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Multivariate Spatial Autoregressive Model with Latent Variables: Application to Economic Growth Modeling

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Abstract
Research involving causal relationships among latent variables that generally use structural equation modeling (SEM) analysis and spatial data simultaneously has an unavoidable impact on using spatial SEM analysis. A region-based spatial SEM model was developed for cases that include spill-over effects across regions. Spatial weights in the inner model offered flexibility and were more informative than conventional techniques. Therefore, this research developed a model that involved multivariate data, latent variables, and spatial data, specifically spatial autoregressive, in the form of a multivariate spatial autoregressive model that involves latent variables (MSAR-VLs). This development integrated latent variable estimation, error distribution of the model, parameter estimation, spatial dependency testing, and partial parameter hypothesis testing, which used the weighted least squares (WLS), the properties of expected and variance value, maximum likelihood estimation (MLE) method, Lagrange multiplier (LM), and Wald test methods, respectively. The results of the MSAR-LVs model development were applied to the economic growth modeling for regencies in East Java, Indonesia. The research findings on developing the MSAR-LVs model are as follows: the error vector distribution in the MSAR-LVs model conformed to a normal distribution. Its mean value was the estimated vector of the exogenous variables. Meanwhile, the standard deviation was represented by a matrix derived from the Kronecker product between the inverse of a diagonal matrix containing the parameters of the outer model for the exogenous variables and the identity matrix. The parameter estimators did not have a closed-form solution; therefore, they were estimated using a numerical approximation approach based on the concentrated log-likelihood function. The estimator matrix that yielded the maximum log-likelihood value was selected. Under the null hypothesis, the LM and Wald test statistics conformed to a chi-square distribution with one degree of freedom. The other findings indicated that the economic growth model had a significant and negative spatial autoregressive coefficient, while the coefficient for the socio-demographic model was not significant. Additionally, human capital exerted a significant effect on economic growth, illustrating that each regency experienced a negative spill-over effect on economic growth from neighboring regencies, influenced by the present human capital in those regions.

Keywords- Multivariate spatial autoregressive, Latent variables, Structural equation modeling, Spatial SEM, Economic growth modeling.

1. Introduction

The research on analysis techniques, including high-dimensional, multivariate data, latent variables, and theoretical construct, was carried out using a sophisticated statistical method, namely structural equation modeling (SEM). This statistical method was adopted for analyzing complex relationships among constructs and between constructs and their indicators (Hair Jr et al., 2021). Bollen (1989) and Whittaker and Schumacker (2022) defined SEM with respect to the two parts of a model, namely outer and inner. The outer model represents the relationship between indicators and latent variables, with the inner depicting the correlation among latent variables.

SEM analysis method had been widely applied in various fields, namely education (Alismaiel, 2021; Atchia and Chinapah, 2023; Kim and Na, 2022; Sun and Liu, 2023; Sun and Xiao, 2022; Yavuzalp and Bahcivan, 2021), social and economic (Abdulrazaq and Ahmad, 2024; Appiah et al., 2023; Borkowski, 2023; Darda and Bhuiyan, 2022; Desiana et al., 2022; Nugraha et al., 2023; Park et al., 2022; Shahid et al., 2024), health (Luo et al., 2024; Movahedi et al., 2023; Nazir and Qureshi, 2023; Nursanyoto et al., 2023; Watanabe et al., 2023), engineering (Punthupeng and Phimolsathien, 2024; Waqar et al., 2023; Wong et al., 2025), environment and housing (Ab Majid et al., 2023; Alhamed and Yusoff, 2023; Astriani et al., 2023; Mohamed et al., 2024), as well as other sectors. SEM denotes modeling analysis that encompasses causal relationships among latent variables. The research conducted by Jeong and Yoon (2018) assessed economic losses attributable to natural disasters, utilizing four latent variables as predictors. The study used spatial analysis, particularly the spatial autoregressive model because the research sample units were locations. Research designed to examine causal relationships involving spatial effect may use spatial analysis.

The developed spatial models by Anselin (1988) had been widely applied in various research fields, namely the human development index (Arum et al., 2024; Astari and Chotib, 2024; Darsyah et al., 2018; Niranjan, 2020; Pramesti and Indrasetianingsih, 2018; Rahma, 2020; Wati and Khikmah, 2020), poverty (Iffah et al., 2023; Kumar et al., 2023; Liu et al., 2023), social and economics (Atalay and Akan, 2023; Cimpoeru and Pisică, 2023; DiRago et al., 2022; Mahran, 2023; Varlamova and Kadochnikova, 2023), health (Abdullah and Law, 2024; Mergenthaler et al., 2022; Spies et al., 2024; Tesema et al., 2023), and other sectors. However, the use of spatial data in SEM requires further development. Jeong and Yoon (2018) conveyed the limitations of their research using latent variables involving spatial sample units that could not answer the relationship between constructs due to the complexity of the spatial SEM model. The research successfully addressed spatial effects but could not evaluate the relationships among latent variables due to the establishment of causal relationships among indicators. Therefore, spatial SEM analysis is essential for modeling research incorporating latent variables and spatial data.

SEM, which represents the relationship among latent variables, has two frameworks in spatial SEM modeling. The first framework is related to distance-based or region-based spatial modeling. Several researchers have carried out the implementation of distance-based spatial modeling (Adrian et al., 2023; Kiani et al., 2024), while distance-based spatial SEM has been developed by Comber et al. (2017). Region-based spatial SEM modeling was developed for cases involving spill-over effects (Anekawati et al., 2020b; Liu et al., 2011; Rahman et al., 2018; Roman and Brandt, 2023) that cannot be interpreted through distance-based spatial SEM. For example, economic growth modeling requires the depiction of spill-over effects. Regions with satisfactory economic growth tend to affect the economic growth of their neighboring regions. Therefore, economic growth modeling is more appropriate using region-based spatial analysis.

The second framework related to spatial weighting was placed either in the inner or outer model. Spatial weights in the outer model generally consist of multivariate spatial data analysis (Wang and Wall, 2003). Preliminary research on spatial SEM and weights in the outer model simultaneously evaluated the relationships among latent variables (Christensen and Amemiya, 2002; Hogan and Tchernis, 2004). Several studies have developed spatial models that incorporate latent variables with spatial weights within the inner model (Congdon, 2008, 2010; Congdon et al., 2007; Hossain and Laditka, 2009; Liu et al., 2005). However, Oud and Folmer (2008) proposed a better approach, placing spill-over effects in the inner model. This approach offers greater flexibility and provides more information compared to the conventional practice of assigning spatial weights to the outer model.

Despite these advancements, a notable limitation was the inability to conduct a spatial dependence test. Such a test is essential to determine the suitability of the dataset for spatial models, such as the spatial autoregressive model (SAR), spatial error model (SERM), spatial autoregressive moving average model (SARMA), among others. Anekawati et al. (2017) further explored spatial modeling with latent variables, applying the framework developed by Oud and Folmer (2008). They estimated the latent variable using a nonparametric approach, specifically the partial least squares (PLS) method as formulated by Trujillo (2009). Model parameters were then estimated using a parametric approach via the maximum likelihood estimation (MLE) method. For the spatial dependence test, a parametric approach, namely the Lagrange multiplier (LM) method, was employed. However, this LM test was developed assuming a model error distribution similar to that of the conventional type designed by Anselin (1988). Meanwhile, the model error distribution can only be identified by estimating latent variables through a parametric approach. Additionally, Anekawati et al. (2020a) advanced the LM test for SERM involving latent variables by utilizing latent variable estimation through a parametric approach, specifically the weighted least squares (WLS) method. This approach facilitated the derivation of a new error distribution for SERM. However, the resulting model remained relatively simple, as it included only a single endogenous latent variable (univariate) and used parameter estimation via a nonparametric method, the generalized method of moments (GMM), as proposed by Kelejian and Prucha (1998, 1999). A limitation of spatial model development with latent variables is the reliance on univariate models and the combination of parametric and nonparametric approaches.

