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Particle Swarm Optimization for Solving Nonlinear Fisher's Equation

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Abstract
The term "optimization" describes a way of increasing the favourable features of a function in a mathematical form while reducing the negative ones. Most of the problems encountered in the real world may be formulated as optimization problems. Particle swarm optimization is a tool which has been used extensively to handle problems across different fields. The purpose of this study is to develop a numerical method for solving nonlinear reaction-diffusion Fisher's equations, whose numerical solutions exhibit the solitons form and have a wide range of applications. This study includes differential quadrature method using exponential basis function and particle swarm optimization. In the work so far documented in the literature, a random number was used to approximate the involved parameter in the basis functions. This value is now determined using the implemented optimization technique. The numerical solutions obtained for the equation is comparable to the data available in the literature. The research presented through figures and tables, yields favourable results.

Keywords- B-spline, Particle swarm optimization (PSO), Fisher's reaction diffusion equation (FRDE), Differential quadrature method (DQM).

1. Introduction

Differential equations (DE) play a vital role in understanding the phenomenon of science and engineering. Researchers are looking for the solution of the modelled DE using various analytical and numerical approaches. However, it is not always possible that the differential equation can be solved by the available analytical techniques to provide the exact solution. There comes the role of numerical approaches that can help in solving the differential equations with the required accuracy.

With the advancement of technology, there are different available options of obtaining the numerical solution of the DEs using the software like MATLAB, Maple and Mathematica. Researchers are continuously working on the modification of the numerical schemes to provide the solution efficiently and accurately. One such attempt to modify the numerical technique is presented in this paper. In this work a hybrid scheme is developed to find more accurate solutions of the well-known Fisher's equations (FEs) using the cubic exponential B-spline basis function with DQM (Expo-MCB-DQM) and an optimization technique based on swarm intelligence known as particle swarm optimization.

FEs is a reaction-diffusion equation that was introduced by Fisher in 1937 (Fisher, 1937) in a study of the growth of advantageous genes due to mutation. After that application of this equation was investigated in many other phenomena such as it describes the pattern formation in wave propagation, it is used to study the growth of cells in tissue engineering, in the modelling of oscillating chemical reactions, in population biology, in healing of a wound and growth of the tumor.

The nonlinear partial Fisher's reaction diffusion equation is given as follows:

$$u_t = vu_{xx} + \rho f(u) \tag{1}$$

where, $u(x,t)$ and x represents the spatial coordinate, t denotes the time, v is a constant that denotes the diffusion coefficient, f is the nonlinear reaction term and ρ is the reaction factor. These terms signified the parameter of the phenomenon that is modelled as Fisher's equations. For example, in the study of the brain tumor, u represents the carrying capacity, the diffusion coefficient v shows the migration of cell and the ρ represents the growth rate.

Similarly, in the study related to the gene propagation, u represents the frequency of the mutant gene and ρ the reaction factor signifies the strength of selection in favour of the mutant gene, with the following considered initial and boundary conditions:

$$u(x,0) = u_0(x),\ x \in [-\infty, \infty] \tag{2}$$

$$\lim_{x\to-\infty} u(x,t) = 1,\ \lim_{x\to\infty} u(x,t) = 0 \tag{3}$$

$$\lim_{x\to\pm\infty} u(x,t) = 0, \tag{4}$$

Based on the requirement of the modelled phenomenon and the parameters, the equation can be solved with the conditions, nonlocal conditions (2) - (3) or the local conditions (2) - (4) with a finite domain $[x_L, x_R]$.

A number of efforts have been made by the researchers to find the accurate solution the equation both analytically and numerically. For instance, Fisher's equation has been solved using the space derivative approach of the Fourier series (Gazdag and Canosa, 1974). The explicit solution for FE has been obtained using (Ablowitz and Zeppetella, 1979). This equation has also been solved using the explicit finite difference approach (Twizell et al., 1990). Researchers have used the Petrov-Galerkin finite element method to find the numerical solution of the equation (Tang and Weber, 1991). A new class of finite-difference schemes (FDS) has been developed by Mickens for solving the Fisher's equation (Mickens, 1994). The numerical study of the equation was performed (Qiu and Sloan, 1998) using the moving mesh method. A comparative study has been done (Rizwan, 2001). For the FDS and the nodal integral method taking the FE numerical solution. The exact solution of the equation has been presented (Wazwaz and Gorguis, 2004) using the Adomian decomposition method. A spectral approach using Chebyshev-Lobatto points is developed (Olmos and Shizgal, 2006) to solve the equation. A differential quadrature approach with FDS is implemented to solve the Fisher's equation numerically (Aswin and Awasthi, 2017). Other numerical techniques, such as the tension spline method, boundary integral equation method, wavelet Galerkin method, quartic B-spline Galerkin method, and DQM have been used to solve the Fisher's problem successfully in recent years (Bellman et al., 1972; Bashan, 2018). Researchers have used a variety of computational techniques to discover the numerical solution to Fisher's reaction diffusion equations, some of them are moving mesh method (Qiu and Sloan, 1998), Petrov-Galerkin finite element method (Tang and Weber, 1991), implicit and explicit FD algorithms (Parekh and Puri, 1990; Twizell et al., 1990), a sinc collocation method (Al-Khaled, 2001), a pseudo-spectral approach (Gazdag and Canosa, 1974; Wang and Jiao, 2017), ADM (Mavoungou and Cherruault, 1994; Wazwaz and Gorguis, 2004), B-spline collocation method (Mittal and Arora, 2010), modified cubic B-spline collocation method (Mittal and Jain, 2013), FDS (Dag and Ersoy, 2016), B-spline Galerkin approach (Şahin et al., 2008), and three alternative numerical approaches for solving FRDE were compared to the discrete singular convolution by Zhao and Wei (2003), pseudo spectral method (Balyan et al., 2020) fractional Fisher equation comparing numerical and approximate solution (Yokus and Yavuz, 2021).

