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Quartic B-Spline Technique for Third-Order Linear Singularly Perturbed Boundary Value Problem with Discontinuous Source Term

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Abstract
In this paper, we developed an effective computational technique for addressing third-order linear singularly perturbed problems having the source term discontinuous. Boundary or interior layers are frequently present in singular perturbation issues, making traditional numerical techniques more challenging. Here, we present a quartic B-spline method (QBSM) for the approximate solution of the third-order singularly perturbed boundary value problem, improving both the accuracy and efficiency of the solutions. In addition, the proposed method's convergence and error are investigated. The performance of the current technique is demonstrated through numerous numerical tests. The numerical findings are compared to other approaches reported in the literature.

Keywords- Singularly perturbed problem, Reaction-diffusion type, Quartic B-spline method, Discontinuous source term, Truncation error.

1. Introduction

Differential equations are the foundation of mathematical modelling in science and engineering, reflecting physical phenomena, including fluid movement, heat transfer, wave propagation, and population dynamics. Singularly perturbed differential equations (SPDEs) are incredibly complex due to a small parameter that multiplies the highest-order derivative. This small parameter produces solutions with sharp transitions, such as boundary or interior layers, requiring specialized approaches for accurate solutions. These problems occur in chemical kinetics, reaction-diffusion processes, aerodynamics, heat conduction, fluid mechanics, elasticity, and other branches of applied mathematics. Reaction-diffusion equations are essential in several application fields, including cellular processes, drug release, ecology, disease propagation, industrial catalysis, environmental pollutant transport, and chemistry in interstellar media. A limitation arises when such problems have higher order and incorporate non-smooth data, such as discontinuous source terms (DST) or sudden changes in boundary conditions. Conventional numerical approaches frequently fail to maintain stability and accuracy under these scenarios, especially when efficiently capturing boundary layer behaviour and discontinuities. Scholars have explored analytical and numerical solutions to singular perturbation issues but have found that standard numerical methods may not provide adequate approximate solutions. That is why they have opted for unconventional methods. Over the past few decades, many scholars have explored the numerical solutions to singular perturbation problems one can refer to (Nahfey, 1981; O'Malley, 1991; Roos et al., 1996).

Several numerical methods have been introduced into the literature throughout the last three decades to solve second and higher-order SPBVP with smooth data. Valarmathi and Ramanujam (2002a, 2002b) introduced a numerical method that combines asymptotic expansions and computational techniques to solve singularly perturbed third-order differential equations, with emphasis on boundary layers. Kumar and Tiwari (2012) developed an efficient initial value approach for solving third-order SPBVP by reducing the third-order differential equation into three unperturbed initial value problems, which are further solved by the Runge-kutta fourth-order method. Khan and Khandelwal (2019) proposed a generalized method based on an exponential quartic spline for solving third-order SPBVP and showed the approach to be second-order convergent. The B-spline method is proposed by Kadalbajoo and Kumar (2007) to solve second-order linear perturbed problems having singular coefficients computationally. Pandya and Doctor (2012) applied a fourth-degree spline to solve third-order SPBVP numerically. Caglar and Caglar (2006) applied B-spline to solve homogeneous and non-homogeneous unique case boundary value problems. A QBSM is proposed by Saini and Mishra (2015) for solving self-adjoint third-order perturbed problems. Goh et al. (2012) presented a QBSM to solve second-order SPBVP. Lodhi and Mishra (2018) developed a septic B-spline technique to improve the maximum absolute error for second-order self-adjoint SPBVP. Weili (1990) considered a class of third-order non-linear ordinary differential equations and shows solutions' existence, uniqueness, and asymptotic approximations using the theory of differential inequalities. Thula and Roul (2018) presented a higher B-spline to solve non-linear singular two-point BVP with Neumann and Robin boundary conditions, and they showed that the method was fourth-order convergent. Malge and Lodhi (2024) found an approximate numerical solution of second-order non-linear SPBVPs with smooth data. Further, Du et al. (2005) and Rufia and Ramos (2022) contributed to solving non-linear type problems. Sun (2005) addressed the existence of positive solutions of the non-linear singular third-order SPBVPs. A class of self-adjoint fourth-order SPBVP is numerically solved using high-order septic B-spline by authors Lodhi and Sahu (2020) and attained fourth-order convergence. A review paper on SPBVP covering splines, turning point problems, and interior layers is addressed by Kumar and Gupta (2010) and Sharma and Patidar (2013).