Meanwhile, in fields such as social sciences and economics, models often demand greater complexity, typically involving multiple response variables (multivariate data). For instance, in economic growth modeling, key dimensions such as human capital (Affandi et al., 2019; Ali et al., 2018; Campbell and Okuwa, 2016; Carillo, 2024; Sarwar et al., 2021; Wau, 2022; Windhani et al., 2023) and socio-economic factors (Rochmatullah et al., 2020) are frequently considered as drivers of economic growth. Moreover, human capital can influence a region's socio-economic conditions (Hanisya et al., 2024; Wau, 2022), requiring the inclusion of multiple response variables or multivariate data. Each construct used in developing an economic growth model is latent, typically measured through multiple indicators and incorporating spill-over effects, thereby necessitating spatial analysis that includes latent variables. The Structural Equation Modeling (SEM) approach has been extensively applied in economic growth studies (Borkowski, 2023; Gyimah et al., 2023; Jayadevan, 2021; Mangir et al., 2020). Modeling economic growth that involves spill-over effects tended to lead to a spatial autoregressive model (SAR) (Ahmad, 2019; Paudel, 2023; Sinu et al., 2024). However, economic growth models that incorporate spill-over effects, particularly those using the Spatial Autoregressive (SAR) model, have been developed without yet integrating SEM analysis (Mahran, 2023). Therefore, this research prioritizes the SAR model over other spatial models.

Accordingly, this research aims to develop a model integrating multivariate data, latent variables, and spatial data-particularly through the relatively new model, namely the Multivariate Spatial Autoregressive Model incorporating latent variables (MSAR-LVs). The MSAR-LVs development encompasses the full process: estimation of factor scores through the weighted least squares (WLS) method, derivation of the error distribution model, parameter estimation using maximum likelihood estimation (MLE), spatial dependence testing through the Lagrange multiplier (LM) test, assessment of statistical significance with the Wald test, and an illustration of the MSAR-LVs application through economic growth modeling.

2. Materials and Methods

The current section discussed the related literature on the spatial autoregressive model with latent variables (SAR-LVs), factor score estimation, and research methodology.

2.1 The Spatial Autoregressive Model with Latent Variables (SAR-LVs)

The spatial SEM was used to substitute each variable with factor scores obtained by estimating latent variables in the outer model (SEM). Endogenous latent variables (η) amount to *q*, and the number of indicators for the *j*-th variable is b_j, where $j=1,2,3,\ldots,q$. The total number of indicators for the endogenous latent variables is $\sum_{j=1}^{q} b_j = B$. Therefore, the outer model equation for the endogenous latent variable (Bollen, 1989) is stated as follows:

$$\underset{B\times 1}{\mathbf{y}} = \underset{B\times q}{\mathbf{\Lambda_y}}\ \underset{q\times 1}{\boldsymbol{\eta}} + \underset{B\times 1}{\boldsymbol{\varepsilon}^*} \text{ with } \underset{B\times 1}{\boldsymbol{\varepsilon}^*} \sim N\left(\underset{B\times 1}{\mathbf{0}}, \underset{B\times B}{\mathbf{\Theta}_{\varepsilon}}\right) \tag{1}$$

where, $\mathbf{\Lambda_y}$ is the coefficient matrix that shows the relationship between $\mathbf{y}$ and $\boldsymbol{\eta}$, with $\boldsymbol{\varepsilon}^*$ depicted as the measurement error vector of $\mathbf{y}$. The covariance matrix of the measurement error of the observed variable $\mathbf{y}$ is $\underset{B\times B}{\mathbf{\Theta}_{\varepsilon}} = diag\left(\sigma^2_{\varepsilon_{(1)1}}, \sigma^2_{\varepsilon_{(2)1}}, \cdots, \sigma^2_{\varepsilon_{(b1)1}}, \sigma^2_{\varepsilon_{(1)2}}, \sigma^2_{\varepsilon_{(2)2}}, \cdots, \sigma^2_{\varepsilon_{(b2)2}}, \cdots, \sigma^2_{\varepsilon_{(1)q}}, \sigma^2_{\varepsilon_{(2)q}}, \cdots, \sigma^2_{\varepsilon_{(bq)q}}\right)$.

The number of exogenous latent variables (ξ) is *p*, the number of indicators for the *i*-th variable is a_i, where $i=1,2,3,\ldots,p$. The total number of indicators for the exogenous latent variables is $\sum_{i=1}^{p} a_i = A$. Therefore, the outer model equation for the exogenous latent variables (Bollen, 1989) is stated as follows

$$\underset{Ax1}{\mathbf{x}} = \underset{Axp}{\mathbf{\Lambda_x}}\ \underset{px1}{\boldsymbol{\xi}} + \underset{Ax1}{\boldsymbol{\delta}^*} \text{ with } \underset{A\times 1}{\boldsymbol{\delta}^*} \sim N\left(\underset{A\times 1}{\mathbf{0}}, \underset{A\times A}{\mathbf{\Theta}_{\delta}}\right) \tag{2}$$

where, $\mathbf{\Lambda_x}$ is the coefficient matrix that shows the relationship between $\mathbf{x}$ and ξ, with $\boldsymbol{\delta}^*$ representing the measurement error vector of $\mathbf{x}$. The covariance matrix of the measurement error of the observed variable $\mathbf{x}$ is $\underset{A\times A}{\mathbf{\Theta}_{\delta}} = diag\left(\sigma^2_{\delta_{(1)1}}, \sigma^2_{\delta_{(2)1}}, \cdots, \sigma^2_{\delta_{(a1)1}}, \sigma^2_{\delta_{(1)2}}, \sigma^2_{\delta_{(2)2}}, \cdots, \sigma^2_{\delta_{(a2)2}}, \cdots, \sigma^2_{\delta_{(1)p}}, \sigma^2_{\delta_{(2)p}}, \cdots, \sigma^2_{\delta_{(ap)p}}\right)$.

The assumptions for Equation (1) and Equation (2) encompass $E\left(\boldsymbol{\varepsilon}^*\right)=\mathbf{0}$, $E\left(\boldsymbol{\delta}^*\right)=\mathbf{0}$, $E\left(\boldsymbol{\eta}\right)=\mathbf{0}$, $E\left(\boldsymbol{\xi}\right)=\mathbf{0}$, $E\left(\boldsymbol{\zeta}\right)=\mathbf{0}$, and $E\left(\boldsymbol{\xi}\boldsymbol{\zeta}'\right)=\mathbf{0}$, with the stipulation that $\boldsymbol{\xi}$ is uncorrelated with $\boldsymbol{\xi}$ and the matrix $\left(\mathbf{I}-\mathbf{B}\right)$ is non-singular.