For solving non-linear Fisher's equation an optimized second order numerical scheme is proposed involving implicit finite difference approximation method (Izadi and Srivastava, 2022). In another study, researchers used Strang splitting technique, Galerkin method in combination of quadratic B-spline function to solve Fisher's equation (Karta, 2023).

In the prior published studies, B-spline basis functions are discussed with DQM, in various forms and modifications. However, the method has not been discussed in coupling with PSO method. In the present work the optimization technique is used to find the minimum errors between the numerical and exact solutions by obtaining the best value of unknown parameter. It is difficult to solve various equations manually which contain nonlinearity and complexity in the order. This study presents to develop a method to solve these kinds of equations with the help of optimization technique which is a requirement to minimize error and increase the efficiency.

This manuscript is arranged in the described manner: Section 2 of the DQM provides an explanation of the numerical method used for combining the PSO with the exponential B-spline. Two test problems for the considered one-dimensional Fisher's equation are illustrated with different kinds of parameters in Section 3. The Section 4 of the paper presents a summary of the topic.

2. Methodology

2.1 Differential Quadrature Method

The Differential Quadrature Method, introduced by Bellman et al. (1972), is an efficient technique to solve partial differential equations (PDEs). DQM offers precise solutions for challenges related to intricate geometries or irregular boundaries. This is accomplished through the adept use of interpolation methods, ensuring accurate approximation of derivatives for unknown functions. DQM is versatile method addressing a variety of issues such as linear and nonlinear problems, both steady-state and transient, and those with variable coefficients. Its applicability spans across one, two, and three dimensions, making it a valuable tool for diverse problem-solving scenarios. In recent years, the DQM is widely used to determine the weighting coefficients using various test functions such as spline functions, Lagrange interpolation polynomials and sinc function in many articles, see (Quan and Chang, 1989; Korkmaz and Dag, 2011; Shu and Richards, 1992), Modified cubic B-spline DQM (Singh and Arora, 2014), cubic B-spline method (Mittal and Arora, 2010), Hyperbolic B-spline based DQM (Kapoor and Joshi, 2020), The standard cubic trigonometric B-spline functions are combined with quadrature technique (Arora and Joshi, 2021) to study the Fisher's equation in 1 and 2 dimensions. The Crank-Nicolson scheme (Arora et al., 2022) is implemented to study the dynamics of the Fisher's equation. In the present study, the Expo-MCB-DQM is incorporated along with a PSO algorithm. The approach is used to find the efficient solution of the PDE by converting them to the ODE by representing the derivatives as the linear sum of the function with the weighting coefficients (Expo-MCB-DQM). This combined approach is assessed successfully by Rani et al. (2023) to enhance the solution accuracy by optimizing the parameters within the basis function.

Let the finite domain of the differential equation $[a, b]$ can be discretized into number of known points as $a = x_0 < x_1 < \cdots\cdots < x_{n-1} < x_n = b$.

The r^{th}derivative of the function can be written as:

$$\left[\frac{d^r U}{dx^r}\right]_{x_i} = \sum_{j=1}^{N} p_{ij}^{(r)} U(x_j), i = 1 \ to \ N, r = 1 \ to \ N - 1 \tag{5}$$

where, $p_{ij}^{(r)}$ represents the coefficients to be calculated using the exponential basis function.

As per the literature, polynomial B-splines are the generalization of the exponential B-spline functions with a free parameter. The third-degree exponential B-spline can be defined as follows:

$$B_i(x) = \begin{cases} b_2\left[(x_{i-2} - x) - \frac{1}{p}\left(\sinh\left(p(x_{i-2} - x)\right)\right)\right] & \text{if } x \in [x_{i-2}, x_{i-1}]; \\ a_1 + b_1(x_i - x) + c_1 e^{p(x_i - x)} + d_1 e^{-p(x_i - x)} & \text{if } x \in [x_{i-1}, x_i]; \\ a_1 + b_1(x - x_i) + c_1 e^{p(x - x_i)} + d_1 e^{-p(x - x_i)} & \text{if } x \in [x_i, x_{i+1}]; \\ b_2\left[(x - x_{i+2}) - \frac{1}{p}\left(\sinh\left(p(x - x_{i+2})\right)\right)\right] & \text{if } x \in [x_{i+1}, x_{i+2}]; \\ 0 & \text{otherwise.} \end{cases}$$

where, $p = \max_{0 \le i \le N} p_i, s = \sinh(ph), c = \cosh(ph)$

$$, \quad b_2 = \frac{p}{2(phc - s)}, a_1 = \frac{phc}{phc - s}, b_1 = \frac{p}{2}\left[\frac{c(c-1) + s^2}{(phc - s)(1 - c)}\right],$$

$$, \quad c_1 = \frac{1}{4}\left[\frac{e^{-ph}(1 - c) + s\left(e^{-ph} - 1\right)}{(phc - s)(1 - c)}\right], d_1 = \frac{1}{4}\left[\frac{e^{ph}(c - 1) + s\left(e^{ph} - 1\right)}{(phc - s)(1 - c)}\right]$$