Though there is a vast literature on SPBVP with smooth data but, researchers also contributed to SPBVP of the non-smooth type data. A computational method is proposed by Babu and Ramanujam (2007) to obtain the approximate numerical solution of third and fourth-order SPBVP with DST. Recently, El- Zahar et al. (2024) proposed a reliable approach for calculating analytical solutions of higher-order SPBVPs having non-smooth data. Valanarasu and Ramanujam (2007b) have developed an asymptotic numerical approach for third-order SPBVPs with a DST where BVP is reduced to a weakly coupled system made up of a first-order ordinary differential equation with a suitable initial condition and a second-order singularly perturbed ODE with boundary conditions. To address this problem, a computational approach consisting of asymptotic expansion and finite difference method has been suggested. Valanarasu and Ramanujam (2007a) and Shanthi and Ramanujam (2008) also addressed the third-order convection-diffusion kind SPBVP with DST using an asymptotic approach. Babu and Ramanujam (2008) addressed the third-order SPBVP with non-smooth data using a finite element approach. Mane and Lodhi (2024) developed the nonpolynomial spline method to solve second-order SPBVP with discontinuity term. Recently, many authors have suggested numerical methods for second and higher-order delay differential equations with DST, to cite a few: Ayele et al. (2022), Daba and Duressa (2022), Rajendran et al. (2025), Subburayan and Mahendran (2020). A fourth-order SPBVP with a discontinuous source term is numerically solved by Mane and Lodhi (2024) using a novel quintic B-spline approach.

Because of excellent precision in capturing solution profiles, B-spline techniques have helped deal with such SPBVPs. However, research on their application to solving third-order SPBVPs is restricted, mainly when dealing with non-smooth data. Hence, the main objective of this research is to build a new computational approach for the numerical solution of the third-order SPBVP without diminishing the order of the differential equation. This approach discretizes the third-order SPBVP using the quartic B-spline collocation method. This approach can handle boundary layers, interior layers, discontinuities, and steep gradients, which makes it more desirable to conventional numerical approaches like finite differences and lower-order splines.

This study emphasizes the application of the quartic B-spline method to approximate the solutions to third-order SPBVP with non-smooth data. The key benefit of this technique over other approaches is that it is stable, efficient, and capable of finding the solution at any mesh point in the interval. Inspired by earlier research (Babu and Ramanujam, 2007; Howes, 1983; Valanarasu and Ramanujam, 2007b), we look into the following SPBVP of third-order reaction-diffusion differential equation with DST on the interval $\mho = (0,1)$.

We introduce the notations for the various intervals as: $\mho^- = (0, z)$, $\mho^+ = (z, 1)$ and $\overline{\mho} = [0,1]$.

To find $p \in C^1(\overline{\mho}) \cap C^2(\mho) \cap C^3(\mho^- \cup \mho^+)$ such that,

$$-\varepsilon p'''(s) + b(s)p'(s) + c(s)p(s) = v(s), s \in \mho^- \cup \mho^+ \tag{1}$$

Subject to boundary conditions:

$$p(0) = \mu_1, p'(0) = \mu_2, p'(1) = \mu_3 \tag{2}$$

Where $b(s)$ and $c(s)$ are continuous functions on $\overline{\mho}$ which satisfies the following conditions:

$$b(s) \geq \sigma_1 > 0 \tag{3}$$

$$0 \geq c(s) \geq -\sigma_2, \sigma_2 > 0 \tag{4}$$

$$\sigma_1 - \theta\sigma_2 \geq \sigma_3 > 0, \text{ for some } \theta > 2 \text{ arbitrarily close to 2, for some } \sigma_3 \tag{5}$$

Further, $v(s)$ is sufficiently smooth on $\mho \backslash \{z\}$, where, $z \in \mho$ is a point of discontinuity and ε is a small positive parameter called the perturbation parameter. When the source term has a sudden change, the solution's first derivative also experiences a sharp transition, forming a narrow region where the solution shifts more abruptly. Because $v(s)$ is discontinuous at the point w, the solution $p(s)$ of Equations (1)-(2) does not necessarily have a continuous third derivative at the point w; that is, $p(s)$ does not belong to the class of $C^3(\mho)$ functions. Here, we represent a jump in any function at the point with $\mathrm{jp}[\mathrm{z}] = \mathrm{jp}(\mathrm{z}, +) - \mathrm{jp}(\mathrm{z}, -)$. Also, here, boundary condition (3) represents that problem (1)-(2) is a non-turning point problem, and boundary condition (4) represents that problem (1)-(2) is a quasi-monotone. A specific actual application of third-order singularly perturbed boundary value problems with DST is in robust control designs such as robotics, aerospace, and power systems. A robotic manipulator with articulated joints experiences rapid (flexible oscillations) and gradual (arm movement) dynamics. The dynamics are governed by a third-order differential equation with a discontinuous source term due to abrupt external disturbances or control input changes. The solution variable represents the displacement or velocity of the robotic manipulator, and a small positive parameter indicates the time scale separation. The boundary condition represents the physical constraints of the system.

The research study is structured as follows: In Section 2, QBSM is clearly laid out as a method for finding an approximate numerical solution. Section 3 deals with mesh selection, which is followed by the computation of the truncation error at nodal points in Section 4. Section 5 provides numerical examples to demonstrate accuracy, whereas Section 6 explores the conclusion.

