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Interval-Valued Fermatean Fuzzy Multi-Criteria Decision-Making for Strategic Partner Ranking in Credit Risk Evaluation

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Abstract
This paper presents a novel multi-criteria decision-making (MCDM) framework based on interval-valued Fermatean fuzzy sets (IVFFSs) to effectively handle ambiguity and imprecision inherent in real-world decision environments. When accurate assessments are not accessible, the suggested model offers a more adaptable and realistic representation of ambiguous information by expanding classical Fermatean fuzzy theory through interval-valued membership and non-membership degrees. In order to aggregate expert evaluations, a new group generalized interval-valued Fermatean fuzzy weighted average (GGIVFFWA) operator is developed. This operator allows for the simultaneous examination of many advisers and deciders perspectives and incorporates group-based parameters. The applicability and efficiency of the proposed framework are demonstrated through a real-world case study on strategic partner selection for credit risk assessment. The obtained outcomes confirm the robustness, stability, and reliability of the proposed technique, which are further validated through a comparative analysis with existing decision-making approaches. General, the proposed framework offers an efficient and practical tool for solving complex MCDM problems under uncertainty.

Keywords- Credit risk evaluation, Fermatean fuzzy sets, Interval-valued Fermatean fuzzy weighted average operator, Group generalized interval-valued Fermatean fuzzy sets, Multi-criteria decision-making.

1. Introduction

The research topic of multi-criteria decision-making (MCDM) is rapidly advancing and provides the optimal choice among limited possibilities based on certain features. In most practical MCDM problems, it is difficult to provide precise data for the available alternatives due to limited information, time constraints, and the inherent ambiguity of decision-makers (DMs). As an addition of regular sets, Zadeh (1965) developed the idea of fuzzy sets (FSs) to overcome this drawback. This MCDM problems are determined by the different methods such as TOPSIS, AHP, DEMATEL, VIKOR, etc. Parida (2018) discussed the MCDM problem for both positive and negative ideal solutions using fuzzy technique order performance by similarity to ideal solution (TOPSIS) technique. Then, Ramalingam (2018) explained an application of MCDM problem for multi-model fusion of features in a 3D face recognition system using fuzzy interval valued TOPSIS technique. Parida (2020) introduced the generalized fuzzy TOPSIS technique in MCDM problem. Sahoo et al. (2023) proposed a generalized fuzzy TOPSIS approach are used in the evaluation of temperature for MCDM problems. In another study by Sahoo et. al. (2024) proposed the fuzzy MCDM problem are explained in the selection of optimal college location under the basis of min-max fuzzy TOPSIS method. Furthermore, Atanassov (1999) proposed the non-membership degree (ND) and membership degree (MD) disclose the notion of intuitionistic fuzzy sets (IFSs), and supply the condition that the total of MD and ND be less than equal to 1. The distinct benefits of utilizing IFSs, by Raji-Lawal et al. (2020) and Zou et al. (2020) been recognized as suitable instruments for characterizing the vagueness and unpredictability of practical issues. It is possible for the DMs to convey their views as MD and ND as $\left(\sqrt{3}/2,\ \sqrt{5}/2\right)$ in a number of claims. Consequently, $\left(\sqrt{3}/2 + \sqrt{5}/2\right) > 1$, IFSs is unable to handle this scenario. Then, extending the order of intuitionistic operations and relations, and also entropy measures under the basis of generalized fuzzy orthopartitions by Boffa et al. (2025). In order to address the issues, Yager (2014) developed the theory of Pythagorean fuzzy sets (PFSs), which are explicated by the MD and ND and satisfy the requirement that the squared sum of the MD and ND less than equal to 1. It is regarded as a spare trustworthy and appropriate instrument as a result for resolving complicated MCDM problem.

For instance, Wei and Lu (2018) discussed the new power fuzzy aggregation operator in Pythagorean fuzzy MCDM problem to develop an MCDM process. After that, Joshi (2019) investigated a range of generalized PFSs aggregation operators. In order to handle hierarchical MCDM situations with Pythagorean fuzzy data, Fei and Deng (2020) presented a novel decision support system (DSS). The AHP technique was recently presented by Shete et al. (2020) for studying the PFSs reliable supply chain transformation enablers. The decision-making trail and evaluation laboratory (DEMATEL) technique for software defined network risk analysis was first presented by Deb and Roy (2021) in the study on PFSs. Then, employing the Choquet integral operator, certain novel operators, that is, Choquet integral aggregation procedures based on Pythagorean fuzzy theory, were introduced by Xing et al. (2018).

Gao et al. (2018) suggested in a MCDM situation, the Hamacher production and prioritized operators were combined to create the Pythagorean Hamacher prioritized operators. Later, Cui et al. (2021) merged the stepwise weight assessment ratio analysis (SWARA) and combined compromise solution representations (CoCoSo) with PFSs to propose a hybrid framework and explore possible obstacles to internet of things (IOT) deployment. Next, Baral et al. (2025) introduced the entropy weight based TOPSIS method for the choice of e-waste recycling partner in an interval number. Then utilizing the TOPSIS technique in IVPFSs, Garg (2017) explained the improved score function (ISF). In accordance with the principles of linear programming, Garg (2018) also introduced the score function and its applications. Then, in compliance with the logarithm laws, several logarithmic aggregation operators were developed by Garg (2019). Peng and Yang (2016) talked about the basis characteristics of IVPFSs aggregation operators. The similarity measures algorithm for parametric value is solved in IVPFSs by Peng and Li (2019). Next, Pamučar et al. (2019) talked about the new weighted aggregated sum product assessment system (WASPAS) for goods movement. Now, expand the PFSs to Pythagorean fuzzy soft sets (PFSSs) and examine their two aggregation operators (AOs) in the context of green supply direction by Zulqarnain et al. (2021). Mahmood et al. (2022) presented a complicated Pythagorean aggregation operator for complex Pythagorean fuzzy sets was presented, taking confidence levels into account. Sahoo et al. (2025) discussed the material selection problem using the two aggregation operators that is, weighted average and geometric of interval-valued Pythagorean fuzzy soft sets.

Moreover, Yager (2016) introduced a more expansive variation of these sets called q-rung orthopair fuzzy sets (q-ROFSs), where the MD and NDs combined 'qth' power is less than 1. Also, discussed the additional orthopair hold the bounding form, the larger dimension of an acceptable orthopair. When $q = 3$, Senapati and Yager (2020) discussed the q-ROFSs as Fermatean fuzzy sets (FFSs). The MD and ND represent the FFSs in such a way that the cube sum of their respective values is less than equal to 1. The restraint relationship between the MD and ND is a crucial distinction across IFSs, PFSs, and FFSs. For managing uncertain MCDM problems, FFSs are therefore a more effective and potent tool than IFSs and PFSs. Despite fuzzy, Pythagorean, and Fermatean fuzzy-based MCDM techniques have been developed extensively, the majority of current models rely on exact or one-valued assessments from decision-makers. However, because of incomplete knowledge, cognitive constraints, and inherent uncertainty, decision-makers are frequently unable to convey their judgments using precise numerical values in real-world decision-making scenarios, especially in group choice contexts. This limitation emphasizes the need for more adaptable representations that can manage both group-based and interval-valued uncertainty at the same time, a need that is not sufficiently met by current methods. A number of academics have recently applied for diverse purposes and concentrated on the FFSs. Also, provided a decision analysis model for FFSs called weighted product measure (WPM). Aydemir and Gunduz (2020) introduced the Dombi AOs to FFSs in order to manage the MCDM issues. Now, Akram et al. (2020) presented a few AOs utilizing the FFSs Einstein t-conorm and t-norm functions. Then, Garg et al. (2020) discussed several AOs on FFSs and applied them to the COVID-19 potential evaluation. In order to address the issue of green supplier evaluation, Keshavarz-Ghorabaee et al. (2020) developed a WASPAS methodology for FFSs, based on Hamacher geometric operators were developed.

For the purpose of ranking the locations of electric car charging stations, Mishra and Rani (2021) investigated a novel Fermatean fuzzy Einstein AOs based MCDM representations. Simic et al. (2023) developed a three-phase Fermatean fuzzy group decision survey to address governments tax-collection issues in order to finance a public transportation network in the FFSs environment. The Hamacher operating norms served as inspiration, Hadi et al. (2021) presented a novel MCDM paradigm for cyclone disaster assessment by defining some Hamacher AOs under the FFSs surroundings. Dempster-Shafer theory and Fermatean fuzzy entropy were combined by Deng and Wang (2021) to create a novel Fermatean fuzzy MCDM method. Kirisci (2023) proposed a TOPSIS based method under Fermatean fuzzy sets (FFS) on the cosine similarity and Euclidean distance measures. Akram et al. (2023) extended the combinative distance-based assessment (CODAS) technique in the Hamacher aggregation operators of 2-tuple linguistic Fermatean fuzzy. In order to measure performance in two MCDM problems such as the best site for a company's facility in the beverage sector and Turkish research universities, the study aims to investigate how well FFSs perform in taking advantage of ambiguity in selection and ranking decisions using TOPSIS and SWARA techniques by Aydogan and Ozkir (2024). In the technology of renewable energy, the extended FFSs are used in group decision-making proposed by Gocer (2024). Qi et al. (2024) proposed the MCDM to provide an explanation of the rough set based FFSs. Then, Kirisci (2024a) discussed the MCDM problem in the application of medical diagnosis using Fermatean fuzzy soft matrices in group decision-making.

Due to inadequate knowledge, DMs find it extremely difficult to properly list their conclusions with crisp values when dealing with several practical decision problems under FFS conditions. It is beneficial for DMs to base their resolution on an interval digit within $[0,1]$ in such situations. On the other hand, several previous works have focused on the creation of FFSs while ignoring their expanded content. Consequently, it is imperative to advance the notion of interval-valued Fermatean fuzzy sets (IVFFSs) which attests to the assumption of interval values by the MD and ND. A new interval type-2 fuzzy sets (IT2FSs) with relative preference relations (RPRs) based on the multi-attributive border approximation area comparison method (MABAC) with the WASPAS method is established for identifying the analytical path of manufacturing projects under group decision-making processes by Dorfeshan and Mousavi (2020). Jeevaraj (2021) discussed the ordering principle of interval-valued Fermatean fuzzy sets, then used to describe its applications of IVFFSs. By defining suitable operational rules, score and accuracy functions, distance measures, and entropy-based weighting schemes, Mandal and Seikh (2022) proposed an interval-valued Fermatean fuzzy set (IVFFS)–based multi-attribute decision-making (MADM) framework for sustainability-oriented decision-making problems. Additionally, they extended the classical TOPSIS method to the IVFF environment.

In the waste recycling problem, Luqman and Shahzadi (2023) introduced the Hamacher aggregation operator for IVFFSs. Similarly, Mishra et al. (2023) proposed an aggregation operator and elaborated the Heronian mean operator for decision-making problems related to climate change transportation policy. In the context of risk assessment, Kirisci (2024b) explained the application of these fuzzy sets in the domain of driving behavior. Bouraima et al. (2023) employed a novel integrated IVFFS approach to identify the causes of accidents and evaluate their significance, aiming to enhance safety performance on a broader scale. Subsequently, Rani and Mishra (2022) introduced the WASPA-based approach for analyzing decision-making problems. Furthermore, Rani et al. (2023), examined a new Einstein aggregation operator within IVFFSs, incorporating an improved scoring function. In order to handle uncertainty in plastic waste management selection problems, Mandal and Seikh (2023) developed a hybrid multi-attribute group decision-making framework under an interval-valued spherical fuzzy environment by combining the MABAC method with entropy-based and deviation-based weighting schemes, supported by Dombi aggregation operators. By combining Dombi aggregation operators with SWARA and PROMETHEE II techniques, Seikh and Mandal (2023) created an interval-valued Fermatean fuzzy MAGDM framework to handle uncertainty in the selection of biomedical waste management organizations. A novel IVFFS approach was proposed in relation to the concept of dominance probability by Qin et al. (2024). Ibrahim et al. (2024) presented an extended rough-based IVFFSs model for autonomous automobile in smart cities. Furthermore, Li et al. (2024) discussed the multiple criteria group decision-making (MCGDM) approach based on interval-valued Fermatean fuzzy sets for solving the new energy vehicle battery supplier (NEVBS) selection problem. In order to handle uncertainty in e-waste management issues, Seikh and Chatterjee (2024a) developed an interval-valued Fermatean fuzzy–based multi-attribute decision-making framework that integrates the SWARA, Best Worst Method (BWM), and VIseKriterijumsko Optimizacijom I Kompromisno Resenje (VIKOR) approaches. The same author Seikh and Chatterjee (2024b) developed a multi-attribute group decision-making framework based on interval-valued Fermatean fuzzy that integrates the additive ratio assessment (ARAS) and SWARA techniques to assess renewable energy sources in the face of uncertainty.

