International Journal of Mathematical, Engineering and Management Sciences
Vol. 10, No. 5, 1248-1265, 2025
https://doi.org/10.33889/IJMEMS.2025.10.5.060

Ram Arti Publishers

Epidemics and Predator-Prey Interactions: A Three-dimensional Discrete Model Perspective

Manisha Yadav
Department of Mathematics,
Faculty of Applied and Basic Sciences, SGT University, Gurugram, 122505, Haryana, India.
E-mail: yadavmanisha98@gmail.com

Pradeep Malik
Department of Mathematics,
Faculty of Applied and Basic Sciences, SGT University, Gurugram, 122505, Haryana, India.
Corresponding author: pradeepmaths@gmail.com

Ashish Tiwari
Department of Mathematics,
Faculty of Applied and Basic Sciences, SGT University, Gurugram, 122505, Haryana, India.
E-mail: ashish.tiwari41@gmail.com

(Received on March 19, 2025; Revised on April 15, 2025; Accepted on April 24, 2025)

Abstract
This research paper comprehensively examines a discretized three-dimensional predator-prey model with disease and prey refuge. The model incorporates vital dynamics such as predation, infection, prey population growth, and refuge population growth to capture the intricate interactions among predators, prey, and refuge in ecological systems. The study investigates the long-term behaviour of predator, prey, and refuge populations through equilibrium analysis, stability analysis, and numerical simulations. Equilibrium points are identified, and their stability properties are assessed, providing insights into the system's critical points. The condition for Neimark-Sacker bifurcation is discussed. Numerical simulations validate the theoretical analysis, providing a deeper understanding of the model's dynamics. Moreover, fractional calculus methods are employed to enhance the modeling and analysis of the system.

Keywords- Three-dimensional discrete model, Stability, Prey refuge, Neimark-Sacker Bifurcation.

1. Introduction

An eco-epidemiological model is a mathematical framework that combines ecological and epidemiological principles to study the interactions between infectious diseases and the population dynamics of their hosts within an ecosystem. It aims to understand how ecological factors, such as species interactions, environmental conditions, and population dynamics, influence the transmission and spread of infectious diseases (Holt, 1984). This type of model considers both the dynamics of the pathogen (disease-causing agent) and the host population. It explores how changes in the environment and host population can affect disease transmission, as well as how the disease itself can impact the dynamics of the host population and the ecosystem as a whole. The analysis and management of disease have both benefited greatly from the use of mathematical models (Power et al., 2015). In an eco-epidemiological model with prey refuge, the interactions between predators and prey are influenced by the presence of an infectious disease. The disease can affect the population sizes and behaviors of both predators and prey, leading to complex dynamics and feedback loops within the ecosystem (Holt et al., 1994). A diseased predator-prey model with prey refuge is a mathematical framework used to examine the interactions between predator and prey populations, taking into account both a disease affecting the predators and a safe refuge for the prey. The foundational dynamics of this model are frequently embodied by differential equations like as the Lotka-Volterra equations, which describe how population changes over time as a function of factors like reproduction and predation. In this more complex model, differential equations incorporate the impacts of the disease and the refuge, illustrating how these elements influence population dynamics over time. These equations are typically solved through numerical methods or computer simulations, allowing researchers to analyze how the refuge and the disease affect both predator and prey populations. The refuge provides a safe area for prey to avoid predators and reproduce, which helps sustain their population. This can result in a larger prey population, potentially supporting a higher predator population. However, the disease affecting predators can reduce their efficiency in hunting, affecting the overall balance. The prey refuge is important for providing resilience, affecting prey availability and, in turn, influencing the dynamics of the predator population. Wang and Ma (2012) studied an eco-epidemiological system with prey refuges and disease in prey. They investigated the stability dynamics of the core system, shifting their focus from a small-scale analysis to a more comprehensive, global understanding. Many researchers have done interesting work on these eco-epidemiological models (Anderson & May, 1986; Hadeler & Freedman, 1989; Venturino, 1995; Chattopadhyay & Arino, 1999; Chattopadhyay & Bairagi, 2001; Venturino, 2002; Hethcote et al., 2004; Liu, 2011; Niu et al., 2011; Shi et al., 2011; Naik et al., 2024). Ma et al. (2009) investigated the effects of refuges used by prey on a predator-prey interaction with a class of functional responses are studied by using the analytical approach and they proved that the effects of refuges can stabilize the interior equilibrium point of the considered model, and destabilize it under a very restricted set of conditions which is disagreement with previous results in this field. Some of the empirical and theoretical work have investigated the effect of prey refuges and drawn a conclusion that the refuges used by prey have a stabilizing effect on the considered interactions and prey extinction can be prevented by the addition of refuges (McNair, 1986; Hochberg & Holt, 1995; Ruxton, 1995; Olivars & Jiliberto, 2003; Kar, 2005; Huang et al., 2006). Recent studies, such as Gaber et al. (2025), have emphasized the role of ecological mechanisms, including the Allee effect and prey refuge, in stabilizing eco-epidemiological models. Their findings underscore the relevance of incorporating such mechanisms in predator-prey dynamics, motivating our approach in the present work (Gaber et al., 2025). Many scientists and researchers are interested in fractional calculus and its applications as it demonstrates more earthy and accurate situations than the system of integer order (Hegazi & Matouk, 2011; Matouk, 2011; Machado & Mata, 2015; Singh et al., 2019; Naik et al., 2020; Singh & Deolia, 2020).

This type of model helps researchers understand the complex dynamics between predators, prey, diseases, and refuges, and can provide insights into strategies for managing ecosystems and controlling disease outbreaks. Fractional calculus methods provide a flexible and potent approach to model and analyze intricate systems. They offer a more precise depiction of system dynamics and contribute to advancements in diverse scientific and engineering fields. These methods enable researchers to capture non-local behaviors, improve accuracy, extend classical results, tackle control and optimization problems, enhance signal and image processing, and analyze time series data with long memory or fractal characteristics. By applying fractional differentiation, the model effectively reflects the memory and long-term dependencies often found in real ecological systems. In prey-predator models, this approach offers a significant benefit by accounting for the historical interactions between species. Unlike conventional methods that focus solely on present population levels, fractional models incorporate past population trends. This makes it possible to better represent extended ecological behaviors, like delayed responses from predators or ongoing environmental influences. Consequently, the models offer a more accurate and comprehensive framework for analyzing complex ecological phenomena such as stability, oscillations, and bifurcation patterns (Mahmoudi & Eskandari, 2024). Fractional-order differential equations have proven effective in modeling various scientific phenomena, particularly those that cannot be adequately described by conventional equation types (Barhoom & Al-Nassir, 2021). In summary, fractional calculus methods offer a versatile and powerful framework for understanding complex systems and driving progress across various disciplines. This study investigated an eco-epidemiological model that involves infection and refuge within the prey population. The model details the interactions between diseased prey and their refuge as follows:

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x+y}{k}\right)-\frac{axz}{\alpha+x}-\phi yx\,,\\
\frac{dy}{dT} &= \phi yx-d_1y-\frac{b(1-n)yz}{\alpha+(1-n)y}\,,\\
\frac{dz}{dT} &= -d_2z+\frac{cb(1-n)yz}{\alpha+(1-n)y}+\frac{ca_1xz}{\alpha+x}
\end{aligned} \tag{1}$$

With positive initial conditions $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$. All parameters are taken non-negative in Equation (1). The detailed information about parameters are provided in **Table 1**.

Table 1. Parameters in the model.

Symbol	Nomenclature
x	Susceptible Prey population
y	Infected Predator population
z	Predator population
r_1	Growth rate of prey
b	Infected Prey Predation rate
k	Environmental Carrying capacity
r_2	Growth rate of predator
α_1	Half saturation constant
c	Prey to Predator Conversion coefficient
a_1	Susceptible Prey Predation rate
d_1	Mortality rate of infected prey
d_2	Mortality rate of Predator Population
ϕ	Rate of infection
n	Refuge constant

Scaling the variables to reduce the number of variables involved in the system 1 as $p=\frac{x}{k}, q=\frac{y}{k}, \eta=\frac{z}{k}$ and to consider the non-dimensional time $t = k\phi T$, the transformation leads to a non-dimensional system. The system 1 becomes,

$$\begin{aligned}
\frac{dp}{dt} &= rp(1-p-q)-pq-\frac{ap\eta}{\alpha+p},\\
\frac{dq}{dt} &= qp-dq-\frac{\omega(1-n)q\eta}{\alpha+(1-n)q}\,,\\
\frac{d\eta}{dt} &= -\delta\eta+\frac{c\omega(1-n)q\eta}{a+(1-n)q}+\frac{cap\eta}{\alpha+p}\,.
\end{aligned}$$

Here conditions are $r=\frac{r_1}{\phi k}$, $\alpha=\frac{\alpha_1}{\phi k}$, $d=\frac{d_1}{\phi k}$, $\omega=\frac{b}{\phi k}$, $a=\frac{a_1}{\phi k}$ and $\delta=\frac{d_2}{\phi k}$ with given initial conditions $p(0) \geq 0, q(0) \geq 0$ and $\eta(0) \geq 0$.