2. Quartic B-Spline Methodology

This section explains the quartic B-spline method to solve third-order SPBVPs with discontinuous source terms. Let $\Lambda = \{0 = s_0 < s_1 < s_2 \ldots < s_{N-1} < s_N = 1\}$ be the partition of the interval $[0,1]$. Let $\bar{h}$ is piecewise uniform spacing described in the next section. Consider X a linear subspace of $L_2\,[0,1]$, a vector space of all square-integrable functions on $\overline{\mho} = [0\ ,1]$.Then, the basis functions of the quartic B-spline are given below:

$$B_{4,i}(s) = \frac{1}{24\bar{h}^4}\begin{cases}(s-s_{i-2})^4, & \text{if } s \in [s_{i-2}, s_{i-1}] \\ \bar{h}^4 + 4\bar{h}^3(s-s_{i-1}) + 6\bar{h}^2(s-s_{i-1})^2 + 4\bar{h}(s-s_{i-1})^3 - 4(s-s_{i-1})^4, & \text{if } s \in [s_{i-1}, s_i] \\ 11\bar{h}^4 + 12\bar{h}^3(s-s_i) - 6\bar{h}^2(s-s_i)^2 - 12\bar{h}(s-s_i)^3 + 6(s-s_i)^4, & \text{if } s \in [s_i, s_{i+1}] \\ \bar{h}^4 + 4\bar{h}^3(s_{i+2}-s) + 6\bar{h}^2(s_{i+2}-s)^2 + 4\bar{h}(s_{i+2}-s)^3 - 4(s_{i+2}-s)^4, & \text{if } s \in [s_{i+1}, s_{i+2}] \\ (s_{i+3}-s)^4, & \text{if } s \in [s_{i+2}, s_{i+3}] \\ 0, & \text{otherwise}\end{cases} \tag{6}$$

We add five more mesh points as $s_{-3} < s_{-2} < s_{-1} < s_0$ and $s_{N+2} > s_{N+1} > s_N$. Each quartic B-spline covers five elements. **Table 1** presents the spline value and their derivatives at the nodal points. Also, the basis function in Equation (6) is three times continuously differentiable functions on the whole real line.

Let $\phi = \{B_{4,-2}, B_{4,-1}, B_{4,0}, B_{4,1}, \ldots, B_{4,N}, B_{4,N+1}\}$ and span $\phi = K_4(\mho)$. Also, ϕ is linearly independent on $[0,1]$ and the dimension of $K_4(\mho) = N + 4$. Let $m(s)$ be the approximate solution of Equation (1)-(2), which is defined as:

$$m(s) = \sum_{i=-2}^{N+1} k_i\, B_{4,i}(s) \tag{7}$$

where, k_i's are unidentified constants, and $B_{4,i}$'s are fourth-degree spline functions. To obtain the approximate solution of BVP (1)-(2), we evaluate the spline functions at mesh points $s = s_i$, (i = 0, 1, 2…, N) by using **Table 1**.

Table 1. Values of $B_{4,i}$,$B'_{4,i}$, $B''_{4,i}$ and $B'''_{4,i}$ at nodal points.

$B_4(s)$	s_{i-2}	s_{i-1}	s_i	s_{i+1}	s_{i+2}	s_{i+3}
$B_{4,i}(s)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$B'_{4,i}(s)$	0	$\frac{1}{6\bar{h}}$	$\frac{3}{6\bar{h}}$	$\frac{-3}{6\bar{h}}$	$\frac{-1}{6\bar{h}}$	0
$B''_{4,i}(s)$	0	$\frac{1}{2\bar{h}^2}$	$\frac{-1}{2\bar{h}^2}$	$\frac{-1}{2\bar{h}^2}$	$\frac{1}{2\bar{h}^2}$	0
$B'''_{4,i}(s)$	0	$\frac{1}{\bar{h}^3}$	$\frac{-3}{\bar{h}^3}$	$\frac{3}{\bar{h}^3}$	$\frac{-1}{\bar{h}^3}$	0

Hence, we get the following relations:

$$m(s_i) = \sum_{i=-2}^{N+1} k_i\, B_{4,i}(s_i) = \frac{1}{24}(k_{i-2} + 11k_{i-1} + 11k_i + k_{i+1}) \tag{8}$$

$$m'(s_i) = \sum_{i=-2}^{N+1} k_i\, B'_{4,i}(s_i) = \frac{1}{6\bar{h}}(-k_{i-2} - 3k_{i-1} + 3k_i + k_{i+1}) \tag{9}$$

$$m''(s_i) = \sum_{i=-2}^{N+1} k_i\, B''_{4,i}(s_i) = \frac{1}{2\bar{h}^2}(k_{i-2} - k_{i-1} - k_i + k_{i+1}) \tag{10}$$

$$m'''(s_i) = \sum_{i=-2}^{N+1} k_i B'''_{4,i}(s_i) = \frac{1}{h^3}(-k_{i-2} + 3k_{i-1} - 3k_i + k_{i+1}) \tag{11}$$