In the MCDM literature, the integration of group generalized interval-valued parameters with Fermatean fuzzy information is still mainly unexplored, despite the introduction of interval-valued Fermatean fuzzy sets (IVFFSs) to improve uncertainty modelling. Group preferences and generalized parameterized information cannot be fully captured by the aggregation methods currently in use under IVFFSs. In order to increase the robustness, adaptability, and realism of decision-making outcomes in challenging MCDM issues like credit risk appraisal, a group generalized interval-valued aggregation method must be developed. Inspired by the idea of IVFFSs, we first present the concept of group generalized interval-valued Fermatean fuzzy weighted average (GGIVFFWA) operator with their respective features. Furthermore, using the group generalized interval-valued parameters (GGIVPs) setup, the GGIVFFWA framework is built to handle the MCDM problems. The main objectives of the present study are as follows:

- To present the concept of group generalized interval-valued Fermatean fuzzy sets (GGIVFFSs) along with their basic features.
- To address decision-making issues involving IVFFSs, a novel GGIVFFWA-based MCDM framework under the group generalized interval-valued parameters (GGIVPs) environment is proposed.
- To illustrate the effectiveness and applicability of the suggested method using a multi-criteria strategic partner selection problem for assessing credit risk.

The remainder of the article is arranged such that Section 2 grants the fundamental ideas concerning FFSs, IVFFSs, and its various operations. Section 3 provides the GGIVFFWA operator and its some operation using their parameters. A unique MCDM approach based on GGIVFFWA operator is introduced and the practicality of the suggested strategy is displayed by the case study of choosing potential strategic partners for credit risk evaluation provided in Section 4. Next, Section 5 discusses the comparison analysis for the efficiency of the method. Section 6 layouts a summary of the paper and future scope.

2. Preliminaries

This section follows contains a few fundamental definitions and theorem.

Definition 2.1 (Atanassov, 1999). If $\theta \in \Omega$ and Ω be a universal set. An intuitionistic fuzzy set I on Ω is recognize with membership $\mu_I(\theta)$ and non-membership $\gamma_I(\theta)$, which associate the functions mapping from each element to the range $[0, 1]$. It is defined in Equation (1) as

$$I = \{\{\langle\theta, \mu_I(\theta), \gamma_I(\theta)\rangle | \theta \in \Omega\}\} \quad (1)$$

where, $\mu_I(\theta), \gamma_I(\theta): \Omega \to [0, 1]$ with $0 \leq \mu_I(\theta), +\gamma_I(\theta) \leq 1$.

The hesitation degrees $\Pi_I(\theta) = \sqrt{1 - \mu_I(\theta) - \gamma_I(\theta)}$.

Definition 2.2 (Yager, 2014). Let Ω be a universe of discourse and $\theta \in \Omega$. A Pythagorean fuzzy set P in Ω is characterized in Equation (2) as

$$P = \{\langle\theta, \mu_P(\theta), \gamma_P(\theta)\rangle | \theta \in \Omega\} \quad (2)$$

where, $(\mu_P(\theta), \gamma_P(\theta)): \Omega \to [0, 1]$ with $0 \leq \mu_P^2(\theta) + \gamma_P^2(\theta) \leq 1$.

The hesitation degrees of function $\Pi_P(\theta) = \sqrt{1 - \mu_P^2(\theta) - \gamma_P^2(\theta)}$.

Definition 2.3 (Senapati and Yager, 2020). A Fermatean fuzzy set F in Ω is act in Equation (3) as

$$F = \{\langle\theta, \mu_F(\theta), \gamma_F(\theta)\rangle | \theta \in \Omega\} \quad (3)$$

where, $(\mu_F(\theta), \gamma_F(\theta)): \Omega \to [0, 1]$ with $0 \leq \mu_F^3(\theta) + \gamma_F^3(\theta) \leq 1$.

The hesitation function $\Pi_F(\theta) = \sqrt[3]{1 - \mu_F^3(\theta) - \mu_F^3(\theta)}$.

Definition 2.4 (Atanassov, 1999). An interval-valued intuitionistic fuzzy set I on Ω is identified with an interval-valued membership $\mu_I(\theta) = [\mu_I^l, \mu_I^u]$, and non-membership degree $\gamma_I(\theta) = [\gamma_I^l, \gamma_I^u]$, that indicate how each element's function mappings to the range $[0, 1]$ are represented in Equation (4) as

$$I = \{\langle\theta, \mu_I(\theta), \mu_I(\theta)\rangle | \theta \in \Omega\} \quad (4)$$

where, $(\mu_I(\theta), \mu_I(\theta)): \Omega \to [0, 1]$ with $0 \leq \mu_I^l, \mu_I^u, \gamma_I^l, \gamma_I^l \leq 1$ and $0 \leq (\sup \mu_I(\theta) + \sup \gamma_I(\theta)) \leq 1$.

Definition 2.5 (Peng and Yang, 2016). An interval-valued Pythagorean fuzzy set P on Ω is identified with MD $\mu_P(\theta) = [\mu_P^l, \mu_P^u]$, and ND $\gamma_P(\theta) = [\gamma_P^l, \gamma_P^u]$. Then, it is denoted in Equation (5) as

$$P = \{\langle\theta, \mu_P(\theta), \mu_P(\theta)\rangle | \theta \in \Omega\} \quad (5)$$

where,
$\left(\mu_P(\theta), \gamma_P(\theta)\right): \Omega \to [0,1]$ with $0 \le \mu_P^l, \mu_P^u, \gamma_P^l, \gamma_P^u \le 1$ and $0 \le (\sup \mu_P(\theta))^2 + (\sup \gamma_P(\theta))^2 \le 1$.

Definition 2.6 (Jeevaraj, 2021). Let $\theta \in \Omega$ and Ω is an inclusive set. An interval-valued Fermatean fuzzy set F on Ω is recognized in Equation (6) as

$$F = \{\langle \theta, \mu_F(\theta), \gamma_F(\theta)\rangle | \theta \in \Omega\} \tag{6}$$

where,
$\left(\mu_F(\theta), \gamma_F(\theta)\right): \Omega \to [0,1]$ with $0 \le \mu_F^l, \mu_F^u, \gamma_F^l, \gamma_F^u \le 1$ and $0 \le (\sup \mu_F(\theta))^3 + (\sup \gamma_F(\theta))^3 \le 1$.

Additionally, an IVFFN is indicate as $\Delta = (\mu_\Delta, \gamma_\Delta) = \{[\mu_\Delta^l, \mu_\Delta^u], [\gamma_\Delta^l, \gamma_\Delta^u]\}$ with $0 \le \mu_\Delta^l \le \mu_\Delta^u \le 1, 0 \le \gamma_\Delta^l \le \gamma_\Delta^u \le 1$ and $0 \le (\mu_\Delta^u)^3 + (\gamma_\Delta^u)^3 \le 1$.

Definition 2.7 (Jeevaraj, 2021). Let $\Delta = \langle [\mu_\Delta^l, \mu_\Delta^u], [\gamma_\Delta^l, \gamma_\Delta^u]\rangle, \Delta_1 = \langle [\mu_{\Delta_1}^l, \mu_{\Delta_1}^u], [\gamma_{\Delta_1}^l, \gamma_{\Delta_1}^u]\rangle$, and $\Delta_2 = \langle [\mu_{\Delta_2}^l, \mu_{\Delta_2}^u], [\gamma_{\Delta_2}^l, \gamma_{\Delta_2}^u]\rangle$ be three IVFFNs, and $\omega > 0$. Then

- $\Delta_1 \oplus \Delta_2 = \left\langle \left[\left((\mu_{\Delta_1}^l)^3 + (\mu_{\Delta_2}^l)^3 - (\mu_{\Delta_1}^l)^3 (\mu_{\Delta_2}^l)^3 \right)^{\frac{1}{3}}, \left((\mu_{\Delta_1}^u)^3 + (\mu_{\Delta_2}^u)^3 - (\mu_{\Delta_1}^u)^3 (\mu_{\Delta_2}^u)^3 \right)^{\frac{1}{3}} \right], [\gamma_{\Delta_1}^l \gamma_{\Delta_2}^l, \gamma_{\Delta_1}^u \gamma_{\Delta_2}^u] \right\rangle$
- $\Delta_1 \otimes \Delta_2 = \left\langle [\mu_{\Delta_1}^l \mu_{\Delta_2}^l, \mu_{\Delta_1}^u \mu_{\Delta_2}^u], \left[\left((\gamma_{\Delta_1}^l)^3 + (\gamma_{\Delta_2}^l)^3 - (\gamma_{\Delta_1}^l)^3 (\gamma_{\Delta_2}^l)^3 \right)^{\frac{1}{3}} \right], \left((\gamma_{\Delta_1}^u)^3 + (\gamma_{\Delta_2}^u)^3 - (\gamma_{\Delta_1}^u)^3 (\gamma_{\Delta_2}^u)^3 \right)^{\frac{1}{3}} \right\rangle$
- $\omega\Delta = \left\langle \left[1 - \left(1 - (\mu_\Delta^l)^3\right)^{\frac{1}{3}}, 1 - \left(1 - (\mu_\Delta^l)^3\right)^{\frac{1}{3}} \right], [(\gamma_\Delta^l)^\omega, (\gamma_\Delta^u)^\omega] \right\rangle$
- $\Delta^\omega = \left\langle [(\mu_\Delta^l)^\omega, (\mu_\Delta^u)^\omega], \left[1 - \left(1 - (\gamma_\Delta^l)^3\right)^{\frac{1}{3}}, 1 - \left(1 - (\gamma_\Delta^l)^3\right)^{\frac{1}{3}} \right] \right\rangle$

Theorem 2.1 (Rani and Mishra, 2022). Let $\Delta = \langle [\mu_\Delta^l, \mu_\Delta^u], [\gamma_\Delta^l, \gamma_\Delta^u]\rangle$, and $\Delta_q = \left\langle \left[\mu_{\Delta_q}^l, \mu_{\Delta_q}^u\right] \left[\gamma_{\Delta_q}^l, \gamma_{\Delta_q}^u\right] \right\rangle$, where, $q = 1, 2$ be three IVFFNs, the $\Delta_1 \oplus \Delta_2, \ \Delta_1 \otimes \Delta_2, \omega\Delta, \Delta^\omega, \omega > 0$ are all IVFFNs.

Theorem 2.2 (Rani and Mishra, 2022). Let $\Delta = \langle [\mu_\Delta^l, \mu_\Delta^u], [\gamma_\Delta^l, \gamma_\Delta^u]\rangle$ and $\Delta_q = \left\langle \left[\mu_{\Delta_q}^l, \mu_{\Delta_q}^u\right], \left[\gamma_{\Delta_q}^l, \gamma_{\Delta_q}^u\right] \right\rangle$, $(where\ q = 1, 2)$ be three IVFFNs and $\omega, \omega_1, \omega_2 > 0$, then

- $\Delta_1 \oplus \Delta_2 = \Delta_2 \oplus \Delta_1$
- $\Delta_1 \otimes \Delta_2 = \Delta_2 \otimes \Delta_1$
- $\omega(\Delta_1 \oplus \Delta_2) = \omega\Delta_1 \oplus \omega\Delta_2$
- $(\Delta_1 \otimes \Delta_2)^\omega = \Delta_1^\omega \otimes \Delta_2^\omega$
- $\Delta^{\omega_1 \oplus \omega_2} = \Delta^{\omega_1} \otimes \Delta^{\omega_2}$

Definition 2.8 (Rani and Mishra, 2022). Suppose $\Delta_q = \left(\mu_{\Delta_q}, \gamma_{\Delta_q}\right) = \left\langle \left[\mu_{\Delta_q}^l, \mu_{\Delta_q}^u\right], \left[\gamma_{\Delta_q}^l, \gamma_{\Delta_q}^u\right] \right\rangle$, $(q = 1, 2, 3, \cdots, n)$ be an IVFFNs. Let ω_q be the weighted vector. The interval-valued Fermatean fuzzy weighted average (IVFFWA) is described in Equation (7) as

$$IVFFWA(\Delta_1, \Delta_2, \cdots, \Delta_n) = \oplus_{q=1}^{n} \omega_q \Delta_q$$
$$= \left\langle \left[\left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}} \right], \left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \right] \right\rangle \quad (7)$$

3. Group Generalized Interval-Valued Fermatean Fuzzy Weighted Average Operator

The interval-valued Fermatean fuzzy weighted average (IVFWA) operator is widely applied in multi-criteria decision-making (MCDM) problems. In this framework, the group generalized IVFFWA (GGIVFFWA) operator plays a crucial rule in consolidating the information associated with each criterion, as GGIVPs allow for the direct integration and processing of expert evaluations.