The above function is defined in R_+^3.

2. Fractional-Order Mathematical Model and its Discretization

Consider the following pre-predator model:

$$\begin{aligned}\frac{dp}{dt} &= rp(1-p-q)-pq-\frac{ap\eta}{\alpha+p},\\ \frac{dq}{dt} &= qp-dq-\frac{\omega(1-n)q\eta}{\alpha+(1-n)q},\\ \frac{d\eta}{dt} &= -\delta\eta+\frac{c\omega(1-n)q\eta}{a+(1-n)q}+\frac{cap\eta}{\alpha+p}\end{aligned} \tag{2}$$

Now, the fractional-order form of the model (2) is given as

$$\begin{aligned}D_t^{\theta}(p) &= rp(1-p-q)-pq-\frac{ap\eta}{\alpha+p},\\ D_t^{\theta}(q) &= qp-dq-\frac{\omega(1-n)q\eta}{\alpha+(1-n)q},\\ D_t^{\theta}(\eta) &= -\delta\eta+\frac{c\omega(1-n)q\eta}{a+(1-n)q}+\frac{cap\eta}{\alpha+p}\end{aligned} \tag{3}$$

where, θ is the fractional order satisfying $\theta \in (0,1)$ and $D = \frac{d^{\theta}}{d\theta}$ is in sense of Caputo derivative.

2.1 Discretization Process

The method given below is a generalization of the forward Euler discretization method in integer order. The process of discretizing a fractional-order system (3) with a piecewise constant argument is outlined as follows:

$$\begin{aligned}D_t^{\theta}(p) &= rp\left(\left[\tfrac{t}{h}\right]h\right)\left(1-p\left(\left[\tfrac{t}{h}\right]h\right)-q\left(\left[\tfrac{t}{h}\right]h\right)\right)-p\left(\left[\tfrac{t}{h}\right]h\right)q\left(\left[\tfrac{t}{h}\right]h\right)-\frac{ap\left(\left[\frac{t}{h}\right]h\right)\eta\left(\left[\frac{t}{h}\right]h\right)}{\alpha+p\left(\left[\frac{t}{h}\right]h\right)},\\ D_t^{\theta}(q) &= q\left(\left[\tfrac{t}{h}\right]h\right)p\left(\left[\tfrac{t}{h}\right]h\right)-dq\left(\left[\tfrac{t}{h}\right]h\right)-\frac{\omega(1-n)q\left(\left[\frac{t}{h}\right]h\right)\eta\left(\left[\frac{t}{h}\right]h\right)}{\alpha+(1-n)q\left(\left[\frac{t}{h}\right]h\right)},\\ D_t^{\theta}(\eta) &= -\delta\eta\left(\left[\tfrac{t}{h}\right]h\right)+\frac{c\omega(1-n)q\left(\left[\frac{t}{h}\right]h\right)\eta\left(\left[\frac{t}{h}\right]h\right)}{a+(1-n)q\left(\left[\frac{t}{h}\right]h\right)}+\frac{cap\eta\left(\left[\frac{t}{h}\right]h\right)}{\alpha+p\left(\left[\frac{t}{h}\right]h\right)}\end{aligned} \tag{4}$$

Now, let t satisfies the inequality $0 \le t \le h$, which implies $0 \le \frac{t}{h} \le 1$. It is obtained as

$$\begin{aligned}D_t^{\theta}(p_1) &= rp_0(1-p_0-q_0)-p_0q_0-\frac{ap_0\eta_0}{\alpha+p_0},\\ D_t^{\theta}(q_1) &= q_0p_0-dq_0-\frac{\omega(1-n)q_0\eta_0}{\alpha+(1-n)q_0},\\ D_t^{\theta}(\eta_1) &= -\delta\eta_0+\frac{c\omega(1-n)q_0\eta_0}{a+(1-n)q_0}+\frac{cap\eta_0}{\alpha+p_0}\end{aligned} \tag{5}$$

Which has solution

$$\begin{aligned}p_1(t) &= p_0+I^{\theta}\left[rp_0(1-p_0-q_0)-p_0q_0-\frac{ap_0\eta_0}{\alpha+p_0}\right],\\ q_1(t) &= q_0+I^{\theta}\left[q_0p_0-dq_0-\frac{\omega(1-n)q_0\eta_0}{\alpha+(1-n)q_0}\right],\\ \eta_1(t) &= \eta_0+I^{\theta}\left[-\delta\eta_0+\frac{c\omega(1-n)q_0\eta_0}{a+(1-n)q_0}+\frac{cap\eta_0}{\alpha+p_0}\right]\end{aligned} \tag{6}$$

where, $I^{\theta} = \frac{h^{\theta}}{\Gamma(1+\theta)}$. Again, let $t \in (h, 2h)$, which makes $1 \le \frac{t}{h} < 2$. It gives

$$
\begin{aligned}
D_t^{\theta}(p_2) &= rp_1(1-p_1-q_1)-p_1q_1-\tfrac{ap_1\eta_1}{\alpha+p_1},\\
D_t^{\theta}(q_2) &= q_1p_1-dq_1-\tfrac{\omega(1-n)q_1\eta_1}{\alpha+(1-n)q_1},\\
D_t^{\theta}(\eta_2) &= -\delta\eta_1+\tfrac{c\omega(1-n)q_1\eta_1}{a+(1-n)q_1}+\tfrac{cap\eta_1}{\alpha+p_1}
\end{aligned}
\tag{7}
$$

The solution to the aforementioned system is presented as follows:

$$
\begin{aligned}
p_2(t) &= p_1+\tfrac{(t-h)^{\theta}}{\Gamma(1+\theta)}\left[rp_1(1-p_1-q_1)-p_1q_1-\tfrac{ap_1\eta_1}{\alpha+p_1}\right],\\
q_2(t) &= q_1+\tfrac{(t-h)^{\theta}}{\Gamma(1+\theta)}\left[q_1p_1-dq_1-\tfrac{\omega(1-n)q_1\eta_1}{\alpha+(1-n)q_1}\right],\\
\eta_2(t) &= \eta_1+\tfrac{(t-h)^{\theta}}{\Gamma(1+\theta)}\left[-\delta\eta_1+\tfrac{c\omega(1-n)q_1\eta_1}{a+(1-n)q_1}+\tfrac{cap\eta_1}{\alpha+p_1}\right]
\end{aligned}
\tag{8}
$$

Similarly, repeating the discretization process i times yields the following result:

$$
\begin{aligned}
p_{(i+1)}(t) &= p_i(ih)+\tfrac{(t-ih)^{\theta}}{\Gamma(1+\theta)}[rp_i(ih)(1-p_i(ih)-q_i(ih))-p_i(ih)q_i(ih)\\
&\quad -\tfrac{ap_i(ih)\eta_i(ih)}{\alpha+p_i(ih)}],\\
q_{(i+1)}(t) &= q_i(ih)+\tfrac{(t-ih)^{\theta}}{\Gamma(1+\theta)}\left[q_i(ih)p_i(ih)-dq_i(ih)-\tfrac{\omega(1-n)q_i(ih)\eta_i(ih)}{\alpha+(1-n)q_i(ih)}\right],\\
\eta_{(i+1)}(t) &= \eta_i(ih)+\tfrac{(t-ih)^{\theta}}{\Gamma(1+\theta)}\left[-\delta\eta_i(ih)+\tfrac{c\omega(1-n)q_i(ih)\eta_1}{a+(1-n)q_i(ih)}+\tfrac{cap\eta_i(ih)}{\alpha+p_i(ih)}\right]
\end{aligned}
\tag{9}
$$

where, $t\in[ih,(i+1)h]$: For $t\in\frac{i+1}{h}$, the Equation (9) is transformed into the discrete-time model given below:

$$
\begin{aligned}
p_{(i+1)}(t) &= p_i+\tfrac{h^{\theta}}{\Gamma(1+\theta)}\left[rp_i(1-p_i-q_i)-p_iq_i-\tfrac{ap_i\eta_i}{\alpha+p_i}\right],\\
q_{(i+1)}(t) &= q_i+\tfrac{h^{\theta}}{\Gamma(1+\theta)}\left[q_ip_i-dq_i-\tfrac{\omega(1-n)q_i\eta_i}{\alpha+(1-n)q_i}\right],\\
\eta_{(i+1)}(t) &= \eta_i+\tfrac{h^{\theta}}{\Gamma(1+\theta)}\left[-\delta\eta_i+\tfrac{c\omega(1-n)q_i\eta_i}{a+(1-n)q_i}+\tfrac{cap\eta_i}{\alpha+p_i}\right]
\end{aligned}
\tag{10}
$$

2.2 Existence and Uniqueness of the Solution

Before proving the existence and uniqueness of theorems, some lemma (Barhoom & Al-Nassir, 2021) are discussed below, which are useful in concerned theorems.

