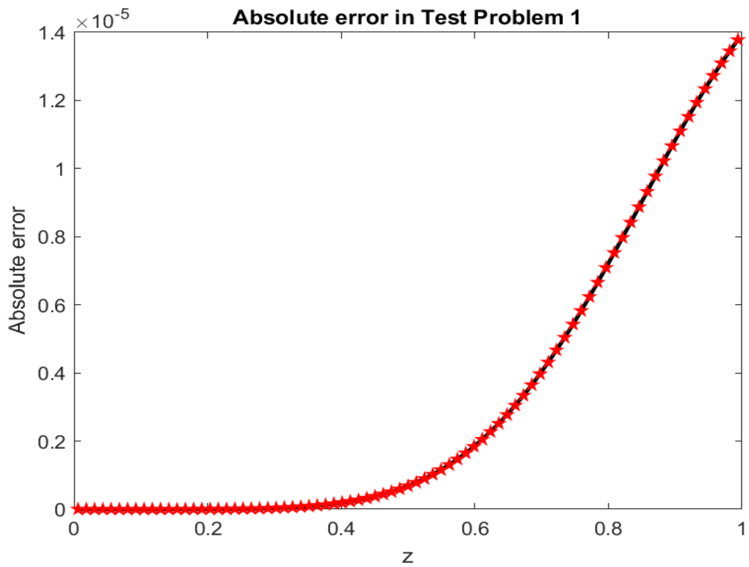

Figure 2. For numerical experiment no.1, value of absolute error for different values of *z*.

***Example* 2.** Third order non-linear ODE (Emden Fowler Equation)

$$u'''(z)+\frac{3}{z}u''(z)=u^3(z)+24e^z+36ze^z+12z^2e^z+z^3e^z-z^9e^{3z} \tag{21}$$

with initial conditions,

$$u(0)=0,u(1)=e,u'(0)=0 \tag{22}$$

Exact solution of the given equation:

$$u(z)=z^3e^z \tag{23}$$

$$u(z)=\sum_{i=1}^{3p} a_i h_i(z)+\frac{3}{z} a_i P_{1,i}(z)-(3u_{r+1}u_r^2-2u_r^3)-(24e^z+36ze^z+12z^2e^z+z^3e^z-z^9e^{3z}).$$

Table 4 shows a comparative study of absolute error of proposed methods with other methods at resolution level j=3 and Table 5 signifies value of L_2 and L_∞ error at different level of resolutions. In Figure 3, comparative study is discussed between the approximate solution and analytical solution derived by proposed method and Figure 4 signifies the value of absolute error at different collocation points.

Table 4. A comparison study of absolute errors of proposed method with other methods at resolution level j=3.

z	Exact solution	Approximate solution by Haar scale 3	Absolute error by Haar scale 3
0.104	0.001283440775195	0.001282586360785	8.54E-07
0.203	0.010362491097857	0.010359072316960	3.41E-06
0.302	0.037445865503173	0.037437985358077	7.88E-06
0.401	0.096482597073391	0.096468138911095	1.44E-05
0.50	0.206090158837516	0.206066773523318	2.33E-05
0.611	0.420496827502144	0.420460286063230	3.65E-05
0.709	0.727518335078141	0.727467043686249	5.12E-05
0.808	1.187018782746865	1.186949558412363	6.92E-05
0.907	1.851351961228259	1.851261220832432	9.07E-05

Table 5. Value of the L_2 and L_∞ error at different number of resolutions.

No. of Resolution	L_2 Error by Haar scale 2 (Singh et al., 2020)	L_∞ error by Haar scale 2 (Singh et al., 2020)	L_2 error by Haar scale 3	L_∞ error at Haar scale 3
J=1	1.8840E-02	2.9323E-02	4.14486E-03	7.8944E-03
J=2	2.1377E-02	1.0697E-02	4.70304E-04	9.8452E-04
J=3	8.1855E-03	2.8254E-03	5.23755E-05	1.1305E-04
J=4	2.9917E-03	7.1846E-04	5.82097E-06	1.2693E-05
J=5	1.0737E-03	1.8035E-04	6.46793E-07	1.4152E-06
J=6	3.8237E-04	4.5133E-05	7.18663E-08	1.5742E-07