Each basis function $B_i(x)$ is twice continuously differentiable. The values of $B_i(x), B_i'(x)$ and $B_i''(x)$ at the knots x_i 's are obtained from the **Table 1**.

Table 1. Exponential B-spline values.

$B_i \backslash x_i,$	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	$\frac{s - ph}{2(ph - s)}$	1	$\frac{s - ph}{2(phc - s)}$	0
$B_i'(x)$	0	$\frac{p(1 - c)}{2(ph - s)}$	0	$\frac{p(c - 1)}{2(phc - s)}$	0
$B_i''(x)$	0	$\frac{p^2 s}{2(phc - s)}$	$\frac{-p^2 s}{phc - s}$	$\frac{p^2 s}{2(phc - s)}$	0

For functions which defined on the interval $[a, b]$, the $B_i(x), i = -1, \ldots, N + 1$ form a basis.

In order to calculate the value of parameter $'p'$ for which the error will be minimised and in order to minimise the error an optimization technique is required. The process of minimizing or maximizing an objective function by choosing the best values for each of its variables from within the permitted range of values is referred to as "optimization." Any real-world application can often be turned into an optimization task. Overall optimization technique has numerous applications across different fields, and they are an essential tool for solving complex problems and improving efficiency and performance. In this manuscript, particle swarm optimization tool is used to optimize the value of errors.

2.2 Particle Swarm Optimization (PSO)

In the class of algorithms inspired by nature, PSO is one of the potential global optimization methods. PSO is a computer technique used in computational science to optimize problems by repeatedly attempting to raise the quality of possible answers.

A computational worldwide optimization technique called PSO was initially suggested by Eberhart and Kennedy (1995). It is based on research into the movement patterns of fish and bird flocks and derives from swarm intelligence. The PSO system finds the solution of an optimization problem by modifying the track of each agent called particles. A stochastic aspect and a deterministic aspect make up the two main parts of particle's movement. It is influenced by the social behaviour of animals, such as flocks of birds, communities of marine animals and the idea of swarms. In terms of the particle swarm optimization approach, the solution swarm is like a flock of birds flying from one place to another. Each participant might work together to discover the most upbeat solution using the PSO method. A particle without quality or volume assumes the role of each individual and a simple behavioural pattern is regulated for each particle to demonstrate the complexity of the complete particle swarm. One may use this strategy to resolve the difficult optimization problems. Swarms are considered as population particles that transmit information to improve the effectiveness of the search solution and find the global optimum in this nature-based swarm optimization technique. Each particle has their best experience which is called their current position. There is a best experience of all the particles that is called their global best position. A particle changes the current best position for particle when it detects a target superior to all previously detected locations. During the iterations, there is a new best solution for each of the n particles. The main purpose of this algorithm is to find the overall best solution from all available solutions. This process goes up to the defined iterations or until the objective is not reached. In this optimization algorithm, the velocity V_i^{k+1} and the position X_i^{k+1} of the ith particle are updated as:

$$V_i^{k+1} = V_i^k + a_1 * random * \left(P_{best_i} - X_i^k\right) + a_2 * random * \left(G_{best} - X_i^k\right),$$
$$X_i^{k+1} = X_i^k + V_i^{k+1},$$

where, V_i^k - is velocity of particle i at k^{th} iteration,
a_1, a_2 – are real parameters,
$random$– is a random number, whose value lies between 0 & 1,
X_i^k = is i^{th}- particle position at k^{th}-iteration,
P_{best_i} – Personal best position of particle i,
G_{best} – Global best position of whole search space.

PSO, a computational technique, iteratively optimizes a problem to minimize errors. This statistical method is employed to determine parameter values effectively. In the pursuit of the best solution to an optimization problem, particles engage in communication, exchange knowledge, and adhere to a straightforward rule. The PSO method offers innovation in determining the optimal shape parameter value of RBF through the analysis of non-linear partial differential equations. This strategy is a global search optimization technique that presents numerous features within the parameter space.