Lemma 1: 1) Suppose that $f(t)\in C[a,b]$ and $D_a^{\alpha}f(t)\in C(a,b]$ with $0<\alpha\le 1$, then we have:

$$
f(t)=f(a)+\frac{1}{\Gamma(\alpha)}(D_a^{\alpha}f)(\xi)(t-a)^{\alpha},
$$

where, $a\le\xi\le x, \forall x\in(a,b]$.

2) Let $u(t)$ be a continuous function on $[t_0,\infty]$ and satisfy

$$
D^{\alpha}{}_a u(t)\le -\lambda u(t)+\mu \tag{11}
$$

where, $0<\alpha<1$, $(\lambda,\mu)\in\mathbb{R}^2$, $\lambda\neq 0$, and $t_0\ge 0$ is the initial time. Then, its solution has the form

$$u(t) \leq \left(u_{t_0} - \frac{\mu}{\lambda}\right) E_\alpha[-\lambda(t-t_0)^\alpha] + \frac{\mu}{\lambda} \tag{12}$$

Lemma 2: *Consider the system* $(D_t^\alpha)x(t) = f(t,x)$ *for* $t > t_0$ *with the initial condition* x_{t_0}*, where* $0 < \alpha \leq 1$*,* $f: [t_0, \infty) \times \Omega \to \mathbb{R}^n$*. If* $f(t,x)$ *satisfies the locally Lipchitz condition with respect to* x*, then there exists a unique solution of the above system on* $[t_0, \infty) \times \Omega$*.*

Theorem 1: *Let* $\Omega_+ = (p,q,\eta)|p \geq 0, q \geq 0$ *and* $\eta \geq 0$ *and denotes all non negative real numbers in* R^3*, and then all solutions of the system* (3) *with* $p_0 \geq 0, q_0 \geq 0$ *and* $\eta_0 \geq 0$ *are uniformly bounded and non-negative.*

Proof: We have to show that $p(t) \geq 0$, $\forall\, t \geq 0$, assuming $p(0) > 0$ for $t = 0$ considering that $p(t) \geq 0, \forall t \geq 0$ is not true. Then there exists a constant $t_1 > 0$ such that

$$\{p(t) > 0, 0 < t < t_1, p(t) = 0, t = t_1, p(t) < 0, t > t_1 \tag{13}$$

From the Equation (3)

$$\begin{aligned} D_t^\theta(p) &= rp(1-p-q) - pq - \frac{ap\eta}{\alpha+p}, \\ D_t^\theta(q) &= qp - dq - \frac{\omega(1-n)q\eta}{\alpha+(1-n)q}, \\ D_t^\theta(\eta) &= -\delta\eta + \frac{c\omega(1-n)q\eta}{a+(1-n)q} + \frac{cap\eta}{\alpha+p} \end{aligned} \tag{14}$$

We have $(D_t^\theta)p(t)|_t = t_1 = 0$ According to Lemma 1, we have $p(t_1^+) = 0$, which contradicts the fact $p(t_1^+) < 0$, that is $p(t) < 0, t > t_1$. Therefore we have $p(t) \geq 0, \forall t \geq 0$. By the same argument $q(t) \geq 0$ $\forall \geq 0$ and $\eta \geq 0$ can get $\forall t \geq 0$. Let $U(t) = \frac{1}{a}p + \frac{1}{w}q + \frac{1}{c}\eta$ then we have

$$D^\theta U(t) = \frac{1}{a}D^\theta p(t) + \frac{1}{w}D^\theta q(t) + \frac{1}{c}D^\theta \eta(t)$$

$$= \frac{1}{a}rp(1-p) + \frac{1}{w}(-dq) + \frac{1}{c}(-\delta\eta) \text{ For each } \psi > 0,$$

we have $D^(\theta)U(t) + \psi u(t) = \frac{1}{a}rp(1-p) - \frac{dq}{w} - \frac{-\delta\eta}{c} + \frac{\psi p}{a} + \frac{\psi q}{w} + \frac{\psi\eta}{c}$

$$\leq \frac{-rp^2}{a} + \frac{(r+\psi)}{a}p + (\psi-d)\frac{q}{w} + (w-\delta)\frac{\eta}{c} \leq \frac{-1}{a}(p^2 - (1+\frac{\psi}{r}p) + (\psi-d)\frac{q}{w} + (w-\delta)\frac{\eta}{c} \leq$$
$$-\frac{1}{a}(p^2 - (1+\frac{\psi}{r}p + \frac{1}{4}(1+\frac{\psi}{r})^2) - \frac{1}{4a}(1+\frac{\psi}{r})^2 + (\psi-d)\frac{q}{w} + (\psi-\delta)\frac{\eta}{c}$$
$$= -\frac{1}{a}(p - \frac{1}{2}(1+\frac{\psi}{r}))^2 + \frac{1}{4a}(1+\frac{\psi}{r})^2 + (\psi-d)\frac{q}{w} + (\psi-\delta)\frac{\eta}{c}$$
$$\leq \frac{1}{4a}(1+\frac{\psi}{r})^2 + (\psi-d)\frac{q}{w} + (\psi-\delta)\frac{\eta}{c}.$$

Therefore $D^\theta U(t) + \psi U(t) \leq l$.

By taking $\psi < min(d,\delta)$ when $l = \frac{1}{4a}(1+\frac{\psi}{r})^2 > 0$. Now using part 2 in Lemma 1, we have,

$$U(t) \leq (U(0) - \frac{l}{\psi})E_\theta(-\psi(t-0)^\theta) + \frac{l}{\psi},$$
$$U(0) \leq E_\theta(-\psi t^\theta) + \frac{l}{\psi}(1 - E_\theta(-\psi t^\theta)).$$
$$\text{Thus } U(t) \to \frac{l}{\psi} \text{ast} \to \infty \text{ and } 0 < U(t) \leq \frac{l}{\psi}.$$

Therefore all solutions of system (3) that starts from Ω_+ are confined in region $\Omega = (p, q, \eta) \in R^3$

$$U(t) \leq (\frac{l}{\psi}) + \epsilon, \text{ for any } \epsilon > 0.$$

Now, the existence and uniqueness of the considered system (3) is discussed in next theorem.

Theorem 2: *Let χ be a sufficiently large, and then for each $S_0 = (p_0, q_0, \eta_0) \in \{(p, q, \eta) \in R^3 | max\{|p|, |q|, |\eta| \leq \chi\}\}$, then there exists a unique solutions $S = (p, q, \eta) \in \Omega$ of the fractional-order system* (3) *with initial conditions S_0, which is defined for all $t \geq 0$.*

Proof: Let $S = (p, q, \eta)$, $(\bar{S}) = (\bar{p}, \bar{q}, \bar{\eta})$ and consider a mapping $H(S) = (H_1(s), H_2(s), H_3(s))$, such that

$$\begin{aligned} H_1(S) &= rp(1-p-q) - pq - \frac{ap\eta}{\alpha+p}, \\ H_2(S) &= qp - dq - \frac{w(1-n)q\eta}{\alpha+(1-n)q}, \\ H_3(S) &= -\delta\eta + \frac{cw(1-n)q\eta}{\alpha+(1-n)q} + \frac{cap\eta}{\alpha+p} \end{aligned} \tag{15}$$

For any $S, \bar{S} \in \Omega$, we have,

$$\begin{aligned} &\| H(S) - H(\bar{S}) \| = \\ &|H_1(S) - H_1(\bar{S})| + |H_2(S) - H_2(\bar{S})| + |H_3(S) - H_3(\bar{S})| \\ &= |rp(1-p-q) - pq - \frac{ap\eta}{\alpha+p} - r\bar{p}(1-\bar{p}-\bar{q}) - \bar{p}\bar{q} - \frac{a\bar{p}\bar{\eta}}{\alpha+\bar{p}}| \\ &+|qp - dq - \frac{w(1-n)q\eta}{\alpha+(1-n)q} - \bar{q}\bar{p} - d\bar{q} - \frac{w(1-n)\bar{q}\bar{\eta}}{\alpha+(1-n)\bar{q}}| \\ &+\left|-\delta\eta + \frac{cw(1-n)q\eta}{\alpha+(1-n)q} + \frac{cap\eta}{\alpha+p} - \delta\bar{\eta} + \frac{cw(1-n)\bar{q}\bar{\eta}}{\alpha+(1-n)\bar{q}} + \frac{ca\bar{p}\bar{\eta}}{\alpha+\bar{p}}\right|. \end{aligned}$$

Since $\frac{1}{(\alpha+p)(\alpha+\bar{p})} < 1$, $\frac{1}{(\alpha+(1-n)q)(\alpha+(1-n)\bar{q})} < 1$, and $\frac{1}{(\alpha+(1-n)q)(\alpha+(1-n)\bar{q})} < 1$, and $|p|, |q|, |\eta| < \chi$, it becomes

$$\| H(S) - H(\bar{S}) \| \leq (r - 2r^2\chi)|p - \bar{p}| + d|q - \bar{q}| + \delta|\eta - \bar{\eta}| = V \| S - \bar{S} \|,$$

where, $V = \max\{r - 2r^2\chi, d, \delta\}$. Therefore, $H(S)$ satisfies the Lipchitz Condition. With the use of Lemma 2, the proof is finished.