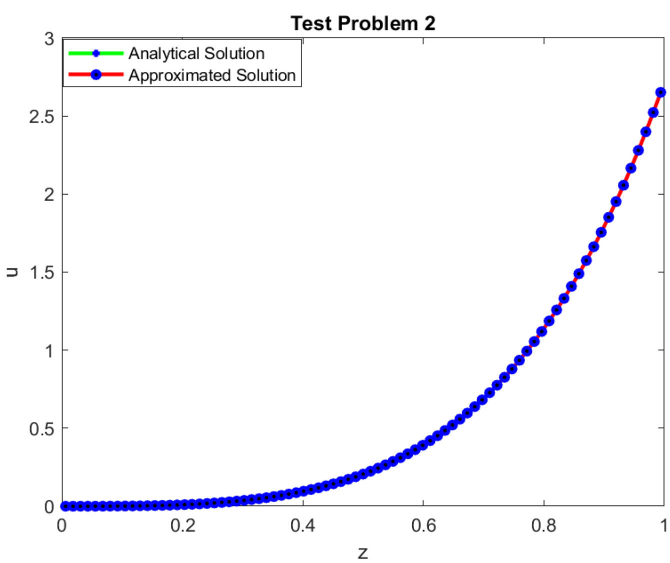

Figure 3. For numerical Experiment no.2, graphical presentation of analytical solution and approximate solution of *u (z, t)* at distinct point of *z*.

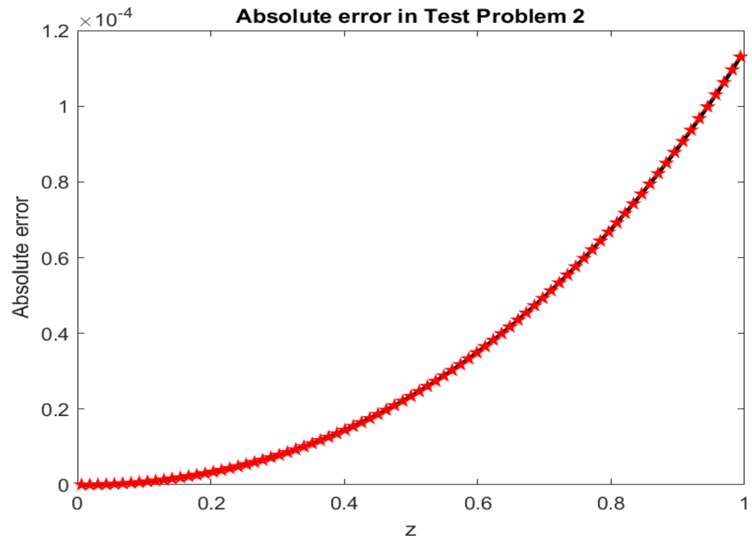

Figure 4. For numerical experiment no. 2: value of absolute error for different values of *z*.

***Example* 3.** A Fourth order linear ODE,

$$u^{iv}(z)+\frac{9}{z}u'''(z)+\frac{18}{z^2}u''(z)+\frac{6}{z^3}u'(z)-32(15+75z^4+54z^8+8z^{12})u=0 \quad (24)$$

with initial conditions,

$$u(0)=1,\ u'(0)=u''(0)=u'''(0)=0 \quad (25)$$

Exact solution of the given equation,

$$u(z)=e^{z^4} \quad (26)$$

$$u(z)=\sum_{i=1}^{3p}a_ih_i(z)+\frac{9}{z}P_{1,i}(z)+\frac{18}{z^2}P_{2,i}(z)+\frac{6}{z^3}P_{3,i}(z)+32(15+75z^4+54z^8+8z^{12})P_{4,i}(z).$$

Table 6. A comparison study of absolute errors of proposed method with other methods at resolution level *j*=3.

z	Exact solution	Approximate solution by VIM (Wazwaz, 2015b)	Approximate solution by Haar scale 2 (Khan et al., 2017)	Approximate solution by Haar scale 3
0.504	1.0667142	1.01574	1.06660	1.0667141
0.605	1.1453966	1.03202	1.14384	1.1453962
0.707	1.2840731	1.05733	1.28388	1.2840730
0.801	1.5117446	1.08968	1.50862	1.5117443
0.902	1.4512819	1.13321	1.94051	1.9459544

Table 7. A comparison study of absolute errors by proposed method at different points for resolution level *j*=2.

z	Exact solution	Approximate solution by HS3WM	Absolute error by HS3WM
0.1	1.000282406409751	1.000282408938148	2.528E-08
0.2	1.001723294678824	1.001723334769499	4.009E-07
0.3	1.009870313695018	1.009870867313081	5.536E-06
0.4	1.033454757045220	1.033458183478689	3.426E-05
0.5	1.064485414542223	1.064494458917860	9.044E-05
0.6	1.149633526509271	1.149664285482661	3.075E-05
0.7	1.312594689586376	1.312681132438711	8.644E-05
0.8	1.619507358288849	1.619719662482946	2.123E-04
0.9	1.969508640506898	1.969871146748689	3.625E-04

Table 8. Value of the L_2 and L_∞ error at different number of resolutions.