As $m(s)$ is the approximate solution to the boundary value problem, hence $m(s)$ will satisfy the Equations (1)-(2); therefore, we have,

$$-\varepsilon m'''(s) + b(s)m'(s) + c(s)m(s) = v(s), s \in [0,1], \tag{12}$$

with boundary conditions,

$$m(0) = \mu_1, m'(0) = \mu_2, m'(1) = \mu_3 \tag{13}$$

Discretizing Equation (12) at the nodal points, we obtain

$$-\varepsilon m'''(s_i) + b_i m'(s_i) + c_i m(s_i) = v_i \tag{14}$$

where $b_i = b\,(s_i)$, $c_i = c\,(s_i)$, and $v_i = v\,(s_i)$.

Using Equations (8)-(11) in (14) and simplifying it we obtain

$$\left(24\varepsilon - 4b_i\bar{h}^2 + c_i\bar{h}^3\right)k_{i-2} + \left(-72\varepsilon - 12b_i\bar{h}^2 + 11c_i\bar{h}^3\right)k_{i-1} + \left(72\varepsilon + 12b_i\bar{h}^2 + 11c_i\bar{h}^3\right)k_i + \left(-24\varepsilon + 4b_i\bar{h}^2 + c_i\bar{h}^3\right)k_{i+1} = 24\bar{h}^3 v_i \tag{15}$$

for $\left\{0 \le i < \frac{N}{2}\right\} \cup \left\{\frac{N}{2} + 1 \le i \le N\right\}$

$$k_{-2} + 11k_{-1} + 11k_0 + k_1 = 24\mu_1 \tag{16}$$

$$-k_{-2} - 3k_{-1} + 3k_0 + k_1 = 6\bar{h}\mu_2 \tag{17}$$

$$-k_{N-2} - 3k_{N-1} + 3k_N + k_{N+1} = 6\bar{h}\mu_3 \tag{18}$$

Correspondingly, at the discontinuity point $s_{\frac{N}{2}} = z$, we shall use a four-point formula for the second derivative approximation of the difference operator, which is given as:

$$L^N m_{\frac{N}{2}} \equiv \frac{-m_{\frac{N}{2}+3} + 4m_{\frac{N}{2}+2} - 5m_{\frac{N}{2}+1} + 2m_{\frac{N}{2}}}{\bar{h}^2} - \frac{-m_{\frac{N}{2}-3} + 4m_{\frac{N}{2}-2} - 5m_{\frac{N}{2}-1} + 2m_{\frac{N}{2}}}{\bar{h}^2} = 0 \tag{19}$$

From Equation (15)-(19), we obtain an ($N+4$) linear equation in ($N+4$) unknowns whose matrix representation is given as follows:

$$CZ = A \tag{20}$$

where, $Z = [k_{-2}, k_{-1}\ldots, k_{-N}, k_{N+1}]^T$ is a column matrix of unknown constants, the right-hand side matrix is given as

$A = \left[24p_1, 6\bar{h}p_2, 24\bar{h}^3 v_0, 24\bar{h}^3 v_1, \ldots, 24\bar{h}^3 v_N, 6\bar{h}p_3\right]^T$ and the coefficient matrix C is given below:

$$C = \begin{bmatrix} 1 & 11 & 11 & 1 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ -1 & -3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ P_0 & Q_0 & R_0 & S_0 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & P_1 & Q_1 & R_1 & S_1 & 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & P_2 & Q_2 & R_2 & S_2 & 0 & 0 & \cdots & & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & & \cdots & & & & P_N & Q_N & R_N & S_N \\ 0 & 0 & 0 & 0 & & \cdots & & & & -1 & -3 & 3 & 1 \end{bmatrix} \tag{21}$$

where,

$P_i = \left(24\varepsilon - 4b_i\bar{h}^{-2} + c_i\bar{h}^{-3}\right)$, $Q_i = \left(-72\varepsilon - 12b_i\bar{h}^{-2} + 11c_i\bar{h}^{-3}\right)$, $R_i = \left(72\varepsilon + 12b_i\bar{h}^{-2} + 11c_i\bar{h}^{-3}\right)$ and

$S_i = \left(-24\varepsilon + 4b_i\bar{h}^{-2} + c_i\bar{h}^{-3}\right)$.

Since matrix C is invertible, we can obtain the values of spline coefficients. Hence, Equation (7) gives a unique numerical solution to BVP (1)-(2).

3. Mesh Selection Approach

This section contributes to the mesh selection technique to generate more points in the region of the boundary layer. On the interval $\mho^- \cup \mho^+$, the following is the construction of a piecewise uniform mesh of N mesh interval.