3. Stability Analysis of the Fractional-order System

The fractional-order system (3) has the following equilibrium points:

1) The $E_0(0,0,0)$be the trivial equilibrium point.

2) $E_1(1,0,0)$be the infected prey-free and Predator-free equilibrium point.

3) $E_2(\bar{p}, 0, \bar{\eta})$ is infection-free equilibrium point
where, $\bar{p} = \frac{\alpha\delta}{ca-\delta}$ and $\bar{\eta} = \frac{\alpha c((ca-\delta)r-\alpha r)}{(ca-\delta)^2}$. The equilibrium point exists for $ca > \delta$ and $(ca-\delta)r > \alpha r$.

4) $E_3(\hat{p}, \hat{q}, 0)$ is predator-free Equilibrium point
where, $\hat{p} = d$ and $\hat{q} = \frac{(1-d)r}{r+1}$. The equilibrium point exists for $d < 1$.

5) The positive equilibrium point is $E_4(p^*, q^*, \eta^*)$ exists for $ac < \delta$, $\frac{\delta\alpha(r+1)}{ac} < p^*$, $(\delta - ac)p^* + \delta\alpha > 0$,

where, $q^* = \frac{(\delta\alpha + p^*(\delta - ac))\alpha}{(acp^* + (\omega c - \delta)(p^* + \alpha))(1-n)}$, $\eta^* = \frac{\alpha c(p^* - d)(p^* + \alpha)}{(acp^* + (\omega c - \delta)(p^* + \alpha))(1-n)}$,

and p^* stands for the quadratic equation i.e.

$$\psi_1 p^2 + \psi_2 p + \psi_3 = 0.$$

Here, $\psi_1 = r(ac + \omega c - \delta)(1-n)$,

$\psi_2 = ((c\omega - \delta)(\alpha r - r) - acr + \alpha(\delta + (\delta - ac)r)(1-n)$,

$\psi_3 = -\alpha((1-n)r(\omega c - \delta) + (cad) - \alpha\delta(1+r))$.

4. Dynamical Behaviour of Discretized Fractional Order

In this section, the dynamics of the discretized fractional-order prey-predator model are explored Equation (10). The Jacobian matrix of the Equation (10) at any equilibrium point (p^*, q^*, η^*) is as follows:

$$J(p^*, q^*, \eta^*) = \begin{pmatrix} a_{11} & a_{22} & a_{33} \\ a_{44} & a_{55} & a_{66} \\ a_{77} & a_{88} & a_{99} \end{pmatrix}.$$

where, $a_{11} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[r - 2rp^* - rq^* - q^* - \frac{\alpha a\eta^*}{(\alpha + p^*)^2}\right]$, $a_{22} = -\frac{h^\theta}{\Gamma(1+\theta)}[rp^* - p^*]$,
$a_{33} = -\frac{h^\theta}{\Gamma(1+\theta)}\left[\frac{ap^*}{\alpha + p^*}\right]$, $a_{44} = -\frac{h^\theta}{\Gamma(1+\theta)}[q^*]$, $a_{55} = 1 + -\frac{h^\theta}{\Gamma(1+\theta)}\left[p^* - d - \frac{\alpha\omega(1-n)\eta^*}{(\alpha + (1-n)q^*)^2}\right]$,
$a_{66} = -\frac{h^\theta}{\Gamma(1+\theta)}\left[\frac{\omega(1-n)q^*}{\alpha + (1-n)q^*}\right]$, $a_{77} = \frac{h^\theta}{\Gamma(1+\theta)}\left[\frac{\alpha ca\eta^*}{(\alpha + p^*)^2}\right]$, $a_{88} = \frac{h^\theta}{\Gamma(1+\theta)}\left[\frac{ac\omega(1-n)\eta^*}{(a + (1-n)q^*)^2}\right]$,
$a_{99} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[-\delta + \frac{c\omega(1-n)q^*}{a + (1-n)q^*} + \frac{cap^*}{\alpha + p^*}\right]$.

4.1 Dynamical Behaviour Around Trivial Point $E_0(0, 0, 0)$

The Jacobian matrix for $E_0(0,0,0)$ is

$$J(0,0,0) = \begin{pmatrix} 1 + \frac{rh^\theta}{\Gamma(1+\theta)} & 0 & 0 \\ 0 & \frac{dh^\theta}{\Gamma(1+\theta)} & 0 \\ 0 & 0 & \frac{-\delta h^\theta}{\Gamma(1+\theta)} \end{pmatrix}.$$

Here, the eigen values are $\lambda_1 = 1 + \frac{rh^\theta}{\Gamma(1+\theta)}$, $\lambda_2 = 1 + \frac{dh^\theta}{\Gamma(1+\theta)}$ and $\lambda_3 = 1 - \frac{\delta h^\theta}{\Gamma(1+\theta)}$.

As, it is so obvious that $|\lambda_1| > 1$ and $|\lambda_2| > 1$ for $\theta \in (0,1)$. So E_1 is source if $|\lambda_3| > 1$ i.e. $|1 - \frac{\delta h^\theta}{\Gamma(1+\theta)}| > 1$ which yields $h > \sqrt[\theta]{\frac{2\Gamma(1+\theta)}{\delta}}$.

4.2 Dynamical Behaviour Around Semi-trivial Point $E_1(1,0,0)$

The Jacobian matrix for $E_1(1,0,0)$ is

$$J(1,0,0) = \begin{pmatrix} 1 + \frac{-rh^\theta}{\Gamma(1+\theta)} & -\frac{(r-1)h^\theta}{\Gamma(1+\theta)} & -\frac{h^\theta}{\Gamma(1+\theta)}\left[\frac{a}{\alpha+1}\right] \\ 0 & 1 + \frac{(1-d)h^\theta}{\Gamma(1+\theta)} & 0 \\ 0 & 0 & 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[-\delta + \frac{ac}{\alpha+1}\right] \end{pmatrix}.$$

Here the eigen values are $\lambda_1 = 1 - \frac{rh^\theta}{\Gamma(1+\theta)}$, $\lambda_2 = 1 + \frac{h^\theta}{\Gamma(1+\theta)}[1-d]$ and $\lambda_3 = 1 + \frac{\delta h^\theta}{\Gamma(1+\theta)}\left[-\delta + \frac{ac}{\alpha+1}\right]$.

1) Sink if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ which yields $h < \sqrt[\theta]{\frac{2\Gamma(1+\theta}{r}}$, $h < \sqrt[\theta]{\frac{2\Gamma(1+\theta}{d-1}}$ and $h < \sqrt[\theta]{\frac{2\Gamma(1+\theta}{\left(\delta - \frac{ac}{\alpha+1}\right)}}$.

2) Source if $|\lambda_1| > 1$, $|\lambda_2| > 1$ and $|\lambda_3| > 1$ which yields $h > \sqrt[\theta]{\frac{2\Gamma(1+\theta}{r}}$, $h > \sqrt[\theta]{\frac{2\Gamma(1+\theta}{d-1}}$ and $h > \sqrt[\theta]{\frac{2\Gamma(1+\theta}{\left(\delta - \frac{ac}{\alpha+1}\right)}}$

3) Saddle if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $|\lambda_3| > 1$ or vice-versa which gives $h < \sqrt[\theta]{\frac{2\Gamma(1+\theta}{r}}$, $h < \sqrt[\theta]{\frac{2\Gamma(1+\theta}{d-1}}$ and $h > \sqrt[\theta]{\frac{2\Gamma(1+\theta}{\left(\delta - \frac{ac}{\alpha+1}\right)}}$.

4) Non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ or $|\lambda_3| = 1$ which yields $h = \sqrt[\theta]{\frac{2\Gamma(1+\theta}{r}}$ or $h = \sqrt[\theta]{\frac{2\Gamma(1+\theta}{d-1}}$ or $h = \sqrt[\theta]{\frac{2\Gamma(1+\theta}{\left(\delta - \frac{ac}{\alpha+1}\right)}}$.