2.3 Algorithm of PSO Technique

The collective activities of birds while searching for food served as the inspiration for the development of the PSO algorithm (Abed and Aladool, 2022). In this technique, particles are considered entities, and their location affects how they behave. There is a requirement of optimization of component of the solution at every location. The search process is driven by the updating of particle positions and velocities at each time step. There is a location for each particle in the swarm that can be resolved in one-dimensional space. Each particle moves according to its best-known locations both locally and across the search space, which are updated when new locations are discovered by other particles. With the use of a simple mathematical formula, the updating guidelines for each particle's location and speed are provided by

$$v_{id}^{t+1} = \chi[v_{id}^{t} + c_1 r_1(p_{id}^{t} - x_{id}^{t}) + c_2 r_2(p_{gd}^{t} - x_{id}^{t})],$$

$$x_{id}^{t+1} = x_{id}^{t} + v_{id}^{t+1},$$

where, x_{id}^{t} , represents c position of i^{th} particle and v_{id}^{t} represents i^{th} particle's velocity in d dimension at time step t, p_{gd} represents the particle having the best fitness value, p_{id} is the particle's best position visited so far, c_1, c_2 represents coefficients of acceleration where c_1 quantifies particle personal experience and c_2 quantifies global experience, χ whose value is represented in the range [0,1] and which is called constriction coefficient and is given by

$$\chi = \frac{2\kappa}{\left|2 - \phi - \sqrt{\phi(\phi - 4)}\right|},$$

With $\phi = \phi_1 + \phi_2, \phi_1 = c_1 r_1, \phi_2 = c_2 r_2$ and $\kappa \approx 1$.

2.4 PSO Algorithm

Input fitness function, lb, ub, Np, T, a_1, a_2
(i) Initialize random population (P) and velocity U_i of particle *i*
(ii) Evaluate objective function value (f)
(iii) Assign P_{best_i} as P and f_{best} as *f*
(iv) Identify the solution with best fitness & assign the solution as G_{best} and fitness as f_{best}
For t=1 to T
For *i*=1 to Np
Determine the velocity U_i
Determine the new position (X_i)
Bound X_i
Find objective function value
Update the population by including X_i & f_i
Update P_{best} and f_{pbest}
Update G_{best} and f_{gbest}
End
End.

The result of using the strategy is a solution that is accurate since it finds out for the no. of repetitions with the no. of iterations across a set population size. Using radial basis functions, this strategy has been effectively used to calculate equation solutions (Koupaei et al., 2018). The parameters taken here are as: size of swarm: 20; no. of maximum iterations: 50; inertia weight is decreased linearly with size $c_1 = c_2 = 2.05$.

2.5 Applications of PSO Technique

A number of papers have been published on applications of PSO. One of the prospective global optimization methods is PSO that has been extensively applied to solve issues in numerous fields such as health care, environmental, industrial, commercial, smart city (Gad, 2022; Zhang et al., 2015), geotechnical engineering (Kalatehjari et al., 2012), civil engineering (Pham et al., 2020), electromagnetics (Robinson and Rahmat-Samii, 2004), and wireless networks (Kulkarni and Venayagamoorthy, 2011) etc.

3. Implementation

To solve the non-linear Fisher's equation

$$u_t = \nu u_{xx} + \rho f(u),$$

an ODE that can be evaluated by any suitable numerical technique is produced by replacing the approximations of the space derivatives obtained by the DQM with exponential B-spline basis functions. Once the solutions to the problem with the known initial condition have been observed, the PSO method is used to minimise the obtained error by comparing exact and numerical answers. Numerical results can be discussed on a predefined domain and time intervals, once the parameter value has been calculated with the least amount of error. The numerical technique can be summarized and visualised in the following way which is explained through the **Figure 1**.

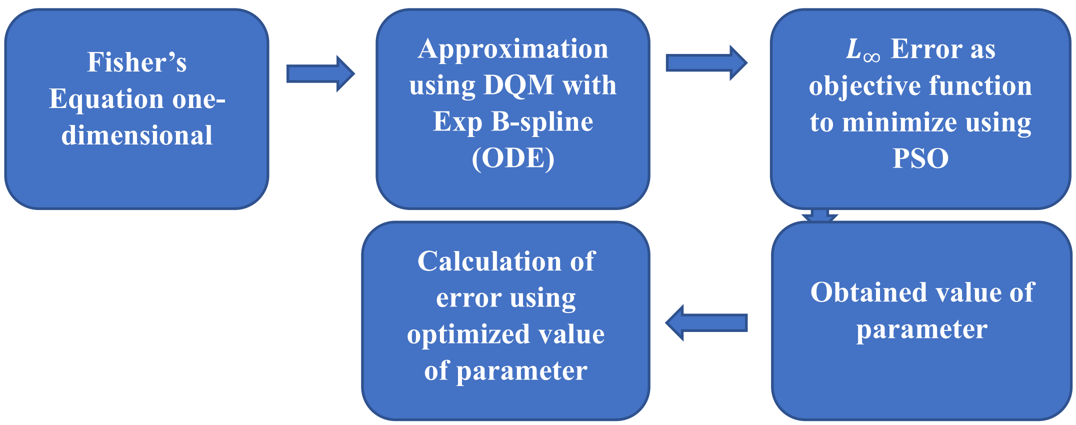

Figure 1. A numerical approach for an approximate solution to Fisher's equation.

4. Numerical Experiments and Discussions

In this section, the numerical solution of the Fisher's equation is considered taking the different set of parameters in form of two numerical examples by the developed hybrid method. The accuracy and the efficiency of the methods are compared by evaluating the error norms.