The domain $\mho^-$ is divided into three subdomains
$[0, \tau_1], [\tau_1, z - \tau_1]$ and $[z - \tau_1, z]$,

Similarly, the domain $\mho^+$ is divided into three subdomains given as:
$[z, z + \tau_2], [z + \tau_2, 1 - \tau_2]$ and $[1 - \tau_2, 1]$,

where, $0 < \tau_1 \leq \frac{z}{4}$ and $0 < \tau_2 \leq \frac{1-z}{4}$ for some τ_1 and τ_2. Here τ_1 and τ_2 are the transition parameters which is chosen for a singularly perturbed problem given as follows:
$\tau_1 = min\left\{\frac{z}{4}, 2\sqrt{\frac{\varepsilon}{\sigma_1}}\ln N\right\}$ and $\tau_2 = min\left\{\frac{1-z}{4}, 2\sqrt{\frac{\varepsilon}{\sigma_1}}\ln N\right\}$.

On $[0, \tau_1], [z - \tau_1, z], [z, z + \tau_2]$ and $[1 - \tau_2, 1]$ we place uniform mesh with $\frac{N}{8}$ mesh intervals while on $[\tau_1, z - \tau_1]$ and $[z + \tau_2, 1 - \tau_2]$ we place a uniform mesh with $\frac{N}{4}$ mesh intervals.

For mesh sizes, we make use of the following notation:

$$\bar{h} = \begin{cases} h_1 = \dfrac{8\tau_1}{N}, & \text{for } i = 1,2 \dots, \dfrac{N}{8}, i = \dfrac{3N}{8}+1, \dots, \dfrac{N}{2}, \\ h_2 = \dfrac{4(d-2\tau_1)}{N}, & \text{for } i = \dfrac{N}{8}+1, \dots, \dfrac{3N}{8}, \\ h_3 = \dfrac{8\tau_2}{N}, & \text{for } i = \dfrac{N}{2}+1, \dots, \dfrac{5N}{8}, i = \dfrac{7N}{8}+1, \dots, N+1, \\ h_4 = \dfrac{4(1-d-2\tau_2)}{N}, & \text{for } i = \dfrac{5N}{8}+1, \dots, \dfrac{7N}{8}. \end{cases}$$

4. Error Analysis

This section outlines a technique for determining error bounds on the solution and its derivatives at nodal points inside the interval $\mho = (0,1)$.
Let $h = max\{h_1, h_2, h_3, h_4\}$.

Using the quartic B-spline basis functions from Equation (6) and also using Equations (8)-(11), the following relationship can be obtained:

$$h[m'(s_{i+1}) + 11m'(s_i) + 11m'(s_{i-1}) + m'(s_{i-2})] = 4[m(s_{i+1}) + 3m(s_i) - 3m(s_{i-1}) - m(s_{i-2})] \tag{22}$$

$$h^2\ m''(s_i) = 2[m(s_{i+1}) - 2m(s_i) + m(s_{i-1})] - \frac{h}{2}[m'(s_{i+1}) - m'(s_{i-1})] \tag{23}$$

$$h^3\ m'''(s_i) = 12[m(s_i) - m(s_{i-1})] - 3h[m'(s_{i+1}) + 6m'(s_i) + m'(s_{i-1})] \tag{24}$$

$$h^4\ m^{iv}(s_{i+}) = 24[m(s_{i-1}) + m(s_i) - 2m(s_{i+1})] + 6h[m'(s_{i-1}) + 8m'(s_i) + 3m'(s_{i+1})] \tag{25}$$

$$h^4\ m^{iv}(s_{i-}) = 24[m(s_{i+1}) + m(s_i) - 2m(s_{i-1})] - 6h[m'(s_{i+1}) + 8m'(s_i) + 3m'(s_{i-1})] \tag{26}$$

where, $m^{iv}(s_{i+})$ denotes the value of $m^{iv}(s_i)$ in the domain $[s_i, s_{i+1}]$. By using the operator notation from Kadalbajoo and Kumar (2007), $E(m(s_i)) = m(s_{i+1})$, Equation (22) can be written as:

$$m'(s_i) = \frac{4[E+3-3E^{-1}-E^{-2}]}{h\ [E+11+11E^{-1}+E^{-2}]} p(s_i) \tag{27}$$

Using the value of $E = e^{hD}$, where $D = \frac{d}{ds}$ is an operator, in Equation (27), we get,

$$m'(s_i) = \frac{4[e^{hD}+3-3e^{-hD}-e^{-2hD}]}{h\ [e^{-2hD}+11e^{-hD}+11+e^{hD}]}\ p(s_i) \tag{28}$$

or

$$24h\left(1 - \frac{1}{2}hD + \frac{1}{3}h^2D^2 - \frac{1}{8}h^3D^3 + \frac{7}{144}h^4D^4 - \cdots\right) m'(s_i) = 24h\left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 - \frac{1}{8}h^3D^4 + \cdots\right) p(s_i) \tag{29}$$