Definition 3.1 Suppose $\Delta_q = \left(\mu_{\Delta_q}, \gamma_{\Delta_q}\right) = \left\langle \left[\mu_{\Delta_q}^{l}, \mu_{\Delta_q}^{u}\right], \left[\gamma_{\Delta_q}^{l}, \gamma_{\Delta_q}^{u}\right] \right\rangle$ where, $q = 1, 2, \cdots, n$ and $\chi_k = \left(\mu_{\chi_k}, \mu_{\chi_k}\right) = \langle [\mu_{\chi_k}^{l}, \mu_{\chi_k}^{u}], [\gamma_{\chi_k}^{l}, \gamma_{\chi_k}^{u}] \rangle$ where, $k = 1, 2, \cdots, t$ be two pairs of IVFFNs. Let ω_q and ρ_k be the weighted vectors with respect to Δ_q and χ_k, with $\sum_{q=1}^{n} \omega_k = 1, \omega_q \in [0, 1]$ and $\sum_{k=1}^{t} \rho_k = 1, \rho_k \in [0, 1]$. Then GGIVFFWA operator such that $GGIVFFWA: \Theta^n \longrightarrow \Theta$ is defined as

$$GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_k\}) = IVFFWA(\Delta_1, \Delta_2, \cdots, \Delta_n) \otimes IVFFWA(\chi_1, \chi_2, \cdots, \chi_k)$$
$$= \left\langle \left[\left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}} \right], \left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \right] \right\rangle$$
$$\otimes \left\langle \left[\left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}} \right], \left[\prod_{k=1}^{t} \left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t} \left(\gamma_{\chi_k}^{u}\right)^{\rho_k} \right] \right\rangle \quad (8)$$

Theorem 3.1 Suppose $\Delta_q = \left(\mu_{\Delta_q}, \gamma_{\Delta_q}\right) = \left\langle \left[\mu_{\Delta_q}^{l}, \mu_{\Delta_q}^{u}\right], \left[\gamma_{\Delta_q}^{l}, \gamma_{\Delta_q}^{u}\right] \right\rangle, q = 1, 2, \cdots, n$ be IVFFNs and $\chi_k = \left(\mu_{\chi_k}, \gamma_{\chi_k}\right) = \langle [\mu_{\chi_k}^{l}, \mu_{\chi_k}^{u}], [\gamma_{\chi_k}^{l}, \gamma_{\chi_k}^{u}] \rangle, k = 1, 2, \cdots, t$ be IVFFNs. The total value produced by the GGIVFFWA operator is defined in Equation (9) as

$$GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\})$$
$$= \left\langle \left[\left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}} \right], \left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \right] \right\rangle \otimes \left\langle \left[\left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}} \right], \left[\prod_{k=1}^{t} \left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t} \left(\gamma_{\chi_k}^{u}\right)^{\rho_k} \right] \right\rangle$$
$$= \left\langle \left[\left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}} \right] . \left[\left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}} \right], \left(\left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \right]^3 + \left[\prod_{k=1}^{t} \left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t} \left(\gamma_{\chi_k}^{u}\right)^{\rho_k} \right]^3 - \left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \right]^3 . \left[\prod_{k=1}^{t} \left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t} \left(\gamma_{\chi_k}^{u}\right)^{\rho_k} \right]^3 \right)^{\frac{1}{3}} \right\rangle \quad (9)$$

Proof: To establish the aforementioned theorem for IVFFNs and the GGIVFFWA operator using the method of induction, we begin with the base case of $n = 1$ and $t = 1$. In this case, determine $\omega_1 = 1 = \rho_1$. By applying the definition 3.1of the GGIVFFWA operator in Equation (8), it can be indicated as
$GGIVFFWA\ (\Delta_1, \chi_1) = (\omega_1\Delta_1) \otimes (\rho_1\chi_1) = \Delta_1 \otimes\ \chi_1$.

As Δ_1 and χ_1 are an IVFFNs, then $\Delta_1 \otimes \chi_1$ is also an IVFFNs by Theorem 2.1. The GGIVPFWA operator is written as
$GGIVFFWA\ (\Delta_1, \chi_1) = \Delta_1 \otimes \chi_1 = (\mu_{\Delta_1}, \gamma_{\Delta_1}) \otimes (\mu_{\chi_1}, \gamma_{\chi_1})$

$$= \left\langle \left[\left(1-\left(1-\left(\mu_{\Delta_1}^l\right)^3\right)^{\omega_1}\right)^{\frac{1}{3}}, \left(1-\left(1-\left(\mu_{\Delta_1}^u\right)^3\right)^{\omega_1}\right)^{\frac{1}{3}}\right]. \left[\left(1-\left(1-\left(\mu_{\chi_1}^l\right)^3\right)^{\rho_1}\right)^{\frac{1}{3}}, \left(1-\left(1-\left(\mu_{\chi_1}^u\right)^3\right)^{\rho_1}\right)^{\frac{1}{3}}\right], \left(\left[\left(\gamma_{\Delta_1}^l\right)^{\omega_1}, \left(\gamma_{\Delta_1}^u\right)^{\omega_1}\right]^3 + \left[\left(\gamma_{\chi_1}^l\right)^{\rho_1}, \left(\gamma_{\chi_1}^u\right)^{\rho_1}\right]^3 - \left[\left(\gamma_{\Delta_1}^l\right)^{\omega_1}, \left(\gamma_{\Delta_1}^u\right)^{\omega_1}\right]^3 \cdot \left[\left(\gamma_{\chi_1}^l\right)^{\rho_1}, \left(\gamma_{\chi_1}^u\right)^{\rho_1}\right]^3\right)^{\frac{1}{3}} \right\rangle$$

$$= \left\langle \left[\mu_{\Delta_1}^l, \mu_{\Delta_1}^u\right] \cdot \left[\mu_{\chi_1}^l, \mu_{\chi_1}^u\right], \left(\left[\gamma_{\Delta_1}^l, \gamma_{\Delta_1}^u\right]^3 + \left[\gamma_{\chi_1}^l, \gamma_{\chi_1}^u\right]^3 - \left[\gamma_{\Delta_1}^l, \gamma_{\Delta_1}^u\right]^3 \cdot \left[\gamma_{\chi_1}^l, \gamma_{\chi_1}^u\right]^3\right)^{\frac{1}{3}} \right\rangle.$$

Therefore, it is true for $n = 1$ and $t = 1$.

Assume that the statement is true for $n = u$ and $t = v$. So
$GGIVFFWA\ (\{\Delta_1, \Delta_2, \cdots, \Delta_u\}, \{\chi_1, \chi_2, \cdots, \chi_v\})$
$= (\omega_1\Delta_1 \oplus \omega_2\Delta_2 \oplus \cdots \oplus \omega_u\Delta_u) \otimes (\rho_1\chi_1 \oplus \rho_1\chi_1 \oplus \cdots \oplus \rho_v\chi_v)$

is an IVFFNs and
$GGIVFFWA\ (\{\Delta_1, \Delta_2, \cdots, \Delta_u\}, \{\chi_1, \chi_2, \cdots, \chi_v\})$

$$= \left\langle \left[\left(1-\prod_{q=1}^u\left(1-\left(\mu_{\Delta_q}^l\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1-\prod_{q=1}^u\left(1-\left(\mu_{\Delta_q}^u\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right], \left[\prod_{q=1}^u\left(\gamma_{\Delta_q}^l\right)^{\omega_q}, \prod_{q=1}^u\left(\gamma_{\Delta_q}^u\right)^{\omega_q}\right] \right\rangle \otimes \left\langle \left[\left(1-\prod_{k=1}^v\left(1-\left(\mu_{\chi_k}^l\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1-\prod_{k=1}^v\left(1-\left(\mu_{\chi_k}^u\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right], \left[\prod_{k=1}^v\left(\gamma_{\chi_k}^l\right)^{\rho_k}, \prod_{k=1}^v\left(\gamma_{\chi_k}^u\right)^{\rho_k}\right] \right\rangle$$

$$= \left\langle \left[\left(1-\prod_{q=1}^u\left(1-\left(\mu_{\Delta_q}^l\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1-\prod_{q=1}^u\left(1-\left(\mu_{\Delta_q}^u\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1-\prod_{k=1}^v\left(1-\left(\mu_{\chi_k}^l\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1-\prod_{k=1}^v\left(1-\left(\mu_{\chi_k}^u\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right], \left(\left[\prod_{q=1}^u\left(\gamma_{\Delta_q}^l\right)^{\omega_q}, \prod_{q=1}^u\left(\gamma_{\Delta_q}^u\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^v\left(\gamma_{\chi_k}^l\right)^{\rho_k}, \prod_{k=1}^v\left(\gamma_{\chi_k}^u\right)^{\rho_k}\right]^3 - \left[\prod_{q=1}^u\left(\gamma_{\Delta_q}^l\right)^{\omega_q} \prod_{q=1}^u\left(\gamma_{\Delta_q}^u\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^v\left(\gamma_{\chi_k}^l\right)^{\rho_k}, \prod_{k=1}^v\left(\gamma_{\chi_k}^u\right)^{\rho_k}\right]^3\right)^{\frac{1}{3}} \right\rangle.$$

To prove the statement for $n = u + 1$ and $t = v + 1$.
For $n = u + 1$ and $t = v + 1$, we aim to demonstrate the true of this assertion.

$$GGIVFFWA\ (\{\Delta_1, \Delta_2, \cdots, \Delta_{u+1}\}, \{\chi_1, \chi_2, \cdots, \chi_{v+1}\})$$
$$= (\omega_1\Delta_1 \oplus \omega_2\Delta_2 \oplus \cdots \oplus \omega_{u+1}\Delta_{u+1}) \otimes (\rho_1\chi_1 \oplus \rho_2\chi_2 \oplus \cdots \oplus \rho_{v+1}\chi_{v+1})$$ is an IVFFNs and

$$GGIVFFWA\ (\{\Delta_1, \Delta_2, \cdots, \Delta_{u+1}\}, \{\chi_1, \chi_2, \cdots, \chi_{v+1}\})$$
$$= \left\langle \left[\left(1 - \prod_{q=1}^{u+1}\left(1 - \left(\mu_{\Delta_q}^l\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{u+1}\left(1 - \left(\mu_{\Delta_q}^u\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1 - \prod_{k=1}^{v+1}\left(1 - \left(\mu_{\chi_k}^l\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^{v+1}\left(1 - \left(\mu_{\chi_k}^u\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right], \left(\left[\prod_{q=1}^{u+1}\left(\gamma_{\Delta_q}^l\right)^{\omega_q}, \prod_{q=1}^{u+1}\left(\gamma_{\Delta_q}^u\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^{v+1}\left(\gamma_{\chi_k}^l\right)^{\rho_k}, \prod_{k=1}^{v+1}\left(\gamma_{\chi_k}^u\right)^{\rho_k}\right]^3 - \left[\prod_{q=1}^{u+1}\left(\gamma_{\Delta_q}^l\right)^{\omega_q}, \prod_{q=1}^{u+1}\left(\gamma_{\Delta_q}^u\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{v+1}\left(\gamma_{\chi_k}^l\right)^{\rho_k}, \prod_{k=1}^{v+1}\left(\gamma_{\chi_k}^u\right)^{\rho_k}\right]^3\right)^{\frac{1}{3}} \right\rangle.$$

The statement $n = u + 1$ and $t = v + 1$ is true. The method of induction, it is concluded that the results are hold for all values of n and t. Hence, completed the proof of the theorem.

The GGIVFFWA operator must satisfy the following requirements.