Test problem 1: The Fisher's Equation (1) was implemented and presented by Gazdag and Canosa (1974) in the study of stability of a moving mesh system taking a large value of non-linear reactive term as compared to the diffusion term with the reaction rate coefficient greater than equal to one. The analytical solution for the above said parametric equation was obtained by Ablowitz and Zepetella (1979) as follows:

$$u(x,t) = \frac{1}{\left[1 + \exp\left(\sqrt{\frac{\rho}{6}}x - \frac{5\rho}{6t}\right)\right]^2}.$$

Solution of the Fisher's equation is obtained considering a finite domain as $[x_L, x_R]$ = [-0.2, 0.8] with nonlocal boundary conditions. The obtained numerical solutions have been presented in form of figures and tables with the comparison of solutions obtained by differential quadrature and the collocation approach.

Here, N are grid points and p is unknown parameter. For different values of N and for different values of p which varies between a range [1, 200]. While changing the value of N and p the value of absolute error changes at different values of t.

Table 2. L_∞ errors obtained by the hybrid scheme at different time levels for $\rho = 2000$ for $\Delta t = 1 \times 10^{-5}$.

N	t = 0.0010			t = 0.0015		
	L_∞	p	L_∞ [40]	L_∞	p	L_∞ [40]
11	3.7880E-06	1	1.2021E-04	1.9631E-06	1	8.8798E-04
21	6.2634E-06	184.3448	6.1551E-06	1.8821E-06	64.9465	3.0570E-06
31	1.6386E-05	200	1.5327E-05	3.9521E-06	200	3.9296E-06

Table 3. L_∞ errors obtained by the hybrid scheme at different time levels for $\rho = 2000$ for $\Delta t = 1 \times 10^{-5}$.

N	t = 0.0020			t = 0.0025		
	L_∞	p	L_∞ [40]	L_∞	p	L_∞ [40]
11	3.2457E-06	12.1442	2.4096E-03	3.0905E-06	16.2559	5.7924E-03
21	5.9916E-07	27.4870	6.5669E-06	3.6021E-07	47.3644	6.2816E-06
31	8.9303E-07	190.5990	8.5750E-07	4.4764E-07	127.5467	3.5690E-07

At different intervals, the exact and approximate solution shown in **Figure 2**. To compare the findings with the exact answer, the results are displayed graphically. The specific approach to the problem is found to be in good accord with the obtained results. To assess whether the numerical solution is accurate for $\rho = 2000$ and $\Delta t = 1 \times 10^{-5}$, results are depicted by L_∞ errors and are presented in **Table 2**, **Table 3** and comparison is done with the results existing in the literature.

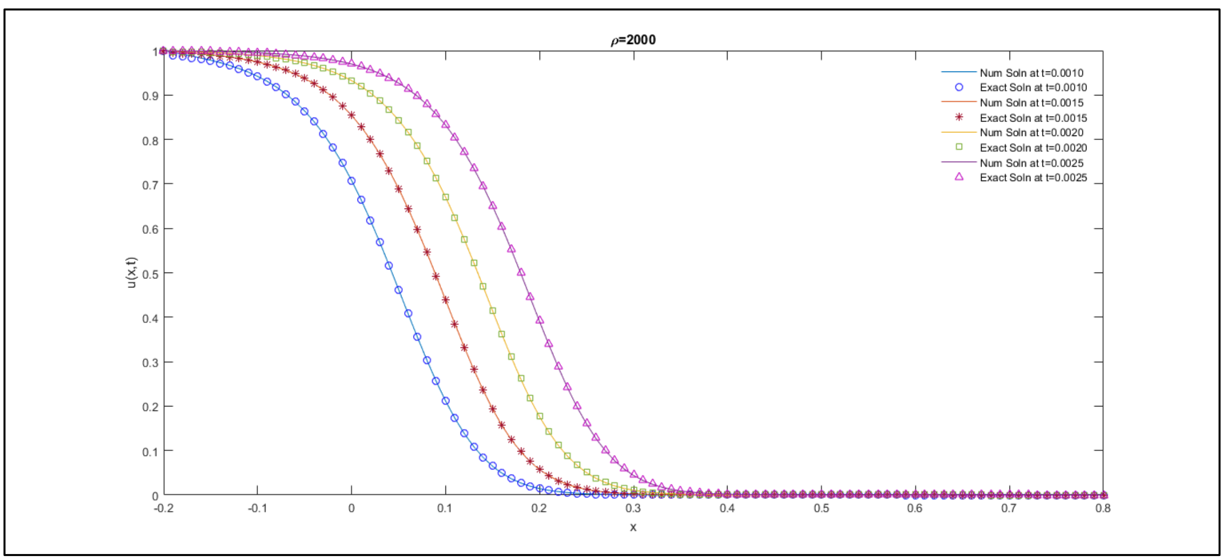

Figure 2. A graphical depiction of x versus time dependent profile of test problem no. 1 for $\rho = 2000$ for $\Delta t = 1 \times 10^{-5}$.

Table 4. L_∞ errors obtained by the hybrid scheme at different time levels for $\rho = 5{,}000$ for $\Delta t = 1 \times 10^{-5}$.