4.3 Dynamical Behaviour Around Semi-trivial Point $E_2(\bar{p}, 0, \bar{\eta})$

The Jacobian matrix for $E_2(\bar{p}, 0, \bar{\eta})$ is

$$J(\bar{p}, 0, \bar{\eta}) = \begin{pmatrix} 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[r - 2r\bar{p} - \frac{\alpha a\bar{\eta}}{(\alpha+\bar{p})^2}\right] & -\frac{h^\theta}{\Gamma(1+\theta)}[r\bar{p} - \bar{p}] & -\frac{h^\theta}{\Gamma(1+\theta)}\left[\frac{a\bar{p}}{\alpha+\bar{p}}\right] \\ 0 & 1 + \frac{h^\theta}{\Gamma(1+\theta)}[\bar{p} - d] & 0 \\ \frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{\alpha ca\bar{\eta}}{(\alpha+\bar{p})^2}\right] & 0 & 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[-\delta + \frac{ac\bar{p}}{\alpha+\bar{p}}\right] \end{pmatrix}.$$

Here the eigen values are $\lambda_1 = 1 + \frac{h^\theta}{\Gamma(1+\theta)}(\bar{p} - d)$ and

$$\lambda_{2,3} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[r - 2r\bar{p} - \frac{\alpha a\bar{\eta}}{(\alpha+\bar{p})^2} - \delta + \frac{ca\bar{p}}{\alpha+\bar{p}} \pm \sqrt{\left(r - 2r\bar{p} - \frac{\alpha a\bar{\eta}}{(\alpha+\bar{p})^2} - \delta + \frac{ca\bar{p}}{\alpha+\bar{p}}\right)^2 - 4\left(r - 2r\bar{p} - \frac{\alpha a\bar{\eta}}{(\alpha+\bar{p})^2}\right)\left(-\delta + \frac{ca\bar{p}}{\alpha+\bar{p}}\right)}\right].$$

$$\lambda_{2,3} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}(A \pm \sqrt{A^2 - 4BC};$$

where, A = $r - 2r\bar{p} - \frac{\alpha a\bar{\eta}}{(\alpha+\bar{p})^2} - \delta + \frac{ca\bar{p}}{\alpha+\bar{p}}$; B = $r - 2r\bar{p} - \frac{\alpha a\bar{\eta}}{(\alpha+\bar{p})^2}$ and C = $-\delta + \frac{ca\bar{p}}{\alpha+\bar{p}}$

1) Sink if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ which yields $d < \frac{2\Gamma(1+\theta)}{h^\theta} + \bar{p}$ and $A < \sqrt{A^2 - 4BC}$.
2) Source if $|\lambda_1| > 1$, $|\lambda_2| > 1$ and $|\lambda_3| > 1$ which yields $d > \frac{2\Gamma(1+\theta)}{h^\theta} + \bar{p}$ and $A > \sqrt{A^2 - 4BC}$.
3) Saddle if $|\lambda_1| > 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ or vice-versa, which gives $d > \frac{2\Gamma(1+\theta)}{h^\theta} + \bar{p}$ and $A < \sqrt{A^2 - 4BC}$.
4) Non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ or $|\lambda_3| = 1$ which yields $d = \frac{2\Gamma(1+\theta)}{h^\theta} + \bar{p}$ or $A < \sqrt{A^2 - 4BC}$.

4.4 Dynamical Behavior Around Semi-trivial Point $E_3(\hat{p}, \hat{q}, 0)$.

The Jacobian matrix for $E_3(\hat{p}, \hat{q}, 0)$ is

$$J(\hat{p}, \hat{q}, 0) = \begin{pmatrix} a_{11} & a_{22} & a_{33} \\ a_{44} & a_{55} & a_{66} \\ a_{77} & a_{88} & a_{99} \end{pmatrix}.$$

where, $a_{11} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{d+1}{r+1}\right)(-rd)$, $a_{22} = -\frac{dh^\theta}{\Gamma(1+\theta)}(r-1)$, $a_{33} = -\frac{ah^\theta}{\Gamma(1+\theta)(\alpha+d)}$, $a_{44} = \frac{rh^\theta}{\Gamma(1+\theta)}\left(\frac{1-d}{r+1}\right)$, $a_{55} = 0$, $a_{66} = -\frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{w(1-n)(r-dr)}{\alpha(r+1)+(1-n)(r-dr)}\right)$, $a_{77} = 0$, $a_{88} = 0$ and $a_{99} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left[-\delta + \frac{cw(1-n)(r-rd)}{a(r+1)+(1-n)(r-rd)} + \left(\frac{cad}{\alpha+d}\right)\right]$.

Here, the eigen values are $\lambda_1 = 1 + \frac{h^\theta}{\Gamma(1+\theta}\left(-\delta + \frac{cw(1-n)(r-rd)}{a(r+1)+(1-n)(r-rd)+\frac{cad}{\alpha+d}}\right)$ and $\lambda_{2,3} = \frac{-A\pm\sqrt{A^2-4B}}{2}$;

where, $A = 1 + \frac{h^\theta}{\Gamma(1+\theta}\left(\frac{d+1}{r+1}\right)(rd)$ and $B = \left(\frac{h^\theta}{\Gamma(1+\theta)}\right)^2 rd(1-d)\frac{r-1}{r+1}$.

1) Sink if $|\lambda_1| < 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ which yields $\delta < \frac{2\Gamma(1+\theta)}{h^\theta} + M$ and $A > 2 \pm \sqrt{A^2 - 4B}$.
2) Source if $|\lambda_1| > 1$, $|\lambda_2| > 1$ and $|\lambda_3| > 1$ which yields $\delta > \frac{2\Gamma(1+\theta)}{h^\theta} + M$ and $A < 2 \pm \sqrt{A^2 - 4B}$.
3) Saddle if $|\lambda_1| > 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$ or vice-versa which gives $\delta > \frac{2\Gamma(1+\theta)}{h^\theta} + M$ and $A < 2 \pm \sqrt{A^2 - 4B}$.
4) Non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ or $|\lambda_3| = 1$ which yields $\delta = \frac{2\Gamma(1+\theta)}{h^\theta} + M$ or $A = 2 \pm \sqrt{A^2 - 4B}$.

Here, $M = \frac{cw(1-n)(r-rd)}{a(r+1)+(1-n)(r-rd)+\frac{cad}{\alpha+d}}$.

4.5 Dynamical Behaviour Around the Positive Fixed Point $E_3(p^*, q^*, \eta^*)$

The Jacobian matrix for $E_3(p^*, q^*, \eta^*)$ is

$$J(p^*, q^*, \eta^*) = \begin{pmatrix} a_{11} & a_{22} & a_{33} \\ a_{44} & a_{55} & a_{66} \\ a_{77} & a_{88} & a_{99} \end{pmatrix}.$$

Here, $a_{11} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left(r - 2rp^* - rq^* - q^* - \frac{\alpha a\eta^*}{(\alpha+p^*)^2}\right)$, $a_{22} = -\frac{h^\theta}{\Gamma(1+\theta)}(rp^* - p^*)$, $a_{33} = -\frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{ap^*}{\alpha+p^*}\right)$, $a_{44} = \frac{h^\theta}{\Gamma(1+\theta)}q^*$, $a_{55} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left(p^* - d - \frac{\alpha w(1-n)\eta^*}{(\alpha+(1-n)q^*)^2}\right)$, $a_{66} = -\frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{w(1-n)q^*}{\alpha+(1-n)q^*}\right)$, $a_{77} = \frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{\alpha ca\eta^*}{(\alpha+p^*)^2}\right)$, $a_{88} = \frac{h^\theta}{\Gamma(1+\theta)}\left(\frac{\alpha cw(1-n)\eta^*}{(\alpha+(1-n)q^*)^2}\right)$, $a_{99} = 1 + \frac{h^\theta}{\Gamma(1+\theta)}\left(-\delta + \frac{cw(1-n)q^*}{\alpha+(1-n)q^*} + \frac{cap^*}{\alpha+P^*}\right)$.

and

$$D_1 = a_{11} + a_{55} + a_{99},$$

$$D_2 = a_{11}a_{55} + a_{11}a_{99} + a_{55}a_{99} - a_{66}a_{88} - a_{44}a_{99} - a_{66}a_{77} \text{ an}$$

$$D_3 = a_{11}a_{55}a_{99} - a_{11}a_{88}a_{66} - a_{22}a_{44}a_{99} + a_{22}a_{66}a_{77} + a_{33}a_{44}a_{88} - a_{33}a_{77}a_{55}\,.$$

The characteristic polynomial for the above-mentioned Jacobian is given as:

$$F(\lambda) = \lambda^3 + D_1\lambda^2 + D_2\lambda + D_3 \tag{16}$$

Lemma 3: *The Equation* (10) *around the positive fixed point* $E_3(p^*, q^*, \eta^*)$ *is sink iff* $|D_1 + D_3| < 1 + D_2$, $|D_1 - 3D_3| < 3 - D_2$ *and* $|D_3^2 + D_2 - D_3D_1 < 1|$ (Grove & Ladas, 2004), where, D_1, D_2 and D_3 are given above.