1) Idempotency

Let $\Delta_q = \left(\mu_{\Delta_q}, \gamma_{\Delta_q}\right) = \left\langle\left[\mu_{\Delta_q}^l, \mu_{\Delta_q}^u\right], \left[\gamma_{\Delta_q}^l, \gamma_{\Delta_q}^u\right]\right\rangle, q = 1, 2, \cdots, n$ and $\chi_k = (\mu_{\chi_k}, \gamma_{\chi_k}) = \langle[\mu_{\chi_k}^l, \mu_{\chi_k}^u], [\gamma_{\chi_k}^l, \gamma_{\chi_k}^u]\rangle, k = 1, 2, \cdots, t$ be two groups of IVFFN. Let ω_q and ρ_k be the weighted vectors with respect to Δ_q and χ_k, with satisfying the conditions of weight. So, all the IVFFNs Δ_q and χ_k are identical, say $\Delta_q = \Delta, \chi_k = \chi$, then Equation (10) represents

$$GGIVFFWA\ (\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\})$$
$$= IVFFWA\ (\Delta_1, \Delta_2, \cdots, \Delta_q) \otimes IVFFWA\ (\chi_1, \chi_2, \cdots, \chi_k) = \Delta \otimes \chi \tag{10}$$

Proof: As $\Delta_q = \Delta = (\mu_\Delta, \gamma_\Delta) = \langle[\mu_\Delta^l, \mu_\Delta^u], [\gamma_\Delta^l, \gamma_\Delta^u]\rangle$ and $\chi_k = \chi = (\mu_\chi, \gamma_\chi) = \langle[\mu_\chi^l, \mu_\chi^u], [\gamma_\chi^l, \gamma_\chi^u]\rangle, \forall\, j, k$, then

$$GGIVFFWA\ (\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\})$$
$$= IVFFWA(\Delta_1, \Delta_2, \cdots, \Delta_n) \otimes IVFFWA(\chi_1, \chi_2, \cdots, \chi_t)$$

$$= \left\langle \left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}\right], \left(\left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^{3} + \left[\prod_{k=1}^{t}\left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t}\left(\gamma_{\chi_k}^{u}\right)^{\rho_k}\right]^{3} - \left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^{3} \cdot \left[\prod_{k=1}^{t}\left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t}\left(\gamma_{\chi_k}^{u}\right)^{\rho_k}\right]^{3}\right)^{\frac{1}{3}} \right\rangle$$

$$= \left\langle \left[\left(1-\left(1-\left(\mu_{\Delta}^{l}\right)^{3}\right)^{\sum_{q=1}^{n}\omega_q}\right)^{\frac{1}{3}}, \left(1-(1-(\mu_{\Delta}^{u})^{3})^{\sum_{q=1}^{n}\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1-\left(1-\left(\mu_{\chi}^{l}\right)^{3}\right)^{\sum_{k=1}^{t}\rho_k}\right)^{\frac{1}{3}}, \left(1-\left(1-\left(\mu_{\chi}^{u}\right)^{3}\right)^{\sum_{k=1}^{t}\rho_k}\right)^{\frac{1}{3}}\right], \left(\left[\left(\gamma_{\Delta}^{l}\right)^{\sum_{q=1}^{n}\omega_q}, (\gamma_{\Delta}^{u})^{\sum_{q=1}^{n}\omega_q}\right]^{3} + \left[\left(\gamma_{\chi}^{l}\right)^{\sum_{k=1}^{t}\rho_k}, \left(\gamma_{\chi}^{u}\right)^{\sum_{k=1}^{t}\rho_k}\right]^{3} - \left[\left(\gamma_{\Delta}^{l}\right)^{\sum_{q=1}^{n}\omega_q}, (\gamma_{\Delta}^{u})^{\sum_{q=1}^{n}\omega_q}\right]^{3} \cdot \left[\left(\gamma_{\chi}^{l}\right)^{\sum_{k=1}^{t}\rho_k}, \left(\gamma_{\chi}^{u}\right)^{\sum_{k=1}^{t}\rho_k}\right]^{3}\right)^{\frac{1}{3}} \right\rangle$$

$$= \left\langle \left[\mu_{\Delta}^{l}, \mu_{\Delta}^{u}\right] \cdot \left[\mu_{\chi}^{l}, \mu_{\chi}^{u}\right], \left(\left[\gamma_{\Delta}^{l}, \gamma_{\Delta}^{u}\right]^{3} + \left[\gamma_{\chi}^{l}, \gamma_{\chi}^{u}\right]^{3} - \left[\gamma_{\Delta}^{l}, \gamma_{\Delta}^{u}\right]^{3} \cdot \left[\gamma_{\chi}^{l}, \gamma_{\chi}^{u}\right]^{3}\right)^{\frac{1}{3}} \right\rangle$$

$$= (\mu_{\Delta}, \gamma_{\Delta}) \otimes (\mu_{\chi}, \gamma_{\chi})$$

$$= \langle [\mu_{\Delta}^{l}, \mu_{\Delta}^{u}], [\gamma_{\Delta}^{l}, \gamma_{\Delta}^{u}] \rangle \otimes \langle [\mu_{\chi}^{l}, \mu_{\chi}^{u}], [\gamma_{\chi}^{l}, \gamma_{\chi}^{u}] \rangle$$

$$= \Delta \otimes \chi.$$

2) Boundedness

Suppose $\Delta_q = \left(\mu_{\Delta_q}, \gamma_{\Delta_q}\right) = \left\langle \left[\mu_{\Delta_q}^{l}, \mu_{\Delta_q}^{u}\right], \left[\gamma_{\Delta_q}^{l}, \gamma_{\Delta_q}^{u}\right] \right\rangle, q = 1, 2, \cdots, n,$ and $\chi_k = (\mu_{\chi_k}, \gamma_{\chi_k}) = \langle [\mu_{\chi_k}^{l}, \mu_{\chi_k}^{u}], [\gamma_{\chi_k}^{l}, \gamma_{\chi_k}^{u}] \rangle, k = 1, 2, \cdots, t$ be two groups of IVFFNs with

$\Delta^{-} = \left[\min_{1 \le q \le n}\left\{\mu_{\Delta_q}\right\}, \max_{1 \le q \le n}\left\{\gamma_{\Delta_q}\right\}\right],$

$\Delta^{+} = \left[\max_{1 \le q \le n}\left\{\mu_{\Delta_q}\right\}, \min_{1 \le q \le n}\left\{\gamma_{\Delta_q}\right\}\right],$

$\chi^{-} = \left[\min_{1 \le k \le t}\{\mu_{\chi_k}\}, \max_{1 \le k \le t}\{\gamma_{\chi_k}\}\right],$

$\chi^{+} = \left[\max_{1 \le k \le t}\{\mu_{\chi_k}\}, \min_{1 \le k \le t}\{\gamma_{\chi_k}\}\right].$

Then, Equation (11) shows

$$\Delta^{-} \otimes \chi^{-} \leq GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\}) \leq \Delta^{+} \otimes \chi^{+} \tag{11}$$

Proof: For any q, we have

$$\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}} \geq \left(1-\prod_{q=1}^{n}\left(1-\min_q\left\{{\mu_{\Delta_q}^{l}}^{3}\right\}\right)^{\omega_p}\right)^{\frac{1}{3}}$$

$$=\left(1-\left(1-\min_q\left\{{\mu_{\Delta_q}^{l}}^{3}\right\}\right)^{\sum_{q=1}^{n}\omega_q}\right)^{\frac{1}{3}}=\left(\min_q\left\{{\mu_{\Delta_q}^{l}}^{3}\right\}\right)^{\frac{1}{3}}=\min_q\left\{\mu_{\Delta_q}^{l}\right\}$$

and

$$\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}} \geq \left(1-\prod_{q=1}^{n}\left(1-\min_q\left\{{\mu_{\Delta_q}^{u}}^{3}\right\}\right)^{\omega_p}\right)^{\frac{1}{3}}$$

$$=\left(1-\left(1-\min_q\left\{{\mu_{\Delta_q}^{u}}^{3}\right\}\right)^{\sum_{q=1}^{n}\omega_q}\right)^{\frac{1}{3}}=\left(\min_q\left\{{\mu_{\Delta_q}^{u}}^{3}\right\}\right)^{\frac{1}{3}}=\min_q\left\{\mu_{\Delta_q}^{u}\right\}.$$

Thus,

$$\left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}},\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \geq \min_q\left\{\mu_{\Delta_q}\right\}.$$

Now,

$$\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}} \leq \left(1-\prod_{q=1}^{n}\left(1-\max_q\left\{{\mu_{\Delta_q}^{l}}^{3}\right\}\right)^{\omega_q}\right)^{\frac{1}{3}}$$

$$=\left(1-\left(1-\max_q\left\{{\mu_{\Delta_q}^{l}}^{3}\right\}\right)^{\sum_{q=1}^{n}\omega_p}\right)^{\frac{1}{3}}$$

$$=\left(\max_q\left\{{\mu_{\Delta_q}^{l}}^{3}\right\}\right)^{\frac{1}{3}}=\max_q\left\{\mu_{\Delta_q}^{l}\right\}.$$

Hence,

$$\left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}},\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \leq \max_q\left\{\mu_{\Delta_q}\right\},$$

and

$$\left(\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q},\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right) \geq \left(\prod_{q=1}^{n}\left(\min_q\left\{\gamma_{\Delta_q}^{l}\right\}\right)^{\omega_q},\prod_{q=1}^{n}\left(\min_q\left\{\gamma_{\Delta_q}^{u}\right\}\right)^{\omega_q}\right)$$

$$=\min_q\left\{\gamma_{\Delta_q}^{l},\gamma_{\Delta_q}^{u}\right\}=\min_q\left\{\gamma_{\Delta_q}\right\}$$

$$\Rightarrow \left(\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right) \geq \min_q\left\{\gamma_{\Delta_q}\right\}.$$

Again,

$$\left(\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right) \leq \max_q\left\{\gamma_{\Delta_q}\right\}.$$

Thus,

$\min_q\left\{\mu_{\Delta_q}\right\} \leq \mu_{\Delta_q} \leq \max_q\left\{\mu_{\Delta_q}\right\}$ and $\min_q\left\{\gamma_{\Delta_q}\right\} \leq \gamma_{\Delta_q} \leq \max_q\left\{\gamma_{\Delta_q}\right\}$.

Similarly, for any k, we found

$\min_k\{\mu_{\chi_k}\} \leq \mu_{\chi_k} \leq \max_k\{\mu_{\chi_k}\}$ and $\min_k\{\gamma_{\chi_k}\} \leq \gamma_{\chi_k} \leq \max_k\{\gamma_{\chi_k}\}$.

Now

$$\left\langle\left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right]\right\rangle \geq \min_q\{\mu_{\chi_k}\}\min_k\{\mu_{\chi_k}\};$$

$$\left\langle\left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right]\right\rangle \leq \max_q\left\{\mu_{\Delta_q}\right\}\max_k\{\mu_{\chi_k}\}$$

and

$$\left(\left[\prod_{q=1}^{n}\left(\gamma_{\chi_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\chi_q}^{u}\right)^{\omega_{q\omega}}\right]^3 + \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3 - \left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3\right)^{\frac{1}{3}}$$

$$\geq \left(\left(\min_q\left\{\gamma_{\Delta_q}\right\}\right)^3 + \left(\min_k\{\gamma_{\chi_k}\}\right)^3 - \left(\min_q\left\{\gamma_{\chi_q}\right\}\right)^3 \cdot \left(\min_k\{\gamma_{\chi_k}\}\right)^3\right)^{\frac{1}{3}}.$$

Now

$GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_q\}, \{\chi_1, \chi_2, \cdots, \chi_t\})$

$$= \left\langle \left[\left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}} \right] \cdot \left[\left(1 - \prod_{k=1}^{t} \left(1 - \left(\mu_{\chi_k}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n} \left(1 - \left(\mu_{\chi_k}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}} \right], \left(\left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^{t} \left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t} \left(\gamma_{\chi_k}^{u}\right)^{\rho_k}\right]^3 - \left[\prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n} \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{t} \left(\gamma_{\chi_k}^{l}\right)^{\rho_k}, \prod_{k=1}^{t} \left(\gamma_{\chi_k}^{u}\right)^{\rho_k}\right]^3 \right)^{\frac{1}{3}} \right\rangle$$

$$\leq \left(\max_q \left\{\mu_{\Delta_q}\right\}\right)\left(\max_k \left\{\mu_{\chi_k}\right\}\right),$$

$$\left(\left(\min_q \left\{\mu_{\Delta_q}\right\}\right)^3 + \left(\min_k \left\{\gamma_{\chi_k}\right\}\right)^3 - \left(\min_q \left\{\mu_{\Delta_q}\right\}\right)^3 \cdot \left(\min_k \left\{\gamma_{\chi_k}\right\}\right)^3 \right)^{\frac{1}{3}} = \Delta^{+} \otimes \chi^{+} \tag{12}$$

and

$$GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\}) \geq \left(\min_q \left\{\mu_{\Delta_q}\right\}\right)\left(\min_k \left\{\mu_{\chi_k}\right\}\right);$$

$$\left(\left(\max_q \left\{\gamma_{\Delta_q}\right\}\right)^3 + \left(\max_k \left\{\mu_{\chi_k}\right\}\right)^3 - \left(\max_q \left\{\gamma_{\Delta_q}\right\}\right)^3 \cdot \left(\max_k \left\{\mu_{\chi_k}\right\}\right)^3 \right)^{\frac{1}{3}} = \Delta^{-} \otimes \chi^{-} \tag{13}$$

Thus, from Equations (12) and (13), we have
$\Delta^{-} \otimes \chi^{-} \leq GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\}) \leq \Delta^{+} \otimes \chi^{+}$.