N	*t* = 0.0010			*t* = 0.0015		
	L_∞	p	L_∞ [40]	L_∞	p	L_∞ [40]
31	1.8794E-06	127.4740	7.1895E-06	8.2982E-07	101.02222	1.9510E-04
41	5.5736E-06	200	7.1122E-07	1.8174E-06	200	1.8788E-06
61	1.7687E-06	200	1.4733E-07	6.2910E-05	200	2.4601E-07
81	2.2558E-05	200	1.1419E-07	9.4953E-05	200	2.7305E-07

Table 5. L_∞ errors obtained by the hybrid scheme at different time levels for $\rho = 2000$ for $\Delta t = 1 \times 10^{-5}$.

N	*t* = 0.0020			*t* = 0.0025		
	L_∞	p	L_∞ [40]	L_∞	p	L_∞ [40]
31	5.2087E-07	94.817065	6.9982E-03	3.1489E-07	92.4473	1.9982E-02
41	6.3480E-07	182.46816	3.8613E-05	4.9421E-07	171.7256	9.6423E-04
61	1.5366E-04	200	1.2951E-07	3.0157E-04	200	1.1264E-06
81	2.5400E-04	200	3.8991E-07	5.1449E-04	200	5.2300E-07

At different intervals, the exact and numerical solution shown in **Figure 3**. In comparison to the exact answer, the results are displayed graphically. The specific approach to the problem is found to be in good accord with the obtained results. To assess whether the numerical solution is accurate for $\rho = 5000$ *and* $\Delta t = 1 \times 10^{-5}$, results are depicted by L_∞ errors and are presented in **Table 4**, **Table 5** and comparison is done with the results existing in the literature.

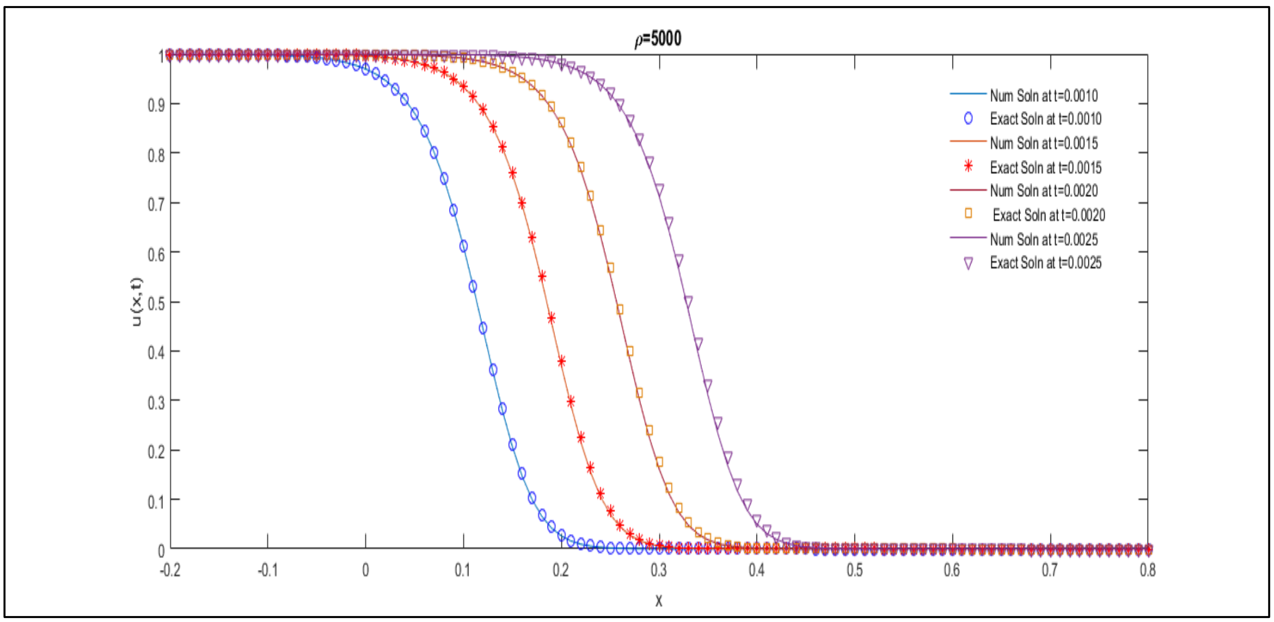

Figure 3. A graphical depiction of *x* versus time dependent profile of Test Problem no. 1 for $\rho = 5000$ for $\Delta t = 1 \times 10^{-5}$.

Similarly at different time levels, **Figure 4** depicts the exact and numerical solution. To compare the exact solution with numerical solution for $\rho = 10{,}000$ *and* $\Delta t = 1 \times 10^{-5}$, results are depicted by L_∞ errors and are presented in **Table 6** and **Table 7**.

Table 6. L_∞ errors obtained by the hybrid scheme at different time levels for $\rho = 10{,}000$ for $\Delta t = 1 \times 10^{-5}$.

N	t = 0.0010		t = 0.0015	
	L_∞	p	L_∞	p
11	0	26.7534	0	28.1728
21	3.2683E-05	59.8813	4.5659E-05	67.7689
31	2.3438E-06	104.4888	3.9461E-06	105.1255
41	1.6684E-06	191.9409	9.7326E-07	179.5408
61	6.1756E-04	200	2.0E-03	200
81	9.7124E-04	200	3.5E-03	200

Table 7. L_∞ errors obtained by the hybrid scheme at different time levels for $\rho = 10{,}000$ for $\Delta t = 1 \times 10^{-5}$.