Oud and Folmer (2008) introduced the incorporation of SAR within SEM by regressing latent variables represented through multiple indicators. This approach utilized a multiple indicators multiple causes (MIMIC) framework, where the observed vector $\mathbf{y}$ for the dependent variable was substituted with $\boldsymbol{\eta}$, and the latent variable $\boldsymbol{\xi}=\mathbf{x}$, incorporating spatial dependence in the inner model. The equation for the SAR-LVs model is expressed as follows:

$$\boldsymbol{\eta} = \rho\mathbf{W}\boldsymbol{\eta} + \mathbf{\Gamma}^*\mathbf{x} + \boldsymbol{\zeta} \tag{3}$$

With reference to Equation (3), we make the following assumptions: 1) Factor scores represent the estimated results of latent variables, serving as substitutes for both response and predictor variables. 2) The MIMIC model is not applied in this framework, indicating the absence of exogenous or endogenous observed variables. 3) Spatial dependence exists in the factor scores of the inner model rather than in the outer model. 4) The $n \times n$ matrix $\mathbf{W}$ illustrates spatial dependence among observations or locations. Consequently, the spatial autoregressive model with a single endogenous latent variable, or univariate (SAR-LVs), can be reformulated as follows:

$$\hat{\boldsymbol{\eta}} = \lambda \mathbf{W} \hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\xi}} \boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{4}$$

Equation (4) is derived based on the assumption that the diagonal elements of the spatial weight matrix $\mathbf{W}$ are zero, the matrix $(\mathbf{I} - \lambda \mathbf{W})$ is nonsingular with $|\lambda| < 1$, and ε_i has identical distribution (has property of independent).

2.2 Factor Score Estimation

The factor score was derived by estimating latent variables using the WLS method. The purpose of WLS was to generate an estimate that converges at the optimum point and can be calculated analytically (Newsom and Smith, 2020). The WLS method is carried out by optimizing the likelihood function of the observation vector ***x***, as Anekawati et al. (2020a) previously did, but that study only involved a single endogenous latent variable. In the same way, the endogenous factor scores are obtained from more than one latent variable.

Based on Equation (1), assuming $\boldsymbol{\Lambda}_{\mathbf{y}}$ and $\boldsymbol{\Theta}_{\varepsilon}$ are constant, the distribution of random variable $\mathbf{y}$ is obtained through the expectation and variance value. The expected value of the random variable $\mathbf{y}$ is $E(\mathbf{y}) = \boldsymbol{\Lambda}_{\mathbf{y}} \boldsymbol{\eta}$ while the variance is $\text{var}(\mathbf{y}) = \text{var}(\boldsymbol{\varepsilon}^*) = \boldsymbol{\Theta}_{\varepsilon}$. Therefore, the distribution of $\mathbf{y}$ is $\underset{B\times 1}{\mathbf{y}} \sim N\left(\underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}} \underset{q\times 1}{\boldsymbol{\eta}}, \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}} \right)$.

Given n random samples of $\mathbf{y}$ with $n = 1, 2, 3, \ldots, N$ $\left(y_{(1)11}, y_{(2)12}, \ldots, y_{(b_1)1n}, y_{(1)21}, y_{(2)22}, \ldots, y_{(b_2)2n}, \ldots, y_{(1)q1}, y_{(2)qn}, \ldots, y_{(b_q)qn}, \eta_{11}, \eta_{12}, \ldots, \eta_{1n}, \eta_{21}, \eta_{22}, \ldots, \eta_{2n}, \ldots, \eta_{q1}, \eta_{q2}, \ldots, \eta_{qn} \right)$ then the distribution is stated as follows:

$$\underset{B\times 1}{\mathbf{y}_n} \sim N\left(\underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}} \underset{q\times 1}{\boldsymbol{\eta}_n}, \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}} \right) \tag{5}$$

The probability function of $\mathbf{y}_n$ is $f(\mathbf{y}_n) = (2\pi)^{-\frac{B}{2}} |\boldsymbol{\Theta}_{\varepsilon}|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\mathbf{y}_n - \boldsymbol{\Lambda}_{\mathbf{y}} \boldsymbol{\eta}_n)^T \boldsymbol{\Theta}_{\varepsilon}^{-1} (\mathbf{y}_n - \boldsymbol{\Lambda}_{\mathbf{y}} \boldsymbol{\eta}_n) \right\}$, while the likelihood function is $L(\boldsymbol{\eta}, \boldsymbol{\Theta}_{\varepsilon}) = \prod_{n=1}^{N} f(\mathbf{y}_n)$. Equation (5) was used to convert the probability function to $L(\boldsymbol{\eta}, \boldsymbol{\Theta}_{\varepsilon}) = (2\pi)^{-\frac{BN}{2}} |\boldsymbol{\Theta}_{\varepsilon}|^{-\frac{N}{2}} \exp\left\{ -\frac{1}{2} \sum_{n=1}^{N} (\mathbf{y}_n - \boldsymbol{\Lambda}_{\mathbf{y}} \boldsymbol{\eta}_n)^T \boldsymbol{\Theta}_{\varepsilon}^{-1} (\mathbf{y}_n - \boldsymbol{\Lambda}_{\mathbf{y}} \boldsymbol{\eta}_n) \right\}$. When simplified the likelihood function can be stated as $L(\boldsymbol{\eta},\boldsymbol{\Theta}_{\varepsilon}) = (2\pi)^{-\frac{BN}{2}} |\boldsymbol{\Theta}_{\varepsilon}|^{-\frac{N}{2}} \exp\left\{-\frac{1}{2}\sum_{n=1}^{N} Q_y\right\}$ with $\underset{1\times1}{Q_y} = \left(\underset{B\times1}{\mathbf{y}_n} - \underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{q\times1}{\boldsymbol{\eta}_n}\right)^T \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}} \left(\underset{B\times1}{\mathbf{y}_n} - \underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\underset{q\times1}{\boldsymbol{\eta}_n}\right)$.

The latent variable $\boldsymbol{\eta}$ was estimated using the WLS method, which adopted $L(\boldsymbol{\eta},\boldsymbol{\Theta}_{\varepsilon})$ optimization, namely $L(\boldsymbol{\eta},\boldsymbol{\Theta}_{\varepsilon}) \Leftrightarrow \min Q_y$. If the partial derivative of Q_y with respect to η and it was set to zero, then the factor score of η is stated as follows:

$$\underset{q\times1}{\hat{\boldsymbol{\eta}}_n} = \left(\underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}}\underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}}\underset{B\times1}{\mathbf{y}_n} \tag{6}$$

Equation (6) conceives random samples of the random variable $\mathbf{y}$, namely $\left(y_{(1)11}, y_{(2)12}, \ldots, y_{(b_1)1n}, y_{(1)21}, y_{(2)22}, \ldots, y_{(b_2)2n}, \ldots, y_{(1)q1}, y_{(2)qn}, \ldots, y_{(b_q)qn}, \eta_{11}, \eta_{12}, \ldots, \eta_{1n}, \eta_{21}, \eta_{22}, \ldots, \eta_{2n}, \ldots, \eta_{q1}, \eta_{q2}, \ldots, \eta_{qn}\right)$, therefore, the endogenous factor score restated in matrix form is:

$$\underset{q\times n}{\hat{\boldsymbol{\eta}}} = \left(\underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}}\underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}}\underset{B\times n}{\mathbf{Y}} \tag{7}$$

where, $\left(\underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T}\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}}\underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)$ is a diagonal matrix whose elements other than the main ones are zero.