After simplification, we get,

$$m'(s_i) = \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 - \frac{1}{8}h^3D^4 + \cdots\right)\left[1 + \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 - \frac{1}{8}h^3D^3 \dots\right)\right]^{-1} p(s_i).$$

$$= \left(D + \frac{1}{720}h^4D^5 - \frac{1}{2016}h^6D^7 + \frac{1}{17280}h^8D^9 + \cdots\right) p(s_i).$$

$$= \left(D - \frac{1}{2}hD^2 + \frac{1}{3}h^2D^3 - \frac{1}{8}h^3D^4 + \cdots\right)\left[1 - \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 \dots\right) + \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 \dots\right)^2 - \left(-\frac{1}{2}hD + \frac{1}{3}h^2D^2 \dots\right)^3 + \cdots\right] p(s_i).$$

Hence,

$$m'(s_i) = p'(s_i) + \frac{1}{720}h^4 p^{(5)}(s_i) - \frac{1}{2016}h^6 p^{(7)}(s_i) + \frac{1}{17280}h^8 p^{(9)}(s_i) + O(h^{10}) \tag{30}$$

Using the similar approach for the Equation (23)-(26) we can prove the following relations:

$$m''(s_i) = p''(s_i) - \frac{1}{240}h^4 p^{(6)}(s_i) + \frac{1}{6048}h^6 p^{(8)}(s_i) + O(h^8) \tag{31}$$

$$m'''(s_i) = p'''(s_i) - \frac{1}{12}h^2 p^{(5)}(s_i) + \frac{1}{240}h^4 p^{(7)}(s_i) - \frac{1}{3024}h^6 p^{(9)}(s_i) + O(h^8) \tag{32}$$

To find the truncation error (TE) at the nodal point, define $e(s_i) = m(s_i) - p(s_i)$ and substitute Equation (30)-(32) in the Taylor's series expansion (TSE) of $e(s_i + \vartheta h)$, we obtain:

$$e(s_i + \vartheta h) = \frac{-(10\vartheta^2 - 1)\vartheta}{720}h^5 p^5(s_i) + \frac{(5\vartheta^2 - 3)\vartheta^2}{1440}h^6 p^6(s_i) + \frac{(7\vartheta^2 - 5)\vartheta}{10080}h^7 p^7(s_i) + O(h^8),$$

where $0 \le \vartheta \le 1$.

Thus, the QBSM is $O(h^5)$ accurate.

The TE at the discontinuity point $s_{\frac{N}{2}} = z$ is obtained by utilizing the four-point formula for the second derivative approximation of the difference operator given by Equation (16). Hence, the TE is given as:

$$e\left(s_{\frac{N}{2}}\right) = \left|-p\left(s_{\frac{N}{2}} + 3h_1\right) + 4p\left(s_{\frac{N}{2}} + 2h_1\right) - 5p\left(s_{\frac{N}{2}} + h_1\right) + p\left(s_{\frac{N}{2}} - 3h_1\right) - 4p\left(s_{\frac{N}{2}} - 2h_1\right) + 5p\left(s_{\frac{N}{2}} - h_1\right)\right|.$$

We simplify and expand the above equation using TSE, and hence we obtain TE as:

$$e\left(s_{\frac{N}{2}}\right) = 2h_1^5 p^{(5)}\left(s_{\frac{N}{2}}\right) + O(h^6).$$

Hence, the TE at the point of the discontinuity $s_{\frac{N}{2}} = z$ is $O(h^5)$ accurate.

As a result, the TE of the innovative method at the nodal point is $O(h^5)$ precise, and the process of convergence is $O(h^2)$.

The following theorem defines the global error bounds.

Theorem 4.1 Let $m(s)$ be the quartic spline approximation of $p(s) \in C^5[0,1]$ satisfying the boundary conditions, then the global error bound for the quartic spline $m(s)$ and its derivatives $m^i(s), 1 \le i \le 4$ is $\left\|p^i(s) - m^i(s)\right\|_\infty = O\left(h^{5-i}\right)$, for $i = 0(1)4$.

5. Numerical Examples and Discussion

In this section, we numerically solved the two examples using the QBSM. Here, the exact solution to the problems is not available, so we find the maximum absolute error (MAE) E^N in the discrete maximum norm $\| \|_\infty$ using the double mesh principle, which is given as follows:

$E_\varepsilon^N = max|m_\varepsilon^N(s_k) - m_\varepsilon^{2N}(s_k)|$ and $E^N = max\, E_\varepsilon^N$, where $m_\varepsilon^N(s_k)$ and $m_\varepsilon^{2N}(s_k)$ represent the approximate solution obtained using N and $2N$ number of mesh points.

The rate of convergence (ROC) r^N is evaluated using the formula:

$$r^N = log_2 \frac{E^N}{E^{2N}}.$$

The outcomes of the two problems are evaluated with MATLAB (R2018a 9.4.0.813654).