3) Monotonicity

Suppose $\Delta_q = \left(\mu_{\Delta_q}, \gamma_{\Delta_q}\right) = \left\langle \left[\mu_{\Delta_q}^{l}, \mu_{\Delta_q}^{u}\right]\left[\gamma_{\Delta_q}^{l}, \gamma_{\Delta_q}^{u}\right] \right\rangle$, and $\Delta_q^{*} = \left(\mu_{\Delta_q^{*}}, \gamma_{\Delta_q^{*}}\right) = \left\langle \left[\mu_{\Delta_q^{*}}^{l}, \mu_{\Delta_q^{*}}^{u}\right]\left[\gamma_{\Delta_q^{*}}^{l}, \gamma_{\Delta_q^{*}}^{u}\right] \right\rangle$, $q = 1, 2, \cdots, n$ be two distinct groups of IVFFNs. Also, $\chi_k = \left(\mu_{\chi_k}, \gamma_{\chi_k}\right) = \langle [\mu_{\chi_k}^{l}, \mu_{\chi_k}^{u}][\gamma_{\chi_k}^{l}, \gamma_{\chi_k}^{u}] \rangle$, and $\chi_k^{*} = \left(\mu_{\chi_k^{*}}, \gamma_{\chi_k^{*}}\right) = \left\langle \left[\mu_{\chi_k^{*}}^{l}, \mu_{\chi_k^{*}}^{u}\right]\left[\gamma_{\chi_k^{*}}^{l}, \gamma_{\chi_k^{*}}^{u}\right] \right\rangle$, $k = 1, 2, \cdots, t$ be other two groups of IVFFNs, with $\mu_{\Delta_q} \leq \mu_{\Delta_q^{*}}$; $\gamma_{\Delta_q} \geq \gamma_{\Delta_q^{*}}$, and $\mu_{\chi_k} \leq \mu_{\chi_k^{*}}$; $\gamma_{\chi_k} \geq \gamma_{\chi_k^{*}}$, for any q and k. Thus, Equation (14) represents

$$GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\}) \leq GGIVFFWA(\{\Delta_1^{*}, \Delta_2^{*}, \cdots, \Delta_n^{*}\}, \{\chi_1^{*}, \chi_2^{*}, \cdots, \chi_t^{*}\}) \tag{14}$$

Proof: For any q, $\mu_{\Delta_q} \leq \mu_{\Delta_q^{*}}$; $\gamma_{\Delta_q} \geq \gamma_{\Delta_q^{*}}$,

that is $\mu_{\Delta_q}^{l} \leq \mu_{\Delta_q^{*}}^{l}$, $\mu_{\Delta_q}^{u} \leq \mu_{\Delta_q^{*}}^{u}$; $\gamma_{\Delta_q}^{l} \geq \gamma_{\Delta_q^{*}}^{l}$, $\mu_{\Delta_q}^{u} \geq \mu_{\Delta_q^{*}}^{u}$, we have

$$\left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q} \geq \left(1 - \left(\mu_{\Delta_q^{*}}^{l}\right)^3\right)^{\omega_q}, \left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q} \geq \left(1 - \left(\mu_{\Delta_q^{*}}^{u}\right)^3\right)^{\omega_q};$$

$$\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q} \geq \left(\gamma_{\Delta_q^{*}}^{l}\right)^{\omega_q}, \left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \geq \left(\gamma_{\Delta_q^{*}}^{u}\right)^{\omega_q}.$$

Thus,

$$\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}} \le \left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q^*}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}},$$

$$\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}} \le \left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q^*}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}};$$

$$\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q} \ge \prod_{q=1}^{n}\left(\gamma_{\Delta_q^*}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q} \ge \prod_{q=1}^{n}\left(\gamma_{\Delta_q^*}^{u}\right)^{\omega_q}.$$

For any k, $\mu_{\chi_k} \le \mu_{\chi_k^*}$; $\gamma_{\chi_k} \le \gamma_{\chi_k^*}$,
that is, $\mu_{\chi_k}^{l} \le \mu_{\chi_k^*}^{l}$, $\mu_{\chi_k}^{u} \le \mu_{\chi_k^*}^{u}$; $\gamma_{\chi_k}^{l} \ge \gamma_{\chi_k^*}^{l}$, $\gamma_{\chi_k}^{u} \ge \gamma_{\chi_k^*}^{u}$, becomes

$$\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{l}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}} \le \left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k^*}^{l}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}},$$

$$\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{u}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}} \le \left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k^*}^{u}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}};$$

$$\prod_{k=1}^{t}\left(\gamma_{\chi_k}^{l}\right)^{\rho_k} \ge \prod_{k=1}^{t}\left(\gamma_{\chi_k^*}^{l}\right)^{\rho_k}, \ \prod_{k=1}^{t}\left(\gamma_{\chi_k}^{u}\right)^{\rho_k} \ge \prod_{k=1}^{t}\left(\gamma_{\chi_k^*}^{u}\right)^{\rho_k}.$$

Then,

$$\left\langle\left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}},\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}\right]\cdot\left[\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{l}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}},\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k}^{u}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}}\right]\right\rangle$$

$$\le\left\langle\left[\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q^*}^{l}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}},\left(1-\prod_{q=1}^{n}\left(1-\left(\mu_{\Delta_q^*}^{u}\right)^{3}\right)^{\omega_q}\right)^{\frac{1}{3}}\right]\cdot\left[\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k^*}^{l}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}},\left(1-\prod_{k=1}^{t}\left(1-\left(\mu_{\chi_k^*}^{u}\right)^{3}\right)^{\rho_k}\right)^{\frac{1}{3}}\right]\right\rangle$$

and

$$\left(\left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3 - \left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3\right)^{\frac{1}{3}}$$

$$\geq \left(\left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3 - \left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3\right)^{\frac{1}{3}}.$$

Now,

$GGIVFFWA(\{\Delta_1, \Delta_2, \cdots, \Delta_n\}, \{\chi_1, \chi_2, \cdots, \chi_t\})$

$$= \left\langle \left[\left(1 - \prod_{q=1}^{n}\left(1 - \left(\mu_{\Delta_q}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n}\left(1 - \left(\mu_{\Delta_q}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1 - \prod_{k=1}^{t}\left(1 - (\mu_{\chi_k}^{l})^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^{t}\left(1 - (\mu_{\chi_k}^{u})^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right], \left(\left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3 - \left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q}^{u}\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{t}(\gamma_{\chi_k}^{l})^{\rho_k}, \prod_{k=1}^{t}(\gamma_{\chi_k}^{u})^{\rho_k}\right]^3\right)^{\frac{1}{3}} \right\rangle$$

$$\leq \left\langle \left[\left(1 - \prod_{q=1}^{n}\left(1 - \left(\mu_{\Delta_q^*}^{l}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^{n}\left(1 - \left(\mu_{\Delta_q^*}^{u}\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\right] \cdot \left[\left(1 - \prod_{k=1}^{t}\left(1 - \left(\mu_{\chi_k^*}^{l}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^{t}\left(1 - \left(\mu_{\chi_k^*}^{u}\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\right], \left(\left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q^*}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q^*}^{u}\right)^{\omega_q}\right]^3 + \left[\prod_{k=1}^{t}\left(\gamma_{\chi_k^*}^{l}\right)^{\rho_k}, \prod_{k=1}^{t}\left(\gamma_{\chi_k^*}^{u}\right)^{\rho_k}\right]^3 - \left[\prod_{q=1}^{n}\left(\gamma_{\Delta_q^*}^{l}\right)^{\omega_q}, \prod_{q=1}^{n}\left(\gamma_{\Delta_q^*}^{u}\right)^{\omega_q}\right]^3 \cdot \left[\prod_{k=1}^{t}\left(\gamma_{\chi_k^*}^{l}\right)^{\rho_k}, \prod_{k=1}^{t}\left(\gamma_{\chi_k^*}^{u}\right)^{\rho_k}\right]^3\right)^{\frac{1}{3}} \right\rangle$$

$= GGIVFFWA(\{\Delta_1^*, \Delta_2^*, \cdots, \Delta_n^*\}, \{\chi_1^*, \chi_2^*, \cdots, \chi_t^*\})$.

4. MCDM Problem Based on GGIVFFWA Operator to Credit Risk Evaluation

This section introduces a distinctive MCDM approach for credit risk evaluation process for potential partners based on the GGIVPFWA operator. The principle of GGIVPFWA operator can be outlined as follows:

(a) The adviser initially provides criterion data through IVFFSs and subsequently presents the interval-valued Fermatean fuzzy (IVFF) decision matrix for various alternatives.

(b) The decision-makers construct the GGIVPs matrix by utilizing GGIVs to assess the exactness of the criterion information.

Subsequently, the criterion values and GGIVPs undergo aggregation through the GGIVFFWA operator, resulting in the determined values. Eventually, the aggregated values corresponding of each alternative are analysed to determine the optimal potential strategic partners.

4.1 Proposed Method

Consider the sets $A = \{A_p\}$, where, $p = 1, 2, \cdots, m$, and $C = \{C_q\}$, where, $q = 1, 2, \cdots, n$, representing the strategic partners of alternatives and criteria, respectively. Additionally, let $a = a_h, h = 1, 2, \cdots, s$, and $d = d_k, k = 1, 2, \cdots, t$, denotes the advisor and decider sets. Define $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T, \varpi = (\varpi_1, \varpi_2, \cdots, \varpi_s)^T$, and $\rho = (\rho_1, \rho_2, \cdots, \rho_k)^T$ as the weighted vectors of criteria, advisor, and decider, respectively. To select the strategic partners using the GGIVFFWA operator, the MCDM technique performs the following steps:

Step 1. The advisor, denoted as a_h, initially transforms the criterion data through the use of IVFFSs representation. The advisor furnishes an IVFF decision matrix, denoted as $\mathbb{F}^h = \left(\Delta_{pq}^h\right)_{m\times n}$, defined as

$$\mathbb{F}^h = \left(\Delta_{pq}^h\right)_{m\times n} = \left(\mu_{\Delta_{pq}^h}, \gamma_{\Delta_{pq}^h}\right)_{m\times n} = \left\langle \left[\mu_{\Delta_{pq}^h}^l, \mu_{\Delta_{pq}^h}^u\right], \left[\gamma_{\Delta_{pq}^h}^l, \gamma_{\Delta_{pq}^h}^u\right] \right\rangle_{m\times n} \tag{15}$$

Step 2. The deciders construct the GGIVPs matrix, denoted as $\mathbb{G}$, with dimension $s \times t$, by transposing the accuracy assessments of each adviser's submission of criteria information, represented as χ_{hk} , defined as

$$\mathbb{G} = (\chi_{hk})_{s\times t}^T = \left(\mu_{\chi_{hk}}, \gamma_{\chi_{hk}}\right)_{s\times t}^T = \left\langle \left[\mu_{\chi_{hk}}^l, \mu_{\chi_{hk}}^u\right], \left[\gamma_{\chi_{hk}}^l, \gamma_{\chi_{hk}}^u\right] \right\rangle_{s\times t}^T \tag{16}$$

Step 3. To normalize the IVFF decision matrix, $\left(\mathbb{F}^h\right)_{m\times n}$ is

$$\overline{\mathbb{F}}^h = \bar{\Delta}_{pq}^h = \left(\bar{\mu}_{\Delta_{pq}^h}, \bar{\gamma}_{\Delta_{pq}^h}\right) = \begin{cases} \left\langle \left[\mu_{\Delta_{pq}^h}^l, \mu_{\Delta_{pq}^h}^u\right], \left[\gamma_{\Delta_{pq}^h}^l, \gamma_{\Delta_{pq}^h}^u\right] \right\rangle, q \in B; \\ \left\langle \left[\gamma_{\Delta_{pq}^h}^l, \gamma_{\Delta_{pq}^h}^u\right], \left[\mu_{\Delta_{pq}^h}^l, \mu_{\Delta_{pq}^h}^u\right] \right\rangle, q \in C. \end{cases} \tag{17}$$

where, B and C are the benefit and cost criterion.