In accordance with Equation (2), including using the same calculations for the endogenous latent variable, assuming that $\boldsymbol{\Lambda}_x$ and $\boldsymbol{\Theta}_{\delta}$ are constant, then the distribution of $\mathbf{x}$ and the exogenous factor score were obtained by applying Equations (8) and (9), respectively.

$$\underset{A\times1}{\mathbf{x}} \sim N\left(\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}\underset{p\times1}{\boldsymbol{\xi}}, \underset{A\times A}{\boldsymbol{\Theta}_{\delta}}\right) \tag{8}$$

$$\underset{p\times n}{\hat{\boldsymbol{\xi}}} = \left(\underset{p\times A}{\boldsymbol{\Lambda}_{\mathbf{x}}^T}\underset{A\times A}{\boldsymbol{\Theta}_{\delta}^{-1}}\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}\right)^{-1} \underset{p\times A}{\boldsymbol{\Lambda}_{\mathbf{x}}^T}\underset{A\times A}{\boldsymbol{\Theta}_{\delta}^{-1}}\underset{A\times n}{\mathbf{X}} \tag{9}$$

where, $\left(\underset{p\times A}{\boldsymbol{\Lambda}_{\mathbf{x}}^T}\underset{A\times A}{\boldsymbol{\Theta}_{\delta}^{-1}}\underset{A\times p}{\boldsymbol{\Lambda}_{\mathbf{x}}}\right)$ is a diagonal matrix whose elements other than the main ones are zero.

2.3 Methodology

This research is structured into two frameworks: developing a mathematical formula for the MSAR-VLs model and applying the model for economic growth modeling. The development of a mathematical formula for the MSAR-VLs model comprised several stages, namely:

(i) The initial phase in MSAR-LVs analysis involves the construction of the spatial weight matrix W_{ij}. Anselin (1988) asserts that the square matrix illustrates the spatial interdependence of cross-sectional units *i* and *j* within the dataset, which is essential for estimating various spatial models. This modeling uses queen contiguity (side-vertex common) spatial weights, defined by Lesage (1999), as $W_{ij} = 1$ for entities that share a common-side or vertex with the regions of interest and $W_{ij} = 0$ for other areas.

(ii) Forming the MSAR-LVs model. The MSAR-VLs model extends to the SAR-VLs model as in Equation (4). The endogenous latent variables are no longer one, but as many as *q*, while the number of exogenous latent variables is *p*. Then, it is written and simplified as a vector matrix.

(iii) Obtaining a mathematical formula for the model error distribution. The error distribution of the MSAR-LVs model is obtained by utilizing the properties of expectation value and variance. These properties state that if *X* is a random variable, *a* and *b* are constant, then $E(a+bX) = a + bE(X)$ and $Var(a+bX) = b^2 Var(X)$.

(iv) Parameter estimation in the model using the MLE method. This was because the MLE was a more accurate and exact estimator than numerous other methods (Dambon et al., 2021). The likelihood function of the MSAR-LVs model is obtained by substituting the model error and multiplying it by the Jacobian value. The estimator of $\boldsymbol{\beta}$ and the variance-covariance matrix are derived by differentiating the ln-likelihood function.

(v) Obtaining mathematical formulas for spatial dependency tests using the LM was easy and practical since it was based on estimating the model using the null hypothesis, which was its simplest form (Anselin, 1988; Breusch and Pagan, 1980). Step 4 required the ln-likelihood function with the model error distribution generated in step 2. The basis of the LM test hypothesis is as follows: $H_0 : \lambda_j = 0$ and $H_1 : \lambda_j \neq 0;\ j = 1, 2, ..., q$. The LM test statistic is defined as $\text{LM} = \hat{\mathbf{D}}_\lambda^T \hat{\boldsymbol{\psi}}^{-1} \hat{\mathbf{D}}_\lambda$ (Breusch & Pagan, 1980), where where $\hat{\mathbf{D}}_\lambda$ is the first derivative of the ln likelihood function to λ at $\lambda = 0$ and $\hat{\boldsymbol{\psi}}^{-1}$ is the (1,1) element of the inverse information matrix $\tilde{\boldsymbol{\psi}}_\theta$ of size 2×2.

(vi) Deriving the mathematical formula for the partial significance test using the Wald test. The Wald test, utilizing estimates from an unrestricted model, is sufficiently versatile to examine multiple distinct and potentially nonlinear parameter restrictions within a single model (Juhl, 2021). At this stage, a Hessian matrix is required, which constitutes the second derivative matrix of the concentrated log-likelihood function.

The second framework of this research is applying the MSAR-VLs model for modeling the causal relationship and spatial effects of regencies' economic growth in East Java Province, Indonesia. This stage will commence with developing a conceptual framework, the operational definitions of latent variables, and the indicators that constitute the latent variables.

Variables and Conceptual Framework

(i) Causality relationship between human capital and economic growth.

Economic growth was influenced by education (Ahmad, 2019; Kusumaningsih et al., 2022; Purnomo and Istiqomah, 2019), life expectancy (Jayadevan, 2021), human development index (Ahmad, 2019; Ciptawaty et al., 2022; Elistia and Syahzuni, 2018; Muslim et al., 2019; Muzzakar et al., 2023; Rizaldi et al., 2024), and health expenditure (Jayadevan, 2021). These indicators were grouped under the human capital dimension. Furthermore, human capital was defined as a set of intangible resources (Goldin, 2019) reflecting all human attributes, including health, morals, education, knowledge, and skills (Teixeira and Queirós, 2016). It was also identified as an element influencing economic growth (Affandi et al., 2019; Ali et al., 2018; Amidi et al., 2020; Carillo, 2024; Sarwar et al., 2021; Teixeira and Queirós, 2016; Windhani et al., 2023). This research states that the human capital factor affected economic growth.

(ii) The causal relationship between social demographics and economic growth.
Economic growth was influenced by population (Ahmad, 2019; Atikah et al., 2021; Rizaldi et al., 2024), unemployment (Kusumaningsih et al., 2022; Muslim et al., 2019; Priambodo, 2021; Soylu et al., 2018), the size of the labor force (Amidi et al., 2020; Atikah et al., 2021; Muslim et al., 2019; Muzzakar et al., 2023; Rizaldi et al., 2024), and the number or percentage of poor people (Jayadevan, 2021; Kusumaningsih et al., 2022; Muzzakar et al., 2023; Priambodo, 2021; Purnomo and Istiqomah, 2019). Meanwhile, these indicators were grouped under social demographics dimension. Social demographics represented the total population, population growth, labor force in the community, and the poor (Pardi et al., 2024). As a result, the social-demographics factor is hypothesized to influence economic growth.

(iii) The causal relationship between social economics and economic growth.
Economic growth was influenced by capital expenditure (Cornevin et al., 2024; Jayadevan, 2021; Nguyen, 2022; Paudel, 2023; Purnomo and Istiqomah, 2019), foreign investment (Asante et al., 2022), per capita expenditure (Arvin et al., 2021; Rasaily and Paudel, 2019; Rizaldi et al., 2024), local revenue (Ciptawaty et al., 2022; Rizaldi et al., 2024), and per capita expenditure index (Hanisya et al., 2024). These indicators were classified under social economics dimensions. Social economics was closely related to regional independence in obtaining capital needs, such as the local acquisition of taxes and levies, legal regional income (Rochmatullah et al., 2020), and community welfare (Lim and Endo, 2016). Therefore, it is hypothesized that the social economic dimension influenced economic growth.

(iv) The causal relationship between human capital and social demographics.
The investment in human capital had an elastic impact on economic growth, resulting in a productive demographic trend (Campbell and Okuwa, 2016). Human capital negatively affected poverty levels (Hanisya et al., 2024) and contributed significantly to the poverty alleviation scheme, particularly in disadvantaged regions (Wau, 2022). As an indicator of human capital, HDI reportedly had a negative effect on poverty, a key socio-demographic indicator (Irawan, 2022). Therefore, human capital is theorized to influence social demographics.