Example 1: Consider three-point SPBVP with DST:

$$-\varepsilon p'''(s) + 4p'(s) - p(s) = \begin{cases} 0.7, & s \leq 0.5 \\ -0.6, & s > 0.5 \end{cases}$$
$$p(0) = 1, p'(0) = 0, p'(1) = 0,$$

Table 2. MAE for the first derivative of Example 1 at $\varepsilon = 10^{-16}$.

N	Quartic B-spline method	Babu and Ramanujam (2007)
16	2.1700E-01	-
32	3.6421E-02	-
64	1.2657E-03	1.6163E-03
128	6.0469E-04	1.0151E-03
256	4.7383E-04	1.0860E-03

For Example 1, MAE is evaluated for the first derivative of the solution for a sufficiently small value of the perturbation parameter $\varepsilon = 10^{-16}$ to demonstrate the singularly perturbed nature using Shishkin mesh. Also, this value of MAE is compared with the existing method given by Babu and Ramanujam (2007) in **Table 2**. Our estimated results are apparently better than in the existing literature Babu and Ramanujam (2007). Also, for Example 1, in **Table 3,** the solution is examined using MAE and ROC by varying the perturbation parameter and the number of mesh points.

However, the ROC of the numerical solution, for Example 1, is almost two. Further, for Example 1, the error graph plot for both the derivative of the solution and numerical solution has been plotted for N = 128 and $\varepsilon = 10^{-16}$, N = 128 and $\varepsilon = 10^{-14}$ in **Figures 1** and **2** respectively.

Table 3. MAE and ROC of the solution in Example 1.

ε	N = 64	N = 128	N = 256	N = 512	N = 1024
10^{-10}	9.0791E-01	1.3481E-01	2.4105E-02	4.2841E-03	1.4041E-03
10^{-12}	6.4785E-02	1.2709E-02	2.8022E-03	9.2504E-04	4.0070E-04
10^{-14}	8.8142E-03	3.1719E-03	1.3280E-03	6.1558E-04	3.0002E-04
10^{-16}	5.0656E-03	2.4122E-03	1.1804E-03	5.8469E-04	2.8994E-04
E^N	9.0791E-01	1.3481E-01	2.4105E-02	4.2841E-03	1.4041E-03
r^N	2.75E+00	2.48E+00	2.49E+00	1.61E+00	

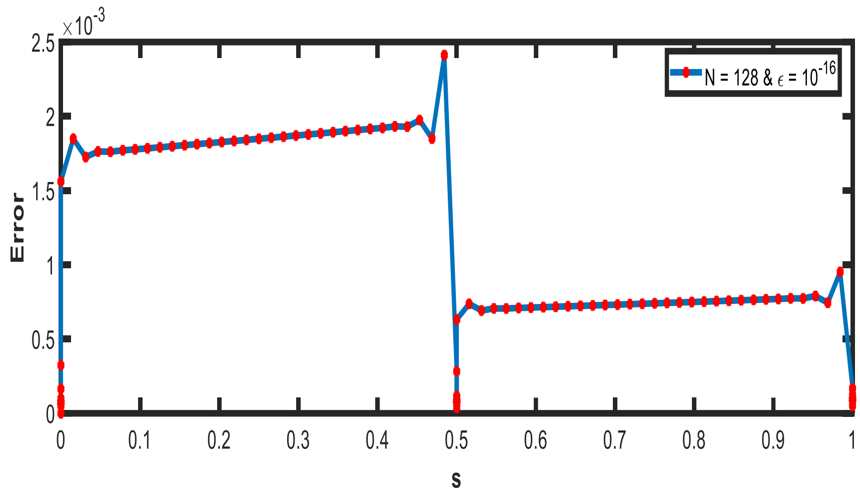

Figure 1. Error plot of Example 1's first derivative solution for $N = 128$ and $\varepsilon = 10^{-16}$.

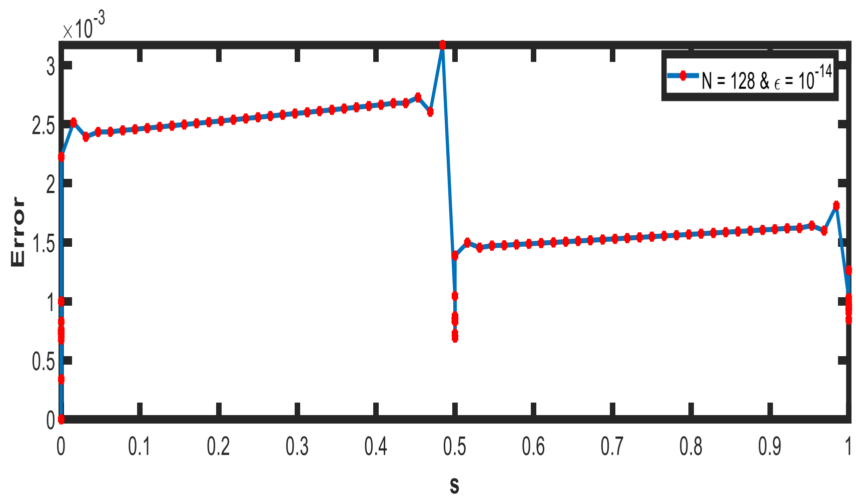

Figure 2. Error plot of Example 1's solution for $N = 128$ and $\varepsilon = 10^{-14}$.