5. Neimark-Sacker Bifurcation

Theorem 3: *The positive equilibrium* $E_3(p^*, q^*, \eta^*)$ *of the system* (3) *undergoes a Neimark-Sacker bifurcation for* $ac < \delta$, $\frac{\delta\alpha(r+1)}{ac} < p^*$, $(\delta - ac)p^* + \delta\alpha > 0$ *if the following conditions are satisfied.*

$$1 - D_2 + D_3(D_1 - D_3) = 0\,, 1 + D_2 - D_3(D_1 + D_3) > 0,$$
$$1 + D_1 + D_2 + D_3 > 0 \text{ and } 1 - D_1 + D_2 - D_3 > 0.$$

where, values of D_1, D_2 and D_3 are given above.

Proof: The characteristic polynomial for the Jacobian matrix of Equation (3) evaluated at the equilibrium point $E_3(p^*, q^*, \eta^*)$ is provided in Equation (16). Based on Lemma 4.1 from (Ali et al., 2019), for $i = 3$, the positive equilibrium $E_3(p^*, q^*, \eta^*)$ of Equation (10) will experience a Neimark-Sacker bifurcation if the conditions mentioned above are satisfied (Khan et al., 2022; Yousef et al., 2022).

6. Numerical Simulation

This section aims to validate the theoretical findings through numerical simulations conducted using MATLAB software. The computations utilize parameters $\theta = 0.4$, $r = 0.5$, $d = 0.1$, $\omega = 0.4$, $n = 0.2$, $\delta = 0.1$, $c = 0.5$, $h = 0.15$, and $\alpha = 0.21$, with initial conditions $p(0) = 0.90511$, $q(0) = 0.06358217$, and $\eta(0) = 0.62543$. **Figure 1** compares two time series in terms of θ for susceptible prey, infected prey, and predator populations. It is evident that higher refuge effect $\theta = 0.9$ leads to faster stabilization with minimal oscillations, indicating that the prey population is better protected from predation. In contrast, when $\theta = 0.6$, the fluctuations last longer before reaching a steady state, suggesting that reduced refuge results in prolonged predator-prey interactions. This comparison highlights the role of prey refuge in stabilizing population dynamics by limiting extreme variations. Ecologically, this finding underscores the importance of protected habitats or refuge zones in natural ecosystems. Refuge areas can buffer prey populations against excessive predation, thus preventing population collapse and promoting resilience. Such mechanisms are observed in coral reefs, dense forest covers, or burrowing behaviors in terrestrial systems, where prey can momentarily escape predators and recover. **Figure 2** further details the time series for these populations individually. This figure presents the time series for each population component separately, providing a detailed view of their behavioral trends. Initially, the susceptible prey population declines due to predation and disease but later stabilizes. The infected prey exhibits damped oscillations as it is influenced by both predation and disease dynamics. The predator population also undergoes fluctuations before settling into equilibrium. A higher prey refuge value $\theta = 0.9$ results in quicker stabilization and reduced predator fluctuations, whereas a lower refuge level $\theta = 0.6$ allows more persistent oscillations, leading to greater instability in predator dynamics. From an ecological viewpoint, these simulations imply that diseases alone may not be sufficient to regulate prey populations effectively if refuge availability is not accounted for. Moreover, instability in predator numbers due to fluctuating prey populations could result in trophic mismatches or eventual predator decline, a common concern in disturbed habitats or fragmented landscapes. **Figure 3** shows how the system transitions between different dynamical states as the predation rate of susceptible prey a varying between $a_{min} = 0$ and $a_{max} = 0.35$. Initially, for smaller values of a, the system displays chaotic behavior, evident from the irregular scattering of points. As a increases, the chaotic behavior temporarily diminishes, and the system transitions into periodic or quasiperiodic oscillations. However, when a exceeds 0.25, chaos reemerges, signifying a Neimark-Sacker bifurcation, where stable oscillations give way to complex dynamics. This figure emphasizes how predation intensity influences the transition between stability and chaotic behavior in the ecosystem. This transition illustrates how changes in predation intensity can act as a bifurcation parameter, fundamentally altering the stability and predictability of an ecosystem. High predation pressure could destabilize prey populations, whereas intermediate levels might foster coexistence through structured oscillations. This has direct implications for ecosystem management, where overpredation (e.g., due to invasive predators or human hunting) may push an otherwise stable ecosystem into a chaotic regime. The phase portraits are illustrated in **Figures 4, 5,** and **6**. **Figure 4** depicts three-dimensional phase plots that illustrate the trajectories of susceptible prey, infected prey, and predator populations under different levels of prey refuge θ. When $\theta = 0.3$, the system exhibits a chaotic attractor, signifying irregular and unpredictable population oscillations. Increasing θ to $\theta = 0.6$ results in more structured trajectories, indicating a transition from chaos to quasiperiodic behavior. At $\theta = 0.9$, the system stabilizes into a limit cycle, demonstrating that stronger prey refuge can suppress chaotic fluctuations and create more predictable population cycles. These results align with observed ecological patterns where ecosystems with adequate prey hiding mechanisms—such as burrows, reefs, or camouflage—tend to exhibit more regular population cycles. This reinforces the idea that enhancing habitat structure can contribute to ecological stability. **Figure 5**, presents phase plots similar to those in **Figure 4** but for a lower predation rate $a = 0.1$. Here, chaotic dynamics are more prominent when $\theta = 0.3$, but as θ increases, the system gradually shifts toward more regular oscillations. Compared to **Figure 4**, the system reaches stability faster due to the lower predation pressure. This result highlights that both reduced predation rate and stronger prey refuge effect contribute to stabilizing the system by minimizing chaotic population fluctuations and promoting predictable ecological patterns. Ecologically, this reflects that ecosystems with lower top-down control (i.e., less aggressive predation) are more likely to reach equilibrium, especially when prey have access to refuge. This can guide conservation strategies aimed at controlling predator populations to avoid destabilizing sensitive prey species. Similarly **Figure 6** presents phase portraits with a fixed at 0.2 and θ values are 0.3, 0.6, and 0.9. The consistent trend observed across simulations confirms the robust stabilizing effect of prey refuge in eco-epidemiological systems. This finding provides a theoretical foundation for ecological interventions such as the establishment of marine protected areas or wildlife corridors, where controlled predation and environmental refuge are key to sustaining biodiversity and population health. To strengthen the numerical exploration of chaotic behavior, the maximal Lyapunov exponent (MLE) over a three-dimensional parameter is drwan. A positive Lyapunov exponent reflects sensitive dependence on initial conditions, characterizing chaotic dynamics, while non-positive values suggest regular, periodic, or equilibrium behavior. This 3D surface plot **in Figure 7** shows how the maximal Lyapunov exponent h_2 varies with the bifurcation parameter a and time. Negative values of h_2 across most regions indicate stable dynamics, while values closer to zero suggest the onset of complex or near-chaotic behavior. Ecologically, the parameter a could represent the predator's attack efficiency or prey vulnerability—meaning changes in a impact how intensely the predator exploits prey. As a increases, the system remains largely stable but starts showing sensitive dependence on initial conditions in some regions, suggesting potential transitions in ecosystem dynamics.

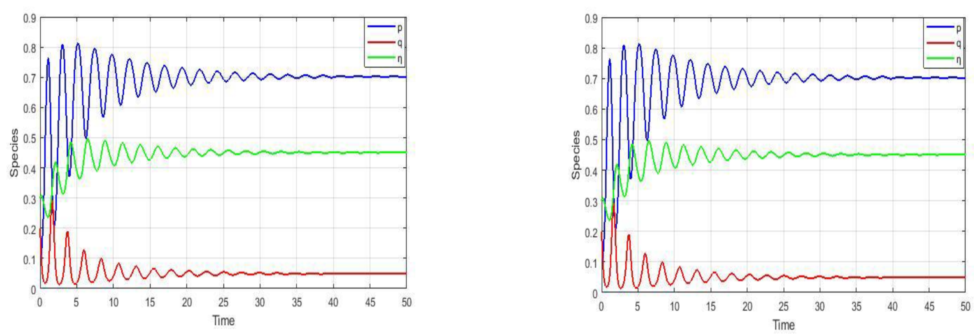

Figure 1. Time series at $\theta = 0.9$ and $\theta = 0.6$.