N	t = 0.0020		t = 0.0025	
	L_∞	p	L_∞	p
11	0	32.908	0	90.4507
21	4.0265E-05	72.1092	3.0399E-05	74.4567
31	4.6042E-06	106.3847	4.7671E-06	107.3924
41	8.3694E-07	175	9.0643E-07	172.7307
61	4.3116E-03	200	7.7256E-03	200
81	8.1045E-03	200	1.4758E-02	200

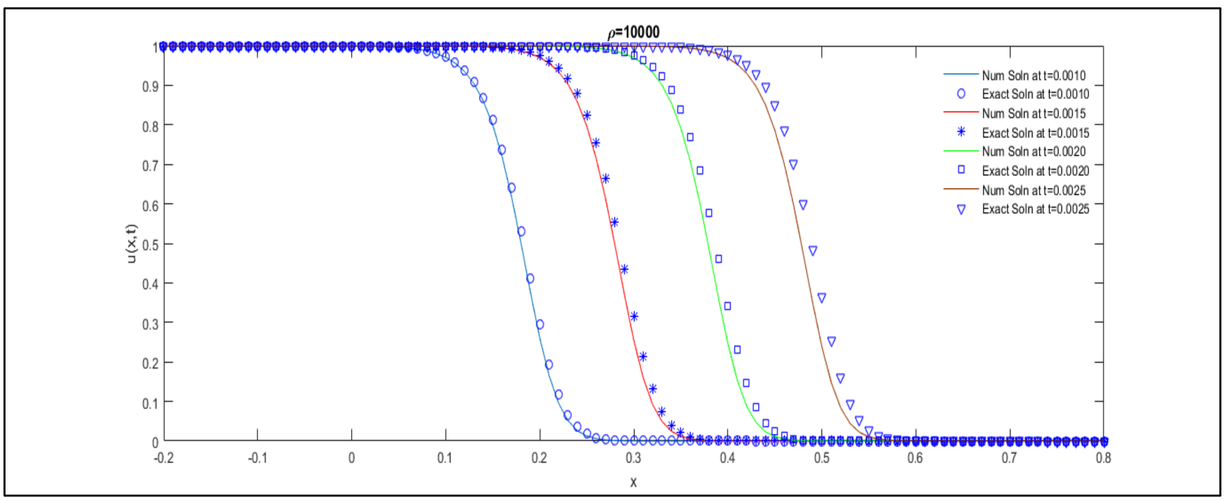

Figure 4. A graphical depiction of x versus time dependent profile of Test Problem no. 1 for $\rho = 10{,}000$ for $\Delta t = 1 \times 10^{-5}$.

Test Problem 2: Now, Solution is obtained by considering the Equation (1) on a finite domain as $[x_L, x_R]$ = [0,1] with nonlocal boundary conditions. Having initial condition as $u(x,0) = \frac{1}{(1+exp(\rho x))^2}$, with boundary conditions $\lim_{x\to-\infty} u(x,t) = 1$, $\lim_{x\to\infty} u(x,t) = 0$ and exact solution is given by:

$$u(x,t) = \frac{1}{\left[1+\exp\left(\sqrt{\frac{\rho}{6}}x - \frac{5\rho t}{6}\right)\right]^2}.$$

Table 8. L_2 and L_∞ errors obtained by the hybrid scheme for the different time levels solutions of test problem no. 2 for $\rho = 6$ with $h = 1 \times 10^{-3}$ and $\Delta t = 1 \times 10^{-6}$ at different levels of time.

Time (*t*)	p	L_2	L_∞	TBC L_2 [51]	TBC L_∞ [51]
0.0001	10	1.5378E-07	1.4354E-13	7.0594E-05	1.2502E-04
0.0002	10	7.7521E-07	4.5552E-12	1.5179E-04	2.5006E-04
0.0003	10	1.8599E-06	2.7389E-11	2.4280E-04	3.7514E-04
0.0004	10	3.3955E-06	9.3024E-11	3.4091E-04	5.0025E-04

Table 9. Value of Absolute error in the PSO-DQM solution of Test Problem no. 2 with $\rho = 6, n = 11$ and $\Delta t = 1 \times 10^{-6}$ at different levels of time.

x	*t* = 0.00010			*t* = 0.0002		
	Numerical value	**Exact value**	**Absolute error**	**Numerical value**	**Exact value**	**Absolute error**
0.1	0.2257628692996	0.2257634281619	3.7886E-7	0.225879623742	0.225881758036	2.1342E-6
0.2	0.2027609522921	0.2027608716127	8.0679E-8	0.202872642577	0.202872349425	2.9315E-7
0.3	0.1812032056058	0.1812032213726	1.5766E-8	0.181307257299	0.181307308866	5.1566E-8
0.4	0.1611480352987	0.1611480329367	2.3620E-9	0.161244518459	0.161244510103	8.3563E-9
0.5	0.1426256974841	0.1426256992945	1.8104E-9	0.142714475916	0.142714480472	4.5555E-9
0.6	0.1256405426468	0.1256405405606	2.0861E-9	0.125721671388	0.125721665786	5.6021E-9
0.7	0.1101729266816	0.1101729400578	1.3376E-8	0.110246528176	0.110246562534	3.4358E-8
0.8	0.0961822267045	0.0961821575567	6.9147E-8	0.096248719771	0.096248528247	1.9152E-7
0.9	0.0836093218458	0.0836096068232	2.8497E-7	0.083667905038	0.083669057269	1.1522E-6