Table 4. The MAE and ROC of the first derivative of the solution in Example 2.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	6.6678E-02	3.3586E-02	1.6863E-02	8.4503E-03	4.2299E-03
2^{-4}	3.1204E-02	1.5626E-02	7.8333E-03	3.9237E-03	1.9639E-03
2^{-6}	2.0444E-03	4.4041E-04	2.1679E-04	1.0501E-04	5.1308E-05
2^{-8}	2.8292E-02	1.5507E-02	8.1082E-03	4.1447E-03	2.0953E-03
2^{-10}	6.0825E-02	3.5474E-02	1.9278E-02	1.0053E-02	5.1343E-03
2^{-12}	1.0164E-01	6.5608E-02	3.7684E-02	2.0302E-02	1.0557E-02
2^{-14}	1.7942E-01	1.0466E-01	6.6955E-02	3.8310E-02	2.0593E-02
2^{-16}	3.9198E-01	1.7882E-01	1.0549E-01	6.7368E-02	3.8502E-02
2^{-18}	5.7241E-01	3.9067E-01	1.7852E-01	1.0575E-01	6.7510E-02
E^N	5.7241E-01	3.9067E-01	1.7852E-01	1.0575E-01	6.7510E-02
r^N	5.5112E-01	1.1299E+00	7.5546E-01	6.4745E-01	

Example 2: Consider three-point SPBVP with DST:

$$-\varepsilon p'''(s) + (1+s)p'(s) = \begin{cases} s, & 0 \leq s \leq 0.5 \\ (1+s)^2, & 0.5 < s \leq 1 \end{cases}$$

$$p(0) = 1, p'(0) = 1, p'(1) = 0.$$

Tables 4 and **5** show the MAE and ROC of Example 2 for the first derivative of the solution and the numerical solution, respectively, for several values of ε and *N*.

Table 5. MAE and ROC of the solution in Example 2.

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	3.0409E-02	1.5145E-02	7.5606E-03	3.7778E-03	1.8883E-03
2^{-4}	1.1202E-02	5.4932E-03	2.7245E-03	1.3574E-03	6.7756E-04
2^{-6}	2.8500E-04	2.2263E-05	2.3862E-05	1.5848E-05	8.9179E-06
2^{-8}	3.3122E-03	1.6943E-03	8.5638E-04	4.3054E-04	2.1589E-04
2^{-10}	4.3488E-03	2.1532E-03	1.0728E-03	5.3581E-04	2.6783E-04
2^{-12}	4.9181E-03	2.3351E-03	1.1421E-03	5.6560E-04	2.8161E-04
2^{-14}	5.9273E-03	2.5241E-03	1.1906E-03	5.8038E-04	2.8682E-04
2^{-16}	1.2829E-02	2.9999E-03	1.2717E-03	5.9894E-04	2.9165E-04
2^{-18}	5.7681E-02	6.4503E-03	1.5055E-03	6.3745E-04	3.0011E-04
E^N	5.7681E-02	1.5145E-02	7.5606E-03	3.7778E-03	1.8883E-03
r^N	1.9293E+00	1.0022E+00	1.0010E+00	1.0382E+00	1.0004E+00

6. Conclusion

This paper presents a distinctive quartic B-spline approach for addressing singularly perturbed third-order boundary value problems with non-smooth data. The suggested method is implemented for two test problems, which has increased MAE and ROC. Also, the numerical outcomes are in contrast to the existing approach proposed by Babu and Ramanujam (2007), and our proposed method results are better. As the value of the perturbation parameter decreases, to study the behaviour of the solution at the boundary and interior layer, a large number of mesh points is needed in that region; hence, the choice of uniform mesh is not favourable for such a problem, so we modified the mesh selection. The truncation error of the QBSM is derived and shown to be $O(h^5)$ accurate, and the method of convergence is $O(h^2)$. Also, an error plot for the solution's derivative and solution has been plotted. Additionally, a hybrid difference operator is used to address the discontinuity point. Future studies could be aimed at developing this methodology for nonlinear cases and coupled systems with more complicated boundary conditions, as well as comparisons with other existing numerical methods. Furthermore, the suggested approach generates a spline function that may be applied to get the solution at any point in the domain.

Conflict of Interest

The authors certify that this article does not include any potential conflict of interest.

AI Disclosure

The authors declare that no assistance is taken from generative AI to write this article.

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