Step 4. To generate the aggregate values $\mathbb{C}^h(A) = \left(\mathbb{C}^h(A_1), \mathbb{C}^h(A_2), \cdots, \mathbb{C}^h(A_m)\right)$, where $\mathbb{C}^h\left(A_p\right)$ represents the aggregate data for each alternative A_p , we employ the GGIVFFWA operator. This operator combines the aggregate $\bar{\Delta}_{pq}^h$, where $p = 1, 2, \cdots, m;\ q = 1, 2, \cdots, n$ and χ_{hk} , where, $h = 1, 2, \cdots, s;\ k = 1, 2, \cdots, t$.

$$\mathbb{C}^h\left(A_p\right) = GGIVFFWA\left(\{\bar{\Delta}_{p1}^h, \bar{\Delta}_{p2}^h, \cdots, \bar{\Delta}_{pn}^h\}, \{\chi_{h1}, \chi_{h2}, \cdots, \chi_{ht}\}\right)$$

$$= \Bigg\langle \Bigg[\Bigg[\left(1 - \prod_{q=1}^n \left(1 - \left(\mu_{\bar{\Delta}_{pq}^h}^l\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}, \left(1 - \prod_{q=1}^n \left(1 - \left(\mu_{\bar{\Delta}_{pq}^h}^u\right)^3\right)^{\omega_q}\right)^{\frac{1}{3}}\Bigg] \cdot \Bigg[\Bigg(1 - \prod_{k=1}^t \Bigg(1 -$$

$$\left(\mu_{\bar{\chi}_{hk}^h}^l\right)^3\Bigg)^{\rho_k}\Bigg)^{\frac{1}{3}}, \left(1 - \prod_{k=1}^t \left(1 - \left(\mu_{\bar{\chi}_{hk}^h}^u\right)^3\right)^{\rho_k}\right)^{\frac{1}{3}}\Bigg]\Bigg], \left(\left[\prod_{q=1}^n \left(\gamma_{\bar{\Delta}_{pq}^h}^l\right)^{\omega_q}, \prod_{q=1}^n \left(\gamma_{\bar{\Delta}_{pq}^h}^u\right)^{\omega_p}\right]^3 +\right.$$

$$\left[\prod_{k=1}^{t}\left(\gamma_{\chi_{hk}^{h}}^{l}\right)^{\rho_k},\prod_{q=1}^{n}\left(\gamma_{\chi_{hk}^{h}}^{u}\right)^{\rho_k}\right]^3-\left[\prod_{q=1}^{n}\left(\gamma_{\bar{\Delta}_{pq}^{h}}^{l}\right)^{\omega_q},\prod_{q=1}^{n}\left(\gamma_{\bar{\Delta}_{pq}^{h}}^{u}\right)^{\omega_p}\right]^3\cdot$$
$$\left.\left[\prod_{k=1}^{t}\left(\gamma_{\chi_{hk}^{h}}^{l}\right)^{\rho_k},\prod_{q=1}^{n}\left(\gamma_{\chi_{hk}^{h}}^{u}\right)^{\rho_k}\right]^3\right)^{\frac{1}{3}}\Bigg\rangle \tag{18}$$

Step 5. Compute the total aggregate data of each alternative, and

$$\mathbb{C}\left(A_p\right)=\sum_{h=1}^{s}\varpi_h\mathbb{C}^h\left(A_p\right),\ p=1,2,\cdots,m \tag{19}$$

Step 6. Calculate the ranking value for each alternative partner using total aggregate values of each alternative. A partner's choice A_p is directly proportional to its ranking value. A higher-ranking value implies a more desirable and superior alternative.

Step 7. Assemble all alternatives in descending order of ranking value, then select the one with the highest rank as the ideal alternative.

4.2 Case Study

In illustrating the credit risk assessment process for strategic potential partners, we employ a case study to elucidate the proposed MCDM technique form on the GGIVFFWA operator within the structure of IVFFSs. To contextualize this, imagine a financial activity aiming to assess the credit risk of three partners, denoted as A_1, A_2, and A_3, as a preliminary investigation.

In alignment with the financial enterprise's development strategy, the manager chooses to evaluate potential strategic partners under three criteria: Character (C_1), Capacity (C_2), and Condition (C_3). Notably, during the decision evaluation process, both the experts i.e. advisers a_h , $h = 1,2,3$ and deciders $d_k, k = 1,2$, may experience uncertainty in articulating precise judgements. Therefore, to model such ambiguity and hesitation, the preference values are expressed using interval-valued Fermatean fuzzy numbers (IVFFNs) in Rani and Mishra (2022).

Considering the operators employed in MCDM techniques, illustrative scenarios can effectively demonstrate the capability of IVFFSs to represent uncertain information. This section emphasizes the effect of the GGIVPs on credit risk evaluation outcomes and the functioning of the GGIVFFWA operator. The weighted vectors for advisers are specified as $\varpi = (0.37, 0.19, 0.44)^T$, for criteria as $\omega = (0.25, 0.35, 0.40)^T$, and for deciders as $\rho = (0.75, 0.25)^T$.

Interval-valued Fermatean fuzzy decision-matrices $\mathbb{F}^h = \left(\Delta_{pq}^h\right)_{3\times3}$, provided by adviser $a_h, h = 1, 2, 3$, are in **Table 1** using Equation (15). The GGIVPs matrix created by two deciders, is represented in **Table 2** as $\mathbb{G} = (\chi_{hk})_{3\times2}^T$ by Equation (16).

The matrix $\mathbb{F}^h = \left(\Delta_{pq}^h\right)_{3\times3}$, does not need to be normalized, as all criteria are benefit criteria. If the criteria are benefit and cost, then it should be normalized by Equation (17). The GGIVFFWA method, based on the parameters follows a straight forward calculation procedure. Initially, aggregated values are computed by Equation (18) to combine the variables χ_{hk} , $k = 1, 2$ and Δ_{pq}^h , $q = 1, 2, 3$. Consequently, the cumulative values are obtained through this process; these are

$\mathbb{C}^1(A_1) = \langle[0.0737, 0.2107], [0.3591, 0.5420]\rangle$, $\mathbb{C}^1(A_2) = \langle[0.0800, 0.2158], [0.3308, 0.5238]\rangle$,

$\mathbb{C}^1(A_3) = \langle[0.1546, 0.3711], [0.2848, 0.4802]\rangle, \mathbb{C}^1(A_4) = \langle[0.0737, 0.2935], [0.3118, 0.5050]\rangle$.

Similarly,
$\mathbb{C}^2(A_1) = \langle[0.1856, 0.3857], [0.2705, 0.4686]\rangle, \mathbb{C}^2(A_2) = \langle[0.0723, 0.2920], [0.2876, 0.5238]\rangle$,

$\mathbb{C}^2(A_3) = \langle[0.1000, 0.3030], [0.2844, 0.5238]\rangle, \mathbb{C}^2(A_4) = \langle[0.0756, 0.3223], [0.3191, 0.4987]\rangle$.

and
$\mathbb{C}^3(A_1) = \langle[0.0433, 0.1651], [0.5131, 0.6791]\rangle, \mathbb{C}^3(A_2) = \langle[0.0311, 0.1743], [0.5111, 0.7265]\rangle$,

$\mathbb{C}^3(A_3) = \langle[0.0460, 0.1862], [0.5781, 0.7152]\rangle, \mathbb{C}^3(A_4) = \langle[0.0401, 0.1507], [0.5679, 0.7187]\rangle$.

By Equation (19), total aggregate values are
$\mathbb{C}(A_1) = \langle[0.0817, 0.2239], [0.4101, 0.5883]\rangle, \mathbb{C}(A_1) = \langle[0.0570, 0.2120], [0.4019, 0.6195]\rangle$,

$\mathbb{C}(A_1) = \langle[0.0964, 0.2768], [0.4138, 0.5919]\rangle, \mathbb{C}(A_1) = \langle[0.0593, 0.2361], [0.4259, 0.5979]\rangle$.

Table 1. Interval-valued Fermatean fuzzy decision-matrix.

Advisers	Alternative\Criteria	C_1	C_2	C_3
a_1	A_1	([0.3, 0.5], [0.2, 0.4])	([0.1. 0.4], [0.4, 0.5])	([0.2, 0.3], [0.3, 0.4])
	A_2	([0.2, 0.4], [0.3, 0.4])	([0.1, 0.2], [0.3, 0.4])	([0.3, 0.5], [0.2, 0.4])
	A_3	([0.1, 0.3], [0.5, 0.6])	([0.2, 0.7], [0.1, 0.3])	([0.6, 0.8], [0.1, 0.2])
	A_4	([0.3, 0.4], [0.5, 0.6])	([0.1, 0.3], [0.3, 0.5])	([0.2, 0.7], [0.1, 0.2])
a_2	A_1	([0.2, 0.4], [0.4, 0.5])	([0.4, 0.5], [0.3, 0.5])	([0.7, 0.8], [0.1, 0.2])
	A_2	([0.3, 0.6], [0.2, 0.7])	([0.1, 0.4], [0.2, 0.3])	([0.2, 0.5], [0.3, 0.6])
	A_3	([0.4, 0.7], [0.2, 0.3])	([0.1, 0.4], [0.3, 0.5])	([0.3, 0.4], [0.2, 0.5])
	A_4	([0.2, 0.3], [0.4, 0.6])	([0.3, 0.6], [0.2, 0.3])	([0.1, 0.6], [0.3, 0.4])
a_3	A_1	([0.2, 0.4], [0.1, 0.6])	([0.4, 0.5], [0.2, 0.4])	([0.2, 0.4], [0.3, 0.5])
	A_2	([0.3, 0.5], [0.4, 0.6])	([0.1, 0.5], [0.2, 0.7])	([0.2, 0.4], [0.1, 0.5])
	A_3	([0.5, 0.7], [0.2, 0.3])	([0.1, 0.4], [0.4, 0.6])	([0.1, 0.2], [0.7, 0.8])
	A_4	([0.4, 0.5], [0.2, 0.5])	([0.2, 0.4], [0.5, 0.6])	([0.2, 0.3], [0.5, 0.6])

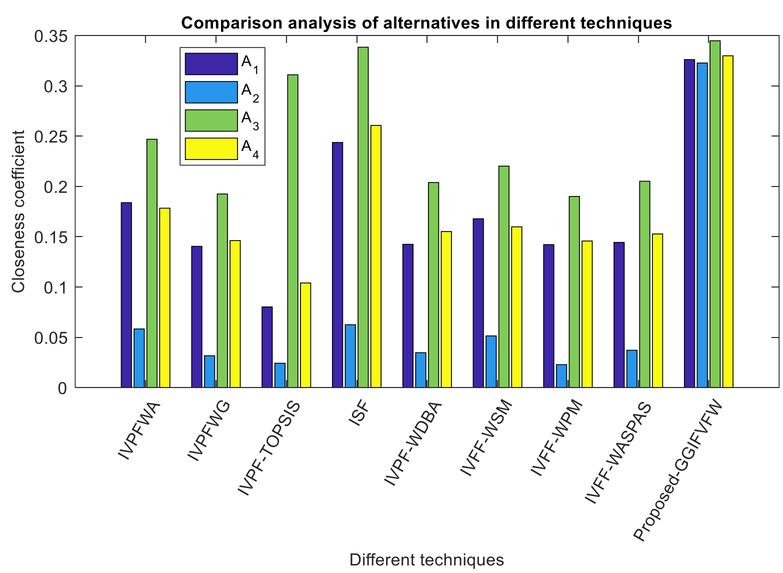

Figure 1. Comparison analysis of alternative in different technique.

Table 2. Group generalized interval-valued parameters.

Deciders\Advisers	a_1	a_2	a_3
d_1	([0.2, 0.4], [0.3, 0.5])	([0.3, 0.6], [0.2, 0.4])	([0.1, 0.3], [0.6, 0.7])
d_2	([0.5, 0.7], [0.2, 0.3])	([0.4, 0.5], [0.3, 0.4])	([0.2, 0.5], [0.3, 0.4])

Table 3. Comparison analysis.