Latent variables, operational definitions, and indicators used to measure the dimensions are shown in **Table 1**. Based on the conceptual framework, the causal relationship among latent variables is represented as shown in **Figure 1**.

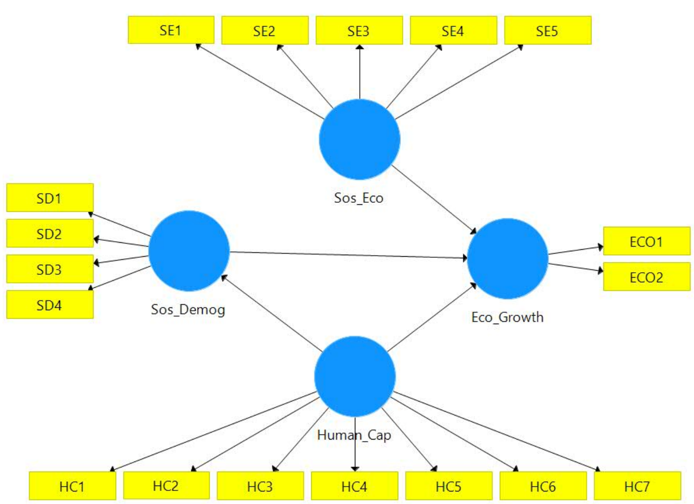

Figure 1. Economic growth model.

The data analysis stage started with evaluating the outer model (in SEM) to obtain the best indicators reflecting the latent. Evaluation of the inner model utilizes Smart-PLS software, taking into account each indicator's factor loading values. Subsequently, latent variables are estimated to derive factor scores, which serve as substitutes for predictor and response variables in the third-stage analysis (first framework). The factor scores are modeled as the results of the second stage (first framework) to estimate respective parameters in the fourth stage (first framework). The model is tested for spatial effects using the LM method in the fifth stage (first framework). The significance of the estimator is determined by using the Wald test as the result in the sixth stage (first framework).

The result of the first framework is mathematical formulas. The mathematical formulas are scripted through a computer program using R-open-source software. Data was acquired from the Central Bureau of Statistics of Indonesia. The sample was obtained from East Java province, comprising 38 regencies.

Table 1. Latent variables, operational definitions, and indicators.

Construct	Operational definition	Indicators	Code
Human capital	The set of intangible resources inherent in human resources (Goldin, 2019) that reflect related attributes, including health, morals, education, knowledge, and skills (Teixeira and Queirós, 2016).	Life expectancy	HC1
		Health index	HC2
		Percentage of the population with health insurance	HC3
		Average years of schooling	HC4
		Expected years of schooling	HC5
		Education index	HC6
		Human development index	HC7
Social demographics	Social demography represents the total population growth, workforce in society, including the poor (Pardi et al., 2024).	Percentage of non-poor population	SD1
		Workforce participation	SD2
		Open unemployment rate	SD3
		Percentage of dependency ratio	SD4
Social economic	Social economics is closely related to the regional autonomy in obtaining capital to meet all relevant needs, such as local tax and retribution acquisition, legitimate regional income (Rochmatullah et al., 2020), as well as community welfare (Lim and Endo, 2016).	Business capital	SE1
		Domestic investment	SE2
		Foreign investment	SE3
		Expenditure index	SE4
		Average per capita expenditure	SE5
Economic growth	Economic growth is a process of continuous improvement in the economic conditions of a country towards a better direction over a certain period (Rizaldi et al., 2024) measured through Gross Regional Domestic Product (Muzzakar et al., 2023).	Gross regional domestic product per capita at constant prices	ECO1
		Growth rate of gross regional domestic product	ECO2

3. Results

The current section focused on the findings from the MSAR-LVs model, model error distribution, spatial dependency test using LM, and partial parameter significance test through the Wald test.

3.1 The Model of a Multivariate Spatial Autoregressive with Latent Variables (MSAR-LVs)

The MSAR-LVs is an extension of the univariate SAR-LVs as stated in Equation (4). This model was used to determine the relationship between the response and predictor variables, regarded as the endogenous ($\hat{\eta}_q$), and exogenous factor scores ($\hat{\xi}_p$) respectively. This was realized by considering the spatial lag effect of the response variable and spatial weight symbolized by λ and $\mathbf{W}$. The MSAR-LVs model with q response and p predictor variables are stated as follows.

$$\begin{bmatrix} \hat{\eta}_{1i} \\ \hat{\eta}_{2i} \\ \vdots \\ \hat{\eta}_{qi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}_{01} + \lambda_1 \sum_{i^*=1, i \neq i^*}^{n} w_{ii^*} \hat{\eta}_{1i^*} + \hat{\boldsymbol{\xi}}^T \boldsymbol{\beta}_1 + \varepsilon_{1i} \\ \boldsymbol{\beta}_{02} + \lambda_2 \sum_{i^*=1, i \neq i^*}^{n} w_{ii^*} \hat{\eta}_{2i^*} + \hat{\boldsymbol{\xi}}^T \boldsymbol{\beta}_2 + \varepsilon_{2i} \\ \vdots \\ \boldsymbol{\beta}_{0q} + \lambda_q \sum_{i^*=1, i \neq i^*}^{n} w_{ii^*} \hat{\eta}_{qi^*} + \hat{\boldsymbol{\xi}}^T \boldsymbol{\beta}_p + \varepsilon_{qi} \end{bmatrix}$$

This equation can be rewritten as $\hat{\boldsymbol{\eta}}_1 = \left[\hat{\eta}_{11} \quad \hat{\eta}_{12} \quad \cdots \quad \hat{\eta}_{1i} \quad \cdots \quad \hat{\eta}_{1n}\right]^T$, $\hat{\boldsymbol{\eta}}_2 = \left[\hat{\eta}_{21} \quad \hat{\eta}_{22} \quad \cdots \quad \hat{\eta}_{2i} \quad \cdots \quad \hat{\eta}_{2n}\right]^T$ to $\hat{\boldsymbol{\eta}}_q = \left[\hat{\eta}_{q1} \quad \hat{\eta}_{q2} \quad \cdots \quad \hat{\eta}_{qi} \quad \cdots \quad \hat{\eta}_{qn}\right]^T$.

$$\hat{\eta}_1 = \lambda_1 W \hat{\eta}_1 + \hat{\boldsymbol{\xi}}^T \boldsymbol{\beta}_1 + \varepsilon_1$$
$$\hat{\eta}_2 = \lambda_2 W \hat{\eta}_2 + \hat{\boldsymbol{\xi}}^T \boldsymbol{\beta}_2 + \varepsilon_2$$
$$\vdots$$
$$\hat{\eta}_q = \lambda_q W \hat{\eta}_q + \hat{\boldsymbol{\xi}}^T \boldsymbol{\beta}_q + \varepsilon_q$$

Let $\hat{\boldsymbol{\eta}} = \left[\hat{\eta}_1 \ \hat{\eta}_2 \ \cdots \ \hat{\eta}_q\right]^T$, $\boldsymbol{\lambda} = diag\left(\lambda_1 \ \lambda_2 \ \cdots \ \lambda_q\right)$, $\boldsymbol{\beta} = \left[\boldsymbol{\beta}_1 \ \boldsymbol{\beta}_2 \ \cdots \ \boldsymbol{\beta}_p\right]^T$, and $\boldsymbol{\varepsilon} = \left[\varepsilon_1 \ \varepsilon_2 \ \cdots \ \varepsilon_q\right]^T$, then the equation was rewritten as follows:

$$\underset{q\times 1}{\hat{\boldsymbol{\eta}}} = \lambda W \underset{q\times 1}{\hat{\boldsymbol{\eta}}} + \underset{q\times(p+1)}{\hat{\boldsymbol{\xi}}} \underset{(p+1)\times 1}{\boldsymbol{\beta}} + \underset{q\times 1}{\boldsymbol{\varepsilon}} \tag{10}$$