No. of Resolution	L_2 error by Haar scale 3	L_∞ error at Haar scale 3
J=1	1.11181E-03	3.93567E-03
J=2	1.34148E-04	5.83032E-04
J=3	1.50753E-05	7.03277E-05
J=4	1.67720E-06	8.00930E-06
J=5	1.86383E-07	8.96960E-07
J=6	2.07096E-08	9.99204E-08

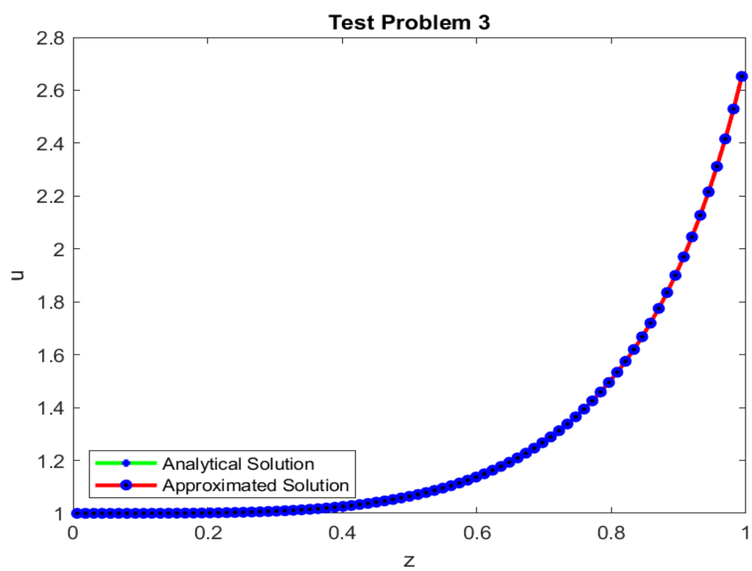

Figure 5. For numerical experiment no. 3, graphical presentation of analytical solution and approximate solution of *u (z, t)* at distinct point of *z.*

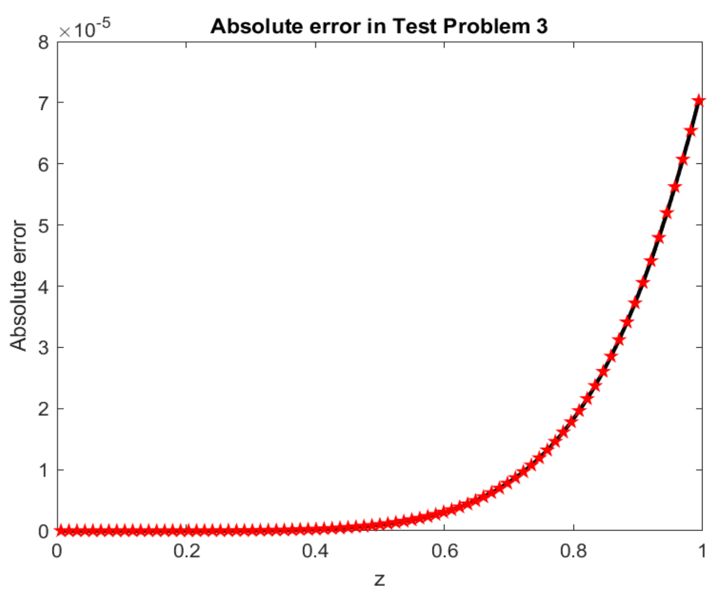

Figure 6. For numerical experiment no. 3: value of absolute error for different values of *z*.

Table 6 shows a comparative study of absolute error of proposed methods with other methods and whereas Table 7 represents a comparative study at resolution level j=2 and Table 8 signifies value of L_2 and L_∞ error at different level of resolutions. In Figure 5, comparative study is discussed between the approximate solution and analytical solution derived by proposed method and Figure 6 signifies the value of absolute error at different collocation points.

7. Conclusion

After going through the results obtained for numerical experiments with proposed technique, we observed that in comparison to other methods a higher order non-linear Emden Fowler equation can be easily solved by using Scale-3 Haar wavelet with less computational cost and high accuracy. Here with the use of MATLAB subprogram, approximated solution of Emden Fowler equation is attained. The solution obtained accuracy of results with higher level of resolution at small number of collocation points by HS3WM in comparison to HS2WM and other existing methods available in the literature.

Therefore, by observing the performance of the method on different numerical experiments it is concluded that, in future the proposed computational method is extended to solve numerous higher order complex ordinary differential equations of same kind having singularity, which has great importance in the field of applied mathematics and astrophysics.

Conflict of Interest

The authors have no conflict of interest.

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