Techniques	A_1	A_2	A_3	A_4	Ranking order	Optimal alternative
IVPFWA [Peng and Yang, 2016]	0.1838	0.0583	0.2468	0.1783	$A_3 > A_1 > A_4 > A_2$	A_3
IVPFWG [Garg, 2017]	0.1404	0.0317	0.1924	0.1461	$A_3 > A_4 > A_1 > A_2$	A_3
IVPF-TOPSIS [Garg, 2017]	0.0802	0.0242	0.3109	0.1040	$A_3 > A_1 > A_4 > A_2$	A_3
ISF [Garg, 2018]	0.2436	0.0625	0.3383	0.2606	$A_3 > A_4 > A_1 > A_2$	A_3
IVPF-WDBA [Peng and Li, 2019]	0.1424	0.0347	0.2038	0.1551	$A_3 > A_4 > A_1 > A_2$	A_3
IVFF-WSM [Rani and Mishra, 2022]	0.1678	0.0514	0.2201	0.1598	$A_3 > A_1 > A_4 > A_2$	A_3
IVFF-WPM [Rani and Mishra, 2022]	0.1421	0.0228	0.1900	0.1457	$A_3 > A_4 > A_1 > A_2$	A_3
IVFF-WASPAS [Rani and Mishra, 2022]	0.1443	0.0371	0.2051	0.1527	$A_3 > A_4 > A_1 > A_2$	A_3
Proposed-GGIVFFWA	0.3260	0.3226	0.3447	0.3298	$A_3 > A_4 > A_1 > A_2$	A_3

The ranking values of each potential strategic partner are as $\mathbb{R}(\mathbb{C}(A_1)) = 0.3260, \mathbb{R}(\mathbb{C}(A_2)) = 0.3226, \mathbb{R}(\mathbb{C}(A_3)) = 0.3447,$ and $\mathbb{R}(\mathbb{C}(A_4)) = 0.3298$. So, the sequence of ranking order is $A_3 > A_4 > A_1 > A_2$.

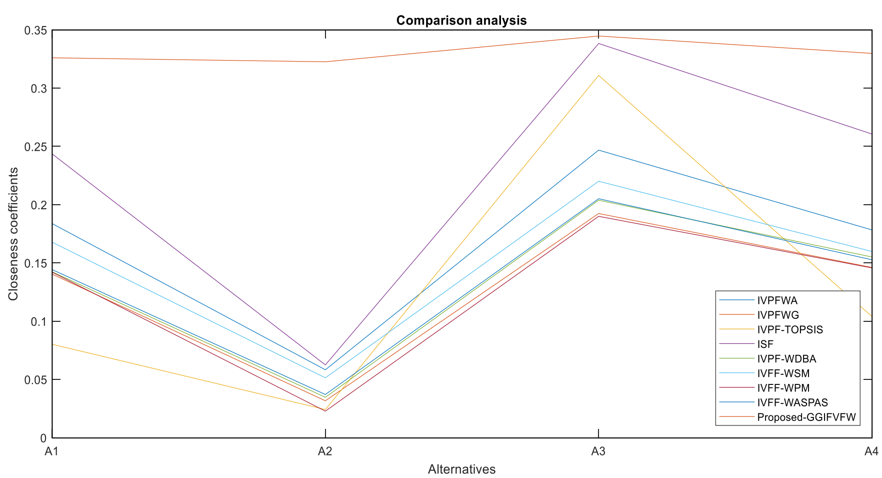

Figure 2. Comparison analysis of alternative in specific techniques.

5. Result Analysis

In order to evaluate the utility of the suggested method, we provide a comparison analysis along with an explanation of the ranking values of each alternative in credit risk evaluation. The purpose of this analysis is to illustrate the proposed approach's effectiveness, consistency, and practical relevance when applied to real-world problems with decision-making that are ambiguous and vague.

5.1 Discussion

In situations where the data are ambiguous or inconsistent, interval-valued sets provide a more accurate representation than traditional fuzzy values. These representations enable decision-makers to incorporate hesitancy and insufficient information in a more realistic way. Therefore, incorporating IVFFSs becomes essential. The presence of membership degrees (MD) and non-membership degrees (ND) imposes limitations on the currently used FFS techniques and restricts their ability to handle parametrizations effectively. As a result, standard FFS-based models may fail to represent the full range of uncertainty present in credit risk evaluation situations.

Consequently, the MCDM problem is directly related to proposed representation. In this paper, we introduce the GGIVFFWA algorithm. A credit risk assessment with strategic partners is used to demonstrate the effectiveness of the proposed operator with the MCDM framework. This application simulates a real-world decision-making environment in which several criteria and expert opinions must be examined simultaneously. Selecting the most suitable strategic partner represents a practical, real-world application that the algorithm can successfully address. These selection problems are important in financial decision-making because they have a direct impact on organizational risk exposure and long-term stability.

The results conform the validity and applicability of the proposed methodology. The generated ranking results clearly discriminate between competing alternatives and provide useful insights for decision-makers.

After ranking all available alternatives using the proposed algorithm, alternative A_3 is identified as the optimal partner. The rankings obtained using existing methods are also compared, and the best option determined by the proposed technique aligns with these results, further confirming the reliability and robustness of the approach. This consistency across different methodologies enhances confidence in the proposed framework.

Using the proposed GGIVFFWA operator, the author deduced the ranking values of the alternatives as 0.3260, 0.3226, 0.3447, and 0.3298, respectively. These values show clear preference differences between the choices, even in an ambiguous decision context. Based on the results, the final rating of the alternatives $A_3 > A_4 > A_1 > A_2$. Therefore, A_3 is identified as the optimal option for credit risk evaluation. The results indicate that A_3 demonstrates exceptional performance across all categories, notably in terms of credibility, repayment capacity, and external conditions.

5.2 Comparison with Existing Techniques

To identify the most suitable alternatives, we compare the proposed method in this section with several established approaches reported in the literature, including those by Garg (2017, 2018), Peng and Yang (2016), Peng and Li (2019), and Rani and Mishra (2022). All the results for the alternatives are summarized in **Table 3**. This comparison study gives a thorough examination of the suggested method's relative performance against well-known decision-making strategies. Also, the relative closeness index exhibits a consistent trend of increase or decreases among the alternatives presented in **Table 3**. Consequently, the proposed method provides consistent clarification of MCDM problems in both FFS and IVFFS environments. This displays the consistency of the ranking results across several fuzzy representations.

The corresponding closeness values, in comparison with the conventional MCDM approach, are illustrated in **Figure 1**. Moreover, the comparison of these methods indicates that alternative A_3 emerges as the optimal choice, presented in **Figures 1** and **Figure 2**. The agreement of the optimal alternative across different methods further confirms the strength of the proposed GGIVFFWA-based framework. Since, this problem includes only four alternatives, the results produced by the proposed method may not appear fully conclusive. However, as the number of alternatives increases, the distinctions in the outcomes will become more pronounced. This designates that the proposed method is particularly appropriate for large-scale decision-making problems.

For large sets of criteria and alternatives, the GGIVFFWA operator methodology outperforms the other techniques reported in Garg (2017, 2018), Peng and Yang (2016), Peng and Li (2019), and Rani and Mishra (2022). The improved performance can be attributed to the suggested operator's ability to successfully aggregate interval-valued data while keeping uncertainty characteristics. The effective and reasonable MCDM problems cab solved with the GGIVFFWA technique due to realistic data processing. As a result, the suggested method provides a dependable and effective decision-making tool for complicated credit risk assessment and strategic partner selection problems.

6. Conclusions

By introducing the hypothesis of IVFFSs, this work aims to increase the scope of uncertain information that decision-making experts can represent by enabling them to specify the MD and ND for a class of possibilities using interval-based descriptions. We have outlined the fundamental operational rules of IVFFSs in terms of IVFFNs. Firstly, with respect to the effective postulates of idempotency, boundedness, and monotonicity, the GGIVFFWA operator has been studied using group generalized parameters. Secondly, a GGIVFFWA-based methodology has been developed to address the MCDM problem in terms of two types of decision-makers that is, decider and adviser. Lastly, a case study on the credit evaluation process for potential partners in risk management has been presented using IVFFSs to illustrate the efficacy and usability of the developed methodology. Lastly, compared the proposed method with several existing techniques, demonstrating its convenience and effectiveness.

We intend to develop additional aggregation operators for IVFFSs in the future research. Furthermore, we will apply the GGIVFFWA operator to introduce new MCDM problems and investigate several practical applications, including image processing, project selection, health care, the energy sector, and transportation, electronic waste management, etc. Additionally, the proposed methodology can be expanded to sustainability-focused decision-making problems like plastic waste management (Mandal and Seikh, 2023) and biomedical waste management (Seikh and Mandal, 2023), thereby representing its potential implication in complex environmental and public health sectors.

Conflicts of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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AI Disclosure

The author(s) declare that no assistance is taken from generative AI to write this article.

References

Akram, M., Niaz, Z., & Feng, F. (2023). Extended CODAS method for multi attribute group decision-making based on 2-tuple linguistic Fermatean fuzzy Hamacher aggregation operators. *Granular Computing, 8*(3), 441-466. https://doi.org/10.1007/s41066-022-00332-3

Akram, M., Shahzadi, G., & Ahmadini, A.A.H. (2020). Decision-making framework for an effective sanitizer to reduce COVID-19 under Fermatean fuzzy environment. *Journal of Mathematics, 2020*(1), 3263407. https://doi.org/10.1155/2020/3263407

Atanassov, K.T. (1999). Interval valued intuitionistic fuzzy sets. In *Intuitionistic Fuzzy Sets. Studies in Fuzziness and Soft Computing* (pp. 139-177). Physica-Verlag HD, Heidelberg. https://doi.org/10.1007/978-3-7908-1870-3_2

Aydemir, S.B., & Gunduz, S.Y. (2020). Fermatean fuzzy TOPSIS method with Dombi aggregation operators and its application in multi-criteria decision making. *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology, 39*(1), 851-869. https://doi.org/10.3233/JIFS-191763

Aydogan, H., & Ozkir, V. (2024). A Fermatean fuzzy MCDM method for selection and ranking problems: case studies. *Expert Systems with Applications*, *237*(Part C), 121628. https://doi.org/10.1016/j.eswa.2023.121628

Baral, S.P., Parida, P.K., & Sahoo, D. (2025). An enhanced TOPSIS-based framework for MCDM with uncertain weights: application to e-waste recycling partner selection. *Results in Control and Optimization*, *19*, 100545. https://doi.org/10.1016/j.rico.2025.100545

Boffa, S., Ciucci, D., & Marsala, C. (2025). Extending intuitionistic operations, orderings, and entropy measures on generalized fuzzy orthopartitions. *Fuzzy Sets and Systems*, *513*, 109381. https://doi.org/10.1016/j.fss.2025.109381

Bouraima, M.B., Gore, A., Ayyildiz, E., Yalcin, S., Badi, I., Kiptum, C.K., & Qiu, Y. (2023). Assessing of causes of accidents based on a novel integrated interval valued Fermatean fuzzy methodology: towards a sustainable construction site. *Neural Computing and Applications*, *35*(29), 21725-21750. https://doi.org/10.1007/s00521-023-08948-5

Cui, Y., Liu, W., Rani, P., & Alrasheedi, M. (2021). Internet of Things (IoT) adoption barriers for the circular economy using Pythagorean fuzzy SWARA-CoCoSo decision-making approach in the manufacturing sector. *Technological Forecasting and Social Change*, *171*, 120951. https://doi.org/10.1016/j.techfore.2021.120951

Deb, R., & Roy, S. (2021). A software defined network information security risk assessment based on Pythagorean fuzzy sets. *Expert Systems with Applications*, *183*, 115383. https://doi.org/10.1016/j.eswa.2021.115383

Deng, Z., & Wang, J. (2021). Evidential Fermatean fuzzy multicriteria decision making based on Fermatean fuzzy entropy. *International Journal of Intelligent Systems*, *36*(10), 5866-5886. https://doi.org/10.1002/int.22534

Dorfeshan, Y., & Mousavi, S.M. (2020). A novel interval type-2 fuzzy decision model based on two new versions of relative preference relation-based MABAC and WASPAS methods (with an application in aircraft maintenance planning), *Neural Computing and Applications*, *32*(8), 3367-3385. https://doi.org/10.1007/s00521-019-04184-y

Fei, L., & Deng, Y. (2020). Multi-criteria decision making in Pythagorean fuzzy environment. *Applied Intelligence*, *50*(2), 537-561. https://doi.org/10.1007/s10489-019-01532-2

Gao, H. (2018). Pythagorean fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, *35*(2), 2229-2245. https://doi.org/10.3233/JIFS-172262

Garg, H. (2017). A new improved score function of an interval-valued Pythagorean fuzzy set based TOPSIS method. *International Journal for Uncertainty Quantification*, *7*(5), 463-474.