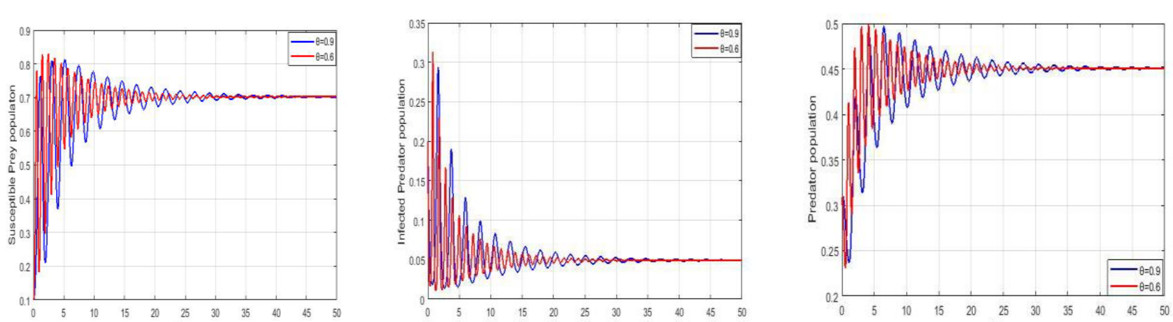

Figure 2. Time series with respect to susceptible prey, infected prey and predator population.

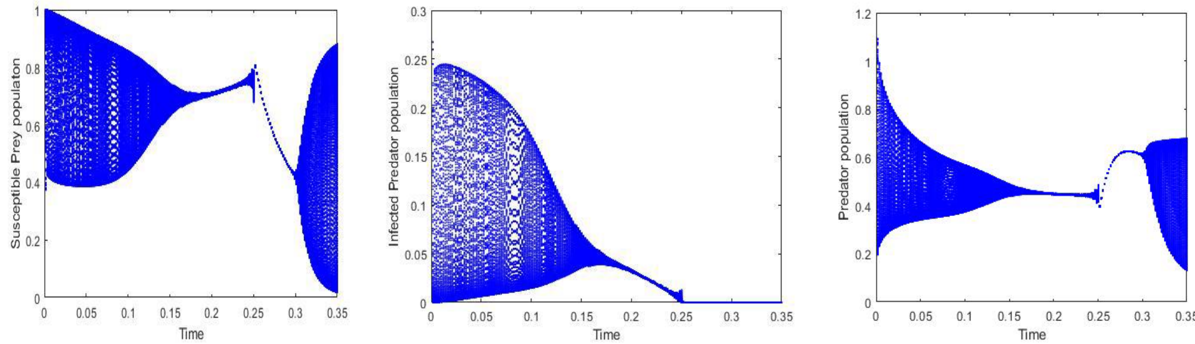

Figure 3. Neimark-Sacker Bifurcation with respect to susceptible prey, infected prey and predator population.

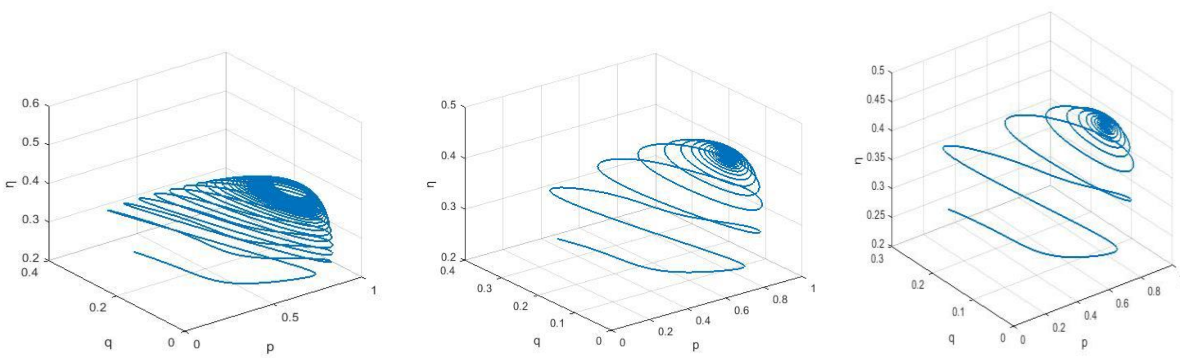

Figure 4. Phase plot for $\theta = 0.3, 0.6$ and 0.9 with a fixed at 0.15.

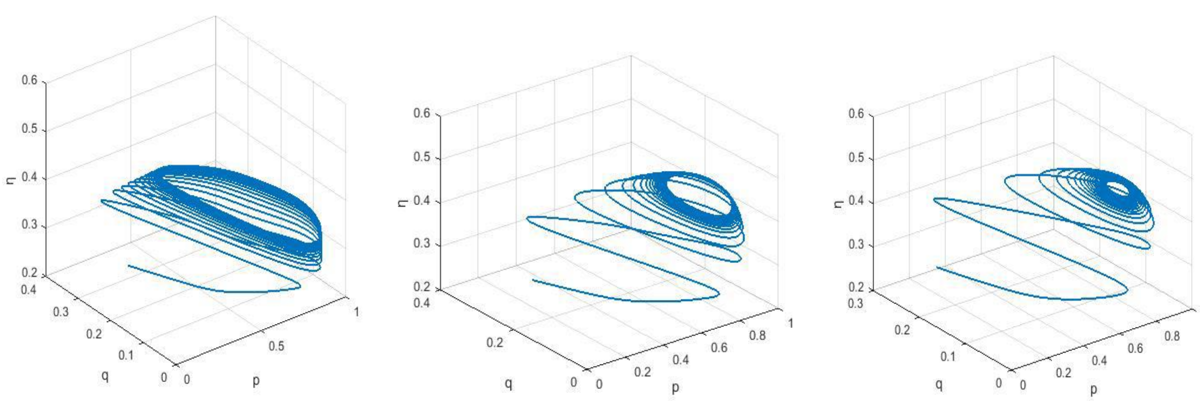

Figure 5. Phase plot for $\theta = 0.3, 0.6$ and 0.9 with a fixed at 0.1.

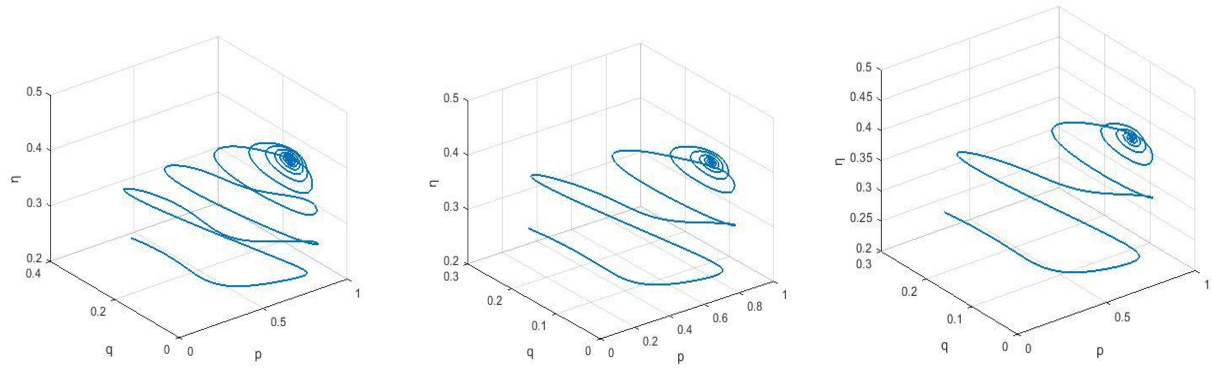

Figure 6. Phase plot for $\theta = 0.3, 0.6$ and 0.9 with a fixed at 0.2.

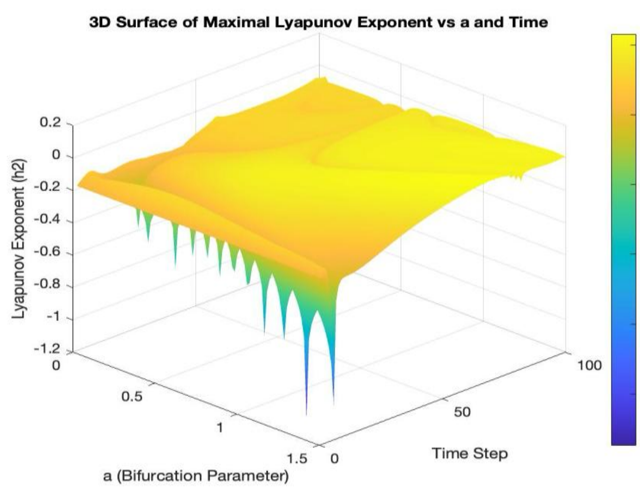

Figure 7. Maximal Lyapunuv exponent.