Supposing a random sample was given, for example the number of areas or locations being observed, totaling *n*, denoted as $i = 1, 2, \ldots, n$. Then the MSAR-LVs model for the response $\hat{\eta}_1, \hat{\eta}_2, \cdots, \hat{\eta}_q$ and predictor variables $\hat{\xi}_1, \hat{\xi}_2, \cdots, \hat{\xi}_p$, considering the spatial lag effect of the response variable and spatial weight matrix symbolized by λ and $\mathbf{W}$ of size $n \times n$, are stated as follows.

$$\hat{\boldsymbol{\eta}}_1 = \lambda_1 \mathbf{W} \hat{\boldsymbol{\eta}}_1 + \hat{\boldsymbol{\xi}} \boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1$$
$$\hat{\boldsymbol{\eta}}_2 = \lambda_2 \mathbf{W} \hat{\boldsymbol{\eta}}_2 + \hat{\boldsymbol{\xi}} \boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2$$
$$\vdots$$
$$\hat{\boldsymbol{\eta}}_q = \lambda_q \mathbf{W} \hat{\boldsymbol{\eta}}_q + \hat{\boldsymbol{\xi}} \boldsymbol{\beta}_p + \boldsymbol{\varepsilon}_q$$

The MSAR-LVs model was written in vector form as follows

$$
\begin{bmatrix} \hat{\eta}_{11} \\ \hat{\eta}_{12} \\ \vdots \\ \hat{\eta}_{1n} \\ \hat{\eta}_{21} \\ \hat{\eta}_{22} \\ \vdots \\ \hat{\eta}_{2n} \\ \vdots \\ \hat{\eta}_{q1} \\ \hat{\eta}_{q2} \\ \vdots \\ \hat{\eta}_{qn} \end{bmatrix}_{qn\times 1}
=
\begin{bmatrix}
\lambda_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \lambda_{12} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \lambda_{21} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \lambda_{22} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda_{2n} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \lambda_{q1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{q2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda_{qn}
\end{bmatrix}^T_{qn\times qn}
\begin{bmatrix}
w_{11} & w_{12} & \cdots & w_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
w_{21} & w_{22} & \cdots & w_{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
w_{n1} & w_{n2} & \cdots & w_{nn} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & w_{11} & w_{12} & \cdots & w_{1n} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & w_{21} & w_{22} & \cdots & w_{2n} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & w_{n1} & w_{n2} & \cdots & w_{nn} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & w_{11} & w_{12} & \cdots & w_{1n} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & w_{21} & w_{22} & \cdots & w_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & w_{n1} & w_{n2} & \cdots & w_{nn}
\end{bmatrix}_{qn\times qn}
\begin{bmatrix} \hat{\eta}_{11} \\ \hat{\eta}_{12} \\ \vdots \\ \hat{\eta}_{1n} \\ \hat{\eta}_{21} \\ \hat{\eta}_{22} \\ \vdots \\ \hat{\eta}_{2n} \\ \vdots \\ \hat{\eta}_{q1} \\ \hat{\eta}_{q2} \\ \vdots \\ \hat{\eta}_{qn} \end{bmatrix}_{qn\times 1}
$$

$$
+\begin{bmatrix}
1 & \hat{\xi}_{11} & \hat{\xi}_{21} & \cdots & \hat{\xi}_{p1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
1 & \hat{\xi}_{12} & \hat{\xi}_{22} & \cdots & \hat{\xi}_{p2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \hat{\xi}_{1n} & \hat{\xi}_{2n} & \cdots & \hat{\xi}_{pn} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & \hat{\xi}_{11} & \hat{\xi}_{21} & \cdots & \hat{\xi}_{p1} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & \hat{\xi}_{12} & \hat{\xi}_{22} & \cdots & \hat{\xi}_{p1} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & \hat{\xi}_{1n} & \hat{\xi}_{2n} & \cdots & \hat{\xi}_{pn} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \hat{\xi}_{11} & \hat{\xi}_{21} & \cdots & \hat{\xi}_{p1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \hat{\xi}_{12} & \hat{\xi}_{22} & \cdots & \hat{\xi}_{p2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \hat{\xi}_{1n} & \hat{\xi}_{2n} & \cdots & \hat{\xi}_{pn}
\end{bmatrix}_{qn\times(pq+q)}
\begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \vdots \\ \beta_{1p} \\ \beta_{20} \\ \beta_{21} \\ \beta_{22} \\ \vdots \\ \beta_{2p} \\ \vdots \\ \beta_{q0} \\ \beta_{q1} \\ \beta_{q2} \\ \vdots \\ \beta_{qp} \end{bmatrix}_{(pq+q)\times 1}
+\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \vdots \\ \varepsilon_{1n} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{2n} \\ \vdots \\ \varepsilon_{q1} \\ \varepsilon_{q2} \\ \vdots \\ \varepsilon_{qn} \end{bmatrix}_{qn\times 1}
$$

or simplified in the following manner:

$$
\underset{qn\times 1}{\begin{bmatrix} \hat{\boldsymbol{\eta}}_1 \\ \hat{\boldsymbol{\eta}}_2 \\ \vdots \\ \hat{\boldsymbol{\eta}}_q \end{bmatrix}}
=
\underset{qn\times qn}{\begin{bmatrix} \lambda_1\mathbf{W} & 0 & \cdots & 0 \\ 0 & \lambda_2\mathbf{W} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_q\mathbf{W} \end{bmatrix}^T}
\underset{qn\times 1}{\begin{bmatrix} \hat{\boldsymbol{\eta}}_1 \\ \hat{\boldsymbol{\eta}}_2 \\ \vdots \\ \hat{\boldsymbol{\eta}}_q \end{bmatrix}}
+
\underset{qn\times(pq+q)}{\begin{bmatrix} \hat{\boldsymbol{\xi}} & 0 & \cdots & 0 \\ 0 & \hat{\boldsymbol{\xi}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{\boldsymbol{\xi}} \end{bmatrix}}
\underset{(pq+q)\times 1}{\begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_q \end{bmatrix}}
+
\underset{qn\times 1}{\begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \vdots \\ \boldsymbol{\varepsilon}_q \end{bmatrix}}
\qquad (11)
$$

where, $\hat{\boldsymbol{\eta}}_1 = \left[\hat{\eta}_{11} \quad \hat{\eta}_{12} \quad \cdots \quad \hat{\eta}_{1i} \quad \cdots \quad \hat{\eta}_{1n}\right]^T$, $\hat{\boldsymbol{\eta}}_2 = \left[\hat{\eta}_{21} \quad \hat{\eta}_{22} \quad \cdots \quad \hat{\eta}_{2i} \quad \cdots \quad \hat{\eta}_{2n}\right]^T$ till $\hat{\boldsymbol{\eta}}_q = \left[\hat{\eta}_{q1} \quad \hat{\eta}_{q2} \quad \cdots \quad \hat{\eta}_{qi} \quad \cdots \quad \hat{\eta}_{qn}\right]^T$, $\boldsymbol{\beta}_1 = \left[1 \quad \beta_{11} \quad \beta_{12} \quad \cdots \quad \beta_{1p}\right]^T$, $\boldsymbol{\beta}_2 = \left[1 \quad \beta_{21} \quad \beta_{22} \quad \cdots \quad \beta_{2p}\right]^T$ till $\boldsymbol{\beta}_q = \left[1 \quad \beta_{q1} \quad \beta_{q2} \quad \cdots \quad \beta_{qp}\right]^T$, matrix $\mathbf{W}$ of dimension $n\times n$, and matrix $\hat{\boldsymbol{\xi}}$ of dimension $n\times(p+1)$.