Table 10. Value of Absolute error in the PSO-DQM solution of Test Problem no. 2 with $\rho = 6, n = 11$ and $\Delta t = 1 \times 10^{-6}$ at different levels of time.

x	*t* = 0.0003			*t* = 0.0004		
	Numerical value	**Exact value**	**Absolute error**	**Numerical value**	**Exact value**	**Absolute error**
0.1	0.2259950683966	0.2260003019396	5.2335E-6	0.2261092349428	0.2261188798587	9.6449E-6
0.2	0.2029844770615	0.2039838634058	6.1365E-7	0.2030964331616	0.2030954135439	1.0196E-6
0.3	0.1814113315203	0.1814114339737	1.0245E-7	0.1815154326416	0.1815155966887	1.6404E-7
0.4	0.1613410425742	0.1613410256438	1.6930E-8	0.1614376067503	0.1614375795559	2.7194E-8
0.5	0.1428032922044	0.1428033001299	7.9225E-9	0.1428921466008	0.1428921582693	1.1668E-8
0.6	0.1258028390791	0.1258028289999	1.0079E-8	0.1258840453382	0.1258840302039	1.5134E-8
0.7	0.1103201611064	0.1103202219756	6.0869E-8	0.1103938272620	0.1103939183868	9.1124E-8
0.8	0.0963152913145	0.0963149344314	3.5688E-7	0.0963819319448	0.0963813761180	5.5582E-7
0.9	0.0837259563702	0.0837285413805	2.5850E-6	0.0837834888559	0.0837880591671	4.5703E-6

At different intervals, the exact and numerical solution can be shown as in **Figure 5**. In comparison to the exact answer, the results are displayed graphically. The specific approach to the problem is found to be in good accord with the obtained results to assess whether the numerical solution is accurate for $\rho = 6, n = 11$ and $\Delta t = 1 \times 10^{-6}$, results are represented in **Table 9** and **Table 10** and in **Table 8**, the L_2 and L_∞ errors are shown.

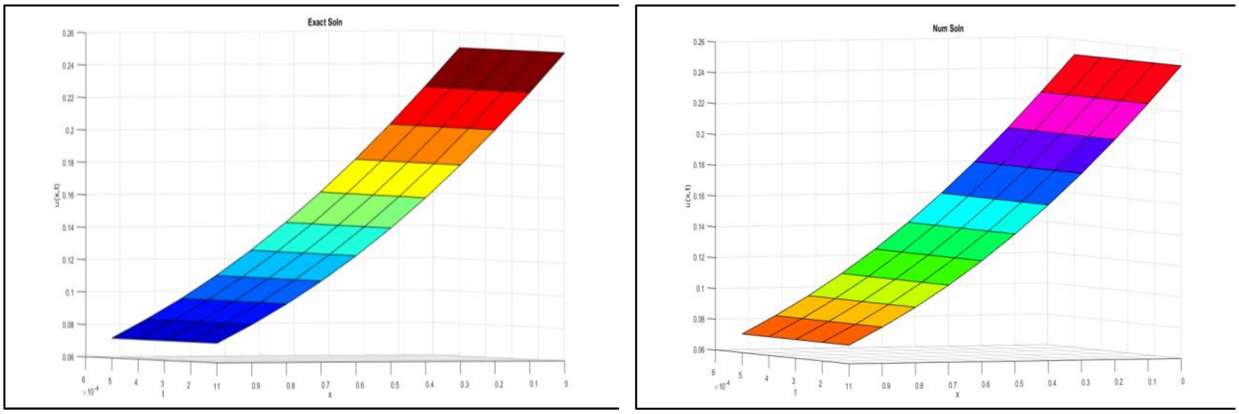

Figure 5. Solution of traveling wave for test problem no. 2 for $\rho = 6$ and $\Delta t = 1 \times 10^{-6}$.

5. Conclusions

To examine the nonlinear Fisher's equation, the exponential B-spline basis function and the PSO approach is used to determine the ideal value for the parameter. The advantage of PSO is that it may identify the particular parameter value that helps the error to be minimized for a given function. The PSO based method can evaluate a parameter in a search space that supports global search. By discussing two test problems, it has been observed that the results obtained are compatible. An algorithm is developed on MATLAB 2015(b) and for the different number of iterations as discussed in the above two test problems, and graphical results are obtained. The data obtained by graphs and tables concluded that results are accurate in comparison to other methods. By considering these results, this technique can be used to obtain the solution of fractional differential equations, higher order derivatives and other partial differential equations in both linear and non-linear form with good accuracy.

Conflict of Interest

The authors have no conflict of interest.

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