7. Conclusion

This paper presented a thorough analysis of a discretized three-dimensional diseased predator-prey model with prey refuge. The model incorporates key ecological dynamics, including predation, infection, prey growth, and refuge growth. The analysis explored the equilibrium points, stability and dynamical behaviour of the system. By identifying equilibrium points and evaluating their stability, it gained insights into the long-term behaviour of predator, prey, and refuge populations. Stability analysis helped us distinguish stable and unstable equilibrium points, providing information on system robustness and critical points. Neimark-Sacker bifurcation is carried out by providing suitable conditions for its existence. Numerical simulations are employed to validate the theoretical analysis and deepen understanding of the model's behaviour. These simulations allowed us to observe dynamic changes over time, study limit cycles or oscillatory behaviour, and explore the effects of parameter variations. The utilization of fractional calculus methods, which extend traditional calculus to fractional orders, offered advantages in modeling and analyzing the system. The findings of this study enhanced the understanding of predator-prey interactions, incorporating disease dynamics and the influence of prey refuge. The analysis and insights gained from this research have implications in various ecological and conservation contexts. Moreover, the application of fractional calculus methods demonstrated their versatility and potential for advancements in scientific and engineering disciplines. In future studies, the model could be extended to include disease dynamics in the predator population, forming a more comprehensive four-dimensional system. This would allow exploration of scenarios where predators are affected by consuming infected prey, potentially leading to outcomes like increased mortality, altered predation rates, or disease-driven trophic cascades. Such an extension may also reveal richer dynamical behaviors and complex bifurcation patterns.

In conclusion, this study offers valuable insights into the dynamics of a discretized three-dimensional diseased predator-prey model with prey refuge. It highlights the importance of considering non-local behaviors showcasing the advantages of fractional calculus methods in ecological modeling and analysis. The results contribute to existing knowledge and provide a foundation for further research in this field.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgments

The authors would like to thank the editor and anonymous reviewers for their comments that help improve the quality of this work.

AI Disclosure

The author(s) declare that no assistance is taken from generative AI to write this article.

References

Ali, I., Saeed, U., & Din, Q. (2019). Bifurcation analysis and chaos control in discrete-time system of three competing species. *Arabian Journal of Mathematics, 8*, 1-14.

Anderson, R.M., & May, R.M. (1986). The invasion, persistence and spread of infectious diseases within animal and plant communities. *Philosophical Transactions of the Royal Society of London. B, 314*(1167), 533-570.

Barhoom, N.S.S., & Al-Nassir, S. (2021). Dynamical behaviors of a fractional-order three dimensional prey-predator model. *Abstract and Applied Analysis*, *2021*(1), 1366797. https://doi.org/10.1155/2021/1366797.

Chattopadhyay, J., & Arino, O. (1999). A predator-prey model with disease in the prey. *Nonlinear Analysis: Theory, Methods & Applications, 36*(6), 747-766.

Chattopadhyay, J., & Bairagi, N. (2001). Pelicans at risk in Salton sea—an eco-epidemiological model. *Ecological Modelling*, *136*(2-3), 103-112.

Gaber, T., Widowati, & Herdiana, R. (2025). The impact of ecological mechanisms on stability in an eco-epidemiological model: allee effect and prey refuge. *Компьютерные исследования и моделирование*, *17*(1), 139-169.

Grove, E.A., & Ladas, G. (2004). *Periodicities in nonlinear difference equations*. CRC Press, New York.

Hadeler, K.P., & Freedman, H.I. (1989). Predator-prey populations with parasitic infection. *Journal of Mathematical Biology, 27*(6), 609-631.

Hegazi, A.S., & Matouk, A.E. (2011). Dynamical behaviors and synchronization in the fractional order hyperchaotic Chen system. *Applied Mathematics Letters*, *24*(11), 1938-1944.

Hethcote, H.W., Wang, W., Han, L., & Ma, Z. (2004). A predator–prey model with infected prey. *Theoretical Population Biology, 66*(3), 259-268.

Hochberg, M.E., & Holt, R.D. (1995). Refuge evolution and the population dynamics of coupled host parasitoid associations. *Evolutionary Ecology, 9*(6), 633-661.

Holt, R.D. (1984). Spatial heterogeneity, indirect interactions, and the coexistence of prey species. *The American Naturalist*, *124*, 377-406.

Holt, R.D., Grover, J., & Tilman, D. (1994). Simple rules for interspecific dominance in systems with exploitative and apparent competition. *The American Naturalist, 144*(5), 741-771.

Huang, Y., Chen, F., & Zhong, L. (2006). Stability analysis of a prey-predator model with Holling type III response function incorporating a prey refuge. *Applied Mathematics and Computation, 182*(1), 672-683.

Kar, T.K. (2005). Stability analysis of a prey predator model incorporating a prey refuge. *Communications in Nonlinear Science and Numerical Simulation, 10*(6), 681-691.

Khan, A.Q., Qureshi, S.M., & Alotaibi, A.M. (2022). Bifurcation analysis of a three species discrete-time predator-prey model. *Alexandria Engineering Journal, 61*(10), 7853-7875.

Liu, X. (2011). Bifurcation of an eco-epidemiological model with a nonlinear incidence rate. *Applied Mathematics and Computation, 218*(6), 2300-2309.

Ma, Z., Li, W., Zhao, Y., Wang, W., Zhang, H., & Li, Z. (2009). Effects of prey refuges on a predator-prey model with a class of functional responses: the role of refuges. *Mathematical Biosciences, 218*(1), 73-79.

Machado, J.A.T., & Mata, M.E. (2015). Pseudo phase plane and fractional calculus modeling of western global economic downturn. *Communications in Nonlinear Science and Numerical Simulation, 22*(1-3), 396-407.

Mahmoudi, M., & Eskandari, Z. (2024). Fractional calculus in ecological systems: bifurcation analysis and continuation techniques for discrete Lotka–Volterra models. *Mathematical Methods in the Applied Sciences*. https://doi.org/10.1002/mma.10603.

Matouk, A.E. (2011). Chaos, feedback control and synchronization of a fractional-order modified autonomous Van der Pol–Duffing circuit. *Communications in Nonlinear Science and Numerical Simulation, 16*(2), 975-986.

McNair, J.N. (1986). The effects of refuges on predator-prey interactions: a reconsideration. *Theoretical Population Biology, 29*(1), 38-63.

Naik, P.A., Eskandari, Z., Yavuz, M., & Huang, Z. (2024). Bifurcation results and chaos in a two-dimensional predator-prey model incorporating Holling-type response function on the predator. *Discrete and Continuous Dynamical Systems, 18*(5), 1212-1229.

Naik, P.A., Yavuz, M., Qureshi, S., Zu, J., & Townley, S. (2020). Modeling and analysis of COVID-19 epidemics with treatment in fractional derivatives using real data from Pakistan. *The European Physical Journal Plus, 135*(3), 1-42.

Niu, X., Zhang, T., & Teng, Z. (2011). The asymptotic behavior of a non-autonomous eco-epidemic model with disease in the prey. *Applied Mathematical Modelling, 35*(1), 457-470.

Olivars, E.G., & Jiliberto, R.R. (2003). Dynamics consequences of prey refuges in a simple model system: more prey, few predators and enhanced stability. *Ecological Modelling, 166*(1-2), 135-146.

Power, D.A., Watson, R.A., Szathmáry, E., Mills, R., Powers, S.T., Doncaster, C.P., & Czapp, B. (2015). What can ecosystems learn? Expanding evolutionary ecology with learning theory. *Biology Direct, 10*, 1-24.

Ruxton, G.D. (1995). Short term refuge use and stability of predator-prey models. *Theoretical Population Biology, 47*(1), 1-17.

Shi, X., Cui, J., & Zhou, X. (2011). Stability and Hopf bifurcation analysis of an eco-epidemic model with a stage structure. *Nonlinear Analysis: Theory, Methods & Applications, 74*(4), 1088-1106.

Singh, A., & Deolia, P. (2020). Dynamical analysis and chaos control in discrete-time prey predator model. *Communications in Nonlinear Science and Numerical Simulation, 90*, 105313. https://doi.org/10.1016/j.cnsns.2020.105313.

Singh, A., Elsadany, A.A., & Elsonbaty, A. (2019). Complex dynamics of a discrete fractional-order Leslie-Gower predator-prey model. *Mathematical Methods in the Applied Sciences, 42*(11), 3992-4007.

Venturino, E. (1995). Epidemics in predator-prey models: diseases in the prey. In *Mathematical Population Dynamics: Analysis of Heterogeneity, Theory of Epidemics, 1*, 381-393.

Venturino, E. (2002). Epidemics in predator–prey models: disease in the predators. *Mathematical Medicine and Biology*, *19*(3), 185-205.

Wang, S., & Ma, Z. (2012). Analysis of an ecoepidemiological model with prey refuges. *Journal of Applied Mathematics*, *2012*(1), 371685. https://doi.org/10.1155/2012/371685.

Yousef, F., Semmar, B., & Al Nasr, K. (2022). Incommensurate conformable-type three-dimensional Lotka–Volterra model: discretization, stability, and bifurcation. *Arab Journal of Basic and Applied Sciences*, *29*(1), 113-120.