Supposing Equation (11) is stated in vector operation form, using the Kronecker product,

$\hat{\mathbf{H}} = \begin{bmatrix} \hat{\boldsymbol{\eta}}_1 & \hat{\boldsymbol{\eta}}_2 & \cdots & \hat{\boldsymbol{\eta}}_q \end{bmatrix}^T$, $\mathbf{B} = \begin{bmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \cdots & \boldsymbol{\beta}_p \end{bmatrix}$, $\boldsymbol{\Xi} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 & \cdots & \boldsymbol{\varepsilon}_q \end{bmatrix}$ and matrix $\boldsymbol{\Lambda} = diag\left(\lambda_1 \;\; \lambda_2 \;\; \ldots \;\; \lambda_q\right)$, then it will become the MSAR-LVs model as in Equation (12)

$$\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} = \left(\underset{q\times q}{\boldsymbol{\Lambda}^T} \otimes \underset{n\times n}{\mathbf{W}}\right)\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} + \left(\underset{q\times q}{\mathbf{I}_q} \otimes \underset{n\times(p+1)}{\hat{\boldsymbol{\xi}}}\right)\underset{(pq+q)\times 1}{vec\left(\mathbf{B}\right)} + \underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)} \tag{12}$$

The MSAR-LVs model stated in Equation (12) is further simplified in Equation (13).

$$\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} = \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)^{-1}} \underset{qn\times(pq+q)}{\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right)} \underset{(pq+q)\times 1}{vec\left(\mathbf{B}\right)} + \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)^{-1}} \underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)} \tag{13}$$

3.2 The Error Distribution of MSAR-LVs

Equations (12) and (13) represent a multivariate spatial regression model in which variable $\hat{\mathbf{H}}$ serves as the response variable and $\hat{\boldsymbol{\xi}}$ is the random independent variable. Hence, the function of $\hat{\mathbf{H}}$ is expressed as $f\left(\hat{\mathbf{H}} \mid \hat{\boldsymbol{\xi}}\right)$. Therefore, variable $\hat{\boldsymbol{\xi}}$ is now fixed rather than random. The error equation of MSAR-LVs as in Equation (12) is:

$$\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)} = \left(\mathbf{I}_{qn} - \underset{q\times q}{\boldsymbol{\Lambda}^T} \otimes \underset{n\times n}{\mathbf{W}}\right)\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \left(\underset{q\times q}{\mathbf{I}_q} \otimes \underset{n\times(p+1)}{\hat{\boldsymbol{\xi}}}\right)\underset{(pq+q)\times 1}{vec\left(\mathbf{B}\right)} \tag{14}$$

The error distribution as in Equation (14) is derived from the expected value and variance, namely:

$$E\left(\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)}\right) = E\left(\underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \underset{qn\times(pq+q)}{\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right)} \underset{(pq+q)\times 1}{vec\left(\mathbf{B}\right)}\right)$$

$$E\left(\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)}\right) = \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \underset{qn\times 1}{vec\left(\boldsymbol{\eta}_n \mathbf{e}^T\right)}$$

The variance value is

$$\operatorname{var}\left(\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)}\right) = \operatorname{var}\left(\underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \underset{qn\times(pq+q)}{\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right)} \underset{(pq+q)\times 1}{vec\left(\mathbf{B}\right)}\right)$$

$$\operatorname{var}\left(\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)}\right) = \operatorname{var}\left(\underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}\right)$$

$$\operatorname{var}\left(\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)}\right) = \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \left(\left(\underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T} \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}} \underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \otimes \underset{n\times n}{\mathbf{I}}\right) \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)^T}$$

Therefore, the error distribution of the MSAR-LVs model is stated as follows

$$\underset{qn\times 1}{vec\left(\boldsymbol{\Xi}\right)} \sim N_{qn}\left(\underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \underset{qn\times 1}{vec\left(\boldsymbol{\eta}_n \mathbf{e}^T\right)}, \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)} \left(\left(\underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T} \underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}} \underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \otimes \underset{n\times n}{\mathbf{I}}\right) \underset{qn\times qn}{\left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)^T}\right) \tag{15}$$

where, it is assumed that the variable $\hat{\xi}$ does not correlate with the error $vec(\boldsymbol{\Xi})$ and $\text{cov}(\hat{\boldsymbol{\eta}}, \hat{\xi}) \neq \mathbf{0}$.

3.3 The Parameter Estimation of MSAR-LVs

The estimated parameters of the MSAR-LVs model are $vec(\hat{\mathbf{B}})$ and $\boldsymbol{\lambda}$ using the MLE method. The estimation process is carried out by proving Theorem 1.

***Theorem* 1:**
Assuming the MSAR-LVs model stated in Equation (12) uses the estimation method of MLE and means of concentrated log-likelihood, then the estimator for parameter $vec(\hat{\mathbf{B}})$ is $vec(\hat{\mathbf{B}})_{initial} = \left(\mathbf{I}_q \otimes \left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\left(\mathbf{I}_{qn} - \hat{\boldsymbol{\Lambda}}^T \otimes \mathbf{W}\right)vec(\hat{\mathbf{H}})$, where $\hat{\boldsymbol{\Lambda}}$ is the matrix $\boldsymbol{\Lambda}$ that produces the maximum value of $\ln L^{con}(\boldsymbol{\Lambda})$. Additionally, $\ln L^{con}(\boldsymbol{\Lambda})$ is the matrix $\boldsymbol{\Lambda}$ whose diagonal contains the values in the sequence $\boldsymbol{\lambda}$.

***Proof*:**
Parameter estimation of MSAR-VLs was performed using the MLE method and means of concentrated log-likelihood to obtain estimates of coefficients and the variance-covariance matrix. The process was based on the likelihood function of the response variable, in this case, $vec(\hat{\mathbf{H}})$.

By assuming $\underset{qn\times qn}{\mathbf{A}} = \left(\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)$ and $\boldsymbol{\Phi}_{\mathbf{y}} = \left(\underset{q\times B}{\boldsymbol{\Lambda}_{\mathbf{y}}^T}\,\underset{B\times B}{\boldsymbol{\Theta}_{\varepsilon}^{-1}}\,\underset{B\times q}{\boldsymbol{\Lambda}_{\mathbf{y}}}\right)^{-1} \otimes \underset{n\times n}{\mathbf{I}}$, the error as in Equation (14) became $\underset{qn\times 1}{vec(\boldsymbol{\Xi})} = \underset{qn\times qn}{\mathbf{A}}\,\underset{qn\times 1}{vec(\hat{\mathbf{H}})} - \underset{qn\times(pq+q)}{\left(\mathbf{I}_q \otimes \hat{\xi}\right)}\,\underset{(pq+q)\times 1}{vec(\mathbf{B})}$, and the error distribution as in Equation (15) was simplified as follows $\underset{qn\times 1}{vec(\boldsymbol{\Xi})} \sim N_{qn}\left(\underset{qn\times qn}{\mathbf{A}}\,\underset{qn\times 1}{vec(\boldsymbol{\eta}_n\mathbf{e}^T)}, \underset{qn\times qn}{\mathbf{A}}\,\underset{qn\times qn}{\boldsymbol{\Phi}_{\mathbf{y}}}\,\underset{qn\times qn}{\mathbf{A}^T}\right)$.