Garg, H. (2018). A linear programming method based on an improved score function for interval-valued Pythagorean fuzzy numbers and its application to decision-making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *26*(1), 67-80. https://doi.org/10.1142/S0218488518500046

Garg, H. (2019). New logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications. *International Journal of Intelligent Systems*, *34*(1), 82-106. https://doi.org/10.1002/int.22043

Garg, H., Shahzadi, G., & Akram, M. (2020). Decision-making analysis based on Fermatean fuzzy Yager aggregation operators with application in COVID 19 testing facility. *Mathematical Problems in Engineering*, *2020*(1), 7279027. https://doi.org/10.1155/2020/7279027

Gocer, F. (2024). A novel extension of Fermatean fuzzy sets into group decision making: a study for prioritization of renewable energy technologies. *Arabian Journal for Science and Engineering*, *49*(3), 4209-4228. https://doi.org/10.1007/s13369-023-08307-5

Hadi, A., Khan, W., & Khan, A. (2021). A novel approach to MADM problems using Fermatean fuzzy Hamacher aggregation operators. *International Journal of Intelligent Systems*, *36*(7), 3464-3499. https://doi.org/10.1002/int.22423

Ibrahim, H.A., Qahtan, S., Zaidan, A.A., Deveci, M., Hajiaghaei-Keshteli, M., Mohammed, R.T., & Alamoodi, A.H. (2024). Sustainability in mobility for autonomous vehicles over smart city evaluation; using interval-valued Fermatean fuzzy rough set-based decision-making model. *Engineering Applications of Artificial Intelligence, 129,* 107609. https://doi.org/10.1016/j.engappai.2023.107609

Jeevaraj, S. (2021). Ordering of interval-valued Fermatean fuzzy sets and its applications. *Expert Systems with Applications*, *185*, 115613. https://doi.org/10.1016/j.eswa.2021.115613

Joshi, B.P. (2019). Pythagorean fuzzy average aggregation operators based on generalized and group-generalized parameter with application in MCDM problems. *International Journal of Intelligent Systems*, *34*(5), 895-919. https://doi.org/10.1002/int.22080

Keshavarz-Ghorabaee, M., Amiri, M., Hashemi-Tabatabaei, M., Zavadskas, E.K., & Kaklauskas, A. (2020). A new decision-making approach based on Fermatean fuzzy sets and WASPAS for green construction supplier evaluation. *Mathematics*, *8*(12), 2202. https://doi.org/10.3390/math8122202

Kirisci, M. (2023). New cosine similarity and distance measures for Fermatean fuzzy sets and TOPSIS approach. *Knowledge and Information Systems*, *65*(2), 855-868. https://doi.org/10.1007/s10115-022-01776-4

Kirisci, M. (2024a). Interval-valued Fermatean fuzzy based risk assessment for self-driving vehicles. *Applied Soft Computing*, *152*, 111265. https://doi.org/10.1016/j.asoc.2024.111265

Kirisci, M. (2024b). Multiple criteria group decision-making using Fermatean fuzzy soft matrices and their medical applications. *Journal of Uncertain Systems*, *17*(2), 2440007. https://doi.org/10.1142/S1752890924400075

Li, J., Gao, K., & Rong, Y. (2024). A hybrid multi-criteria group decision method ology based on fairly operators and EDAS method under interval-valued Fermatean fuzzy environment. *Granular Computing, 9*(2), 41. https://doi.org/10.1007/s41066-024-00463-9

Luqman, A., & Shahzadi, G. (2023). Multi-attribute decision-making for electronic waste recycling using interval-valued Fermatean fuzzy Hamacher aggregation operators. *Granular Computing*, *8*(5), 991-1012. https://doi.org/10.1007/s41066-023-00363-4

Mandal, U., & Seikh, M.R. (2022). Interval-valued fermatean fuzzy TOPSIS method and its application to sustainable development program. In: Saraswat, M., Sharma, H., Balachandran, K., Kim, J.H., Bansal, J.C. (eds) *Congress on Intelligent Systems*. Springer Nature, Singapore, pp. 783-796. https://doi.org/10.1007/978-981-16-9113-3_57

Mandal, U., & Seikh, M.R. (2023). Interval-valued spherical fuzzy MABAC method based on Dombi aggregation operators with unknown attribute weights to select plastic waste management process. *Applied Soft Computing*, *145*, 110516. https://doi.org/10.1016/j.asoc.2023.110516

Mishra, A.R., & Rani, P. (2021). Multi-criteria healthcare waste disposal location se lection based on Fermatean fuzzy WASPAS method. *Complex and Intelligent Systems*, *7*(5), 2469-2484.

Mishra, A.R., Rani, P., Deveci, M., Gokasar, I., Pamucar, D., & Govindan, K. (2023). Interval-valued Fermatean fuzzy heronian mean operator-based decision making method for urban climate change policy for transportation activities. *Engineering Applications of Artificial Intelligence*, *124*, 106603. https://doi.org/10.1016/j.engappai.223.106603

Mohmood, T., Ali, Z., Ullah, K., Khan, Q., AISalman, H., Gumaei, A., & Rahman, S.M.M. (2022). Complex Pythagorean fuzzy aggregation operators based on confidence levels and their applications. *Mathematical Biosciences and Engineering*, *19*(1), 1078-1107. https://doi.org/10.3934/mbe.2022050

Pamučar, D., Sremac, S., Stević, Ž., Ćirović, G., & Tomić, D. (2019). New multi-criteria LNN WASPAS model for evaluating the work of advisors in the transport of hazardous goods. *Neural Computing and Applications*, *31*(2), 5045-5068. https://doi.org/10.1007/s00521-018-03997-7

Parida, P.K. (2018). A multi-attributes decision making model based on fuzzy TOPSIS for positive and negative ideal solutions with ranking order. *International Journal of Civil Engineering and Technology*, *9*(6), 190-198.

Parida, P.K. (2020). Some generalized results on multi-criteria decision making model using fuzzy TOPSIS technique. In: Dehuri, S., Mishra, B., Mallick, P., Cho, S.B., Favorskaya, M. (eds) *Biologically Inspired Techniques in Many-Criteria Decision Making*. Springer International Publishing, Cham, pp. 189-199. https://doi.org/10.1007/978-3-030-39033-4_18

Peng, X., & Li, W. (2019). Algorithms for interval-valued Pythagorean fuzzy sets in emergency decision making based on multiparametric similarity measures and WDBA. *IEEE Access*, *7*, 7419-7441. https://doi.org/10.1109/ACCESS.2018.2890097

Peng, X., & Yang, Y. (2016). Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. *International Journal of Intelligent Systems, 31*(5), 444-487. https://doi.org/10.1002/int.21790

Qi, G., Atef, M., & Yang, B. (2024). Fermatean fuzzy covering-based rough set and their applications in multi-attribute decision-making. *Engineering Applications of Artificial Intelligence*, *127*(Part A), 107181. https://doi.org/10.1016/j.engappi.2023.107181

Qin, H., Peng, Q., & Ma, X. (2024). A novel interval-valued Fermatean fuzzy three-way decision making method with probability dominance relations. *Expert Systems with Applications*, *242*, 122727. https://doi.org/10.1016/j.eswa.2023.122727

Raji-Lawal, H.Y., Akinwale, A.T., Folorunsho, O., & Mustapha, A.O. (2020). Decision support system for dementia patients using intuitionistic fuzzy similarity measure. *Soft Computing Letters*, *2*, 100005. https://doi.org/10.1016/j.socl.2020.100005

Ramalingam, S. (2018). Fuzzy interval-valued multi-criteria based decision making for ranking features in multi-model 3D face recognition. *Fuzzy Sets and Systems*, *337*, 25-51. https://doi.org/10.1016/j.fss. 2017.06.002

Rani, P., & Mishra, A.R. (2022). Interval-valued Fermatean fuzzy sets with multi criteria weighted aggregated sum product assessment-based decision analysis framework. *Neural Computing and Applications*, *34*(10), 8051-8067. https://doi.org/10.1007/s00521-021-06782-1

Rani, P., Mishra, A.R., Deveci, M., & Antucheviciene, J. (2022). New complex proportional assessment approach using Einstein aggregation operators and improved score function for interval-valued Fermatean fuzzy sets. *Computers & Industrial Engineering, 169*, 108165. https://doi.org/10.1016/j.cie.2022.108165

Sahoo, D., Parida, P.K., & Pati, B. (2024). Efficient fuzzy multi-criteria decision-making for optimal college location selection: a comparative study of min-max fuzzy TOPSIS approach. *Results in Control and Optimization*, *15*, 100422. https://doi.org/10.1016/j.rico.2024.100422

Sahoo, D., Parida, P.K., Baral, S.P., & Pati, B. (2025). An innovative aggregation operator for enhanced decision-making: a study on interval-valued Pythagorean fuzzy soft sets in material selection. *Applied Soft Computing*, *172*, 112888. https://doi.org/10.1016/j.asoc.2025.112888

Sahoo, D., Parida, P.K., Baral, S.P., & Sahoo, S.K. (2023). A generalized fuzzy TOPSIS technique in multi-criteria decision-making for evaluation of temperature. In: Buyya, R., Misra, S., Leung, YW., Mondal, A. (eds) *Proceedings of International Conference on Advanced Communications and Machine Intelligence*. Springer Nature, Singapore, pp. 47-58. https://doi.org/10.1007/978-981-99-2768-5_7

Seikh, M.R., & Chatterjee, P. (2024a). Determination of best renewable energy sources in India using SWARA-ARAS in confidence level-based interval-valued Fermatean fuzzy environment. *Applied Soft Computing*, *155*, 111495. https://doi.org/10.1016/j.asoc.2024.111495

Seikh, M.R., & Chatterjee, P. (2024b). Identifying sustainable strategies for electronic waste management utilizing confidence-based group decision-making method in interval valued Fermatean fuzzy environment. *Engineering Applications of Artificial Intelligence*, *135*, 108701. https://doi.org/10.1016/j.engappai.2024.108701

Seikh, M.R., & Mandal, U. (2023). Interval-valued Fermatean fuzzy Dombi aggregation operators and SWARA based PROMETHEE II method to bio-medical waste management. *Expert Systems with Applications*, *226*, 120082. https://doi.org/10.1016/j.eswa.2023.120082

Senapati, T., & Yager, R.R. (2020). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, *11*(2), 663-674. https://doi.org/10.1007/s12652-019-01377-0

Shete, P.C., Ansari, Z.N., & Kant, R. (2020). A Pythagorean fuzzy AHP approach and its application to evaluate the enablers of sustainable supply chain innovation. *Sustainable Production and Consumption*, *23*, 77-93. https://doi.org/10.1016/j.spc.2020.05.001

Simic, V., Gokasar, I., Deveci, M., & Isik, M. (2023). Fermatean fuzzy group decision making based CODAS approach for taxation of public transit investments. *IEEE Transactions on Engineering Management*, *70*(12), 4233-4248. https://doi.org/10.1109/TEM.2021.3109038

Wei, G., & Lu, M. (2018). Pythagorean fuzzy power aggregation operators in multiple attribute decision making. *International Journal of Intelligent Systems*, *33*(1), 169-186. https://doi.org/10.1002/int.21946

Xing, Y., Zhang, R., Wang, J., & Zhu, X. (2018). Some new Pythagorean fuzzy Choquet-Frank aggregation operators for multi-attribute decision making. *International Journal of Intelligent System*, *33*(11), 2189-2215.

Yager, R.R. (2014). Pythagorean membership grades in multicriteria decision making, *IEEE Transaction on Fuzzy System*, *22*(4), 958-965. https://doi.org/10.1109/TFUZZ.2013.2278989

Yager, R.R. (2016). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems, 25*(5), 1222-1230.

Zadeh, L.A. (1965). Fuzzy Sets. *Information and Control*, *8*(3), 338-353.

Zou, X.-Y., Chen, S.-M., & Fan, K.-Y. (2020). Multiple attribute decision making using improved intuitionistic fuzzy weighted geometric operators of intuitionistic fuzzy values. *Information Sciences*, *535*, 242-253.

Zulqarnain, R.M., Xin, X.L., Garg, H., & Khan, W.A. (2021). Aggregation operators of Pythagorean fuzzy soft sets with their application for green supplier chain management. *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, *40*(3), 5545-5563. https://doi.org/10.3233/JIFS-202781