The Jacobian for the error equation as in (14) is represented as $J = \left|\frac{\partial Vec(\boldsymbol{\Xi})}{\partial \hat{\mathbf{H}}}\right| = \left|\mathbf{I}_{qn} - \boldsymbol{\Lambda}^T \otimes \mathbf{W}\right| = |\mathbf{A}|$. Using the Gaussian function, the likelihood function for the error was derived as follows: $L(\sigma^2, \boldsymbol{\varepsilon}) = c(\boldsymbol{\varepsilon})|V|^{-1/2}\exp\left[-\frac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{V}^{-1}\boldsymbol{\varepsilon}\right]$, where $\mathbf{V}$ is the variance-covariance matrix of $vec(\boldsymbol{\Xi})$, denoted as $\mathbf{V} = \underset{qn\times qn}{\mathbf{A}}\,\underset{qn\times qn}{\boldsymbol{\Phi}_{\mathbf{y}}}\,\underset{qn\times qn}{\mathbf{A}^T}$.

Likelihood function $vec(\hat{\mathbf{H}})$ of MSAR-LVs model is obtained by substituting error $vec(\boldsymbol{\Xi})$ and multipliying the value by Jacobian, stated as follows $L(\boldsymbol{\Lambda}, \mathbf{B}, \mathbf{V}; \hat{\mathbf{H}}) = |\mathbf{V}|^{-1/2}|\mathbf{A}|\exp\left[-\frac{1}{2}\left(\mathbf{A}vec(\hat{\mathbf{H}}) - \left(\mathbf{I}_q \otimes \hat{\xi}\right)vec(\mathbf{B})\right)^T\mathbf{V}^{-1}\left(\mathbf{A}vec(\hat{\mathbf{H}}) - \left(\mathbf{I}_q \otimes \hat{\xi}\right)vec(\mathbf{B})\right)\right]$, while ln likelihood function of MSAR-LVs model is stated in Equations (16) and (17).

$$\mathcal{L}\left(\boldsymbol{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)=-\frac{1}{2}\ln\left|\mathbf{V}\right|+\ln\left|\mathbf{A}\right|-\frac{1}{2}\left(\left(vec\left(\boldsymbol{\Xi}\right)\right)^{T}\mathbf{V}^{-1}vec\left(\boldsymbol{\Xi}\right)\right) \tag{16}$$

$$\mathcal{L}\left(\boldsymbol{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)=-\frac{1}{2}\ln\left|\mathbf{V}\right|+\ln\left|\mathbf{A}\right|-\frac{1}{2}\left[\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\left(\mathbf{I}_q\otimes\hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^{T}\mathbf{V}^{-1}\times\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\left(\mathbf{I}_q\otimes\hat{\xi}\right)vec\left(\mathbf{B}\right)\right)\right] \tag{17}$$

The next step was to determine the first derivative of Equations (16) and (17) in respect to parameter $vec\left(\mathbf{B}\right)$ (Appendix). The first derivative of $vec\left(\mathbf{B}\right)$ was set to zero to obtain an estimate, resulting in the following: $\frac{\partial\ln\mathcal{L}\left(\boldsymbol{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial vec\left(\mathbf{B}\right)}=\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)\left(\mathbf{I}_q\otimes\hat{\xi}\right)vec\left(\mathbf{B}\right)=\mathbf{0}$, moved side to become $\underset{(pq+q)\times qn}{\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)}\underset{qn\times(pq+q)}{\left(\mathbf{I}_q\otimes\hat{\xi}\right)}\underset{(pq+q)x1}{vec\left(\mathbf{B}\right)}\Big|_{vec\left(\mathbf{B}\right)=vec\left(\hat{\mathbf{B}}\right)_{initial}}=\underset{(pq+q)\times qn}{\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)}\underset{qn\times qn}{\mathbf{A}}\underset{qn\times1}{vec\left(\hat{\mathbf{H}}\right)}$, left and right side multiplied by $\left(\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)\left(\mathbf{I}_q\otimes\hat{\xi}\right)\right)^{-1}$ to become $\underset{(pq+q)\times1}{vec\left(\hat{\mathbf{B}}\right)_{initial}}=\underset{(pq+q)\times(pq+q)}{\left(\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)\left(\mathbf{I}_q\otimes\hat{\xi}\right)\right)^{-1}}\underset{(pq+q)\times1}{\left(\mathbf{I}_q\otimes\hat{\xi}^{T}\right)\mathbf{A}\,vec\left(\hat{\mathbf{H}}\right)}$, and then simplified to $\underset{(pq+q)\times1}{vec\left(\hat{\mathbf{B}}\right)_{initial}}=\underset{(pq+q)\times qn}{\left(\mathbf{I}_q\otimes\left(\hat{\xi}^{T}\hat{\xi}\right)^{-1}\hat{\xi}^{T}\right)}\underset{qn\times qn}{\mathbf{A}}\underset{qn\times1}{vec\left(\hat{\mathbf{H}}\right)}$.

Therefore, the estimator $vec\left(\mathbf{B}\right)$ was rewritten as stated in Equation (18).

$$vec\left(\hat{\mathbf{B}}\right)_{initial}=\underset{(pq+q)\times qn}{\left(\mathbf{I}_q\otimes\left(\hat{\xi}^{T}\hat{\xi}\right)^{-1}\hat{\xi}^{T}\right)}\underset{qn\times qn}{\left(\mathbf{I}_{qn}-\boldsymbol{\Lambda}^{T}\otimes\mathbf{W}\right)}\underset{qn\times1}{vec\left(\hat{\mathbf{H}}\right)} \tag{18}$$

The estimator of the variance-covariance matrix $\mathbf{V}$ was obtained by differentiating the ln likelihood function as shown in Equation (16), which was substituted with the $vec\left(\boldsymbol{\beta}\right)$ estimator

$$\ln\mathcal{L}\left(\boldsymbol{\Lambda},vec\left(\hat{\mathbf{B}}\right)_{initial},\mathbf{V};\hat{\mathbf{H}}\right)=-\frac{1}{2}\ln\left|\underset{qn\times qn}{\mathbf{V}}\right|+\ln\left|\underset{qn\times qn}{\mathbf{A}}\right|-\frac{1}{2}\left(\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\left(\mathbf{I}_q\otimes\hat{\xi}\right)\underset{qn\times1}{\left(\mathbf{I}_q\otimes\left(\hat{\xi}^{T}\hat{\xi}\right)^{-1}\hat{\xi}^{T}\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)}\right)^{T}\underset{qn\times qn}{\mathbf{V}^{-1}}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\left(\mathbf{I}_q\otimes\hat{\xi}\right)\underset{qn\times1}{\left(\mathbf{I}_q\otimes\left(\hat{\xi}^{T}\hat{\xi}\right)^{-1}\hat{\xi}^{T}\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)}\right)\right).$$

Based on the properties of the Kronecker product, the ln likelihood equation was rewritten as follows in Equation (19).

$$\ln\mathcal{L}\left(\boldsymbol{\Lambda},vec\left(\hat{\mathbf{B}}\right)_{initial},\mathbf{V};\hat{\mathbf{H}}\right)=-\frac{1}{2}\ln\left|\underset{qn\times qn}{\mathbf{V}}\right|+\ln\left|\underset{qn\times qn}{\mathbf{A}}\right|-\frac{1}{2}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\underset{qn\times1}{\left(\mathbf{I}_q\otimes\hat{\xi}\left(\hat{\xi}^{T}\hat{\xi}\right)^{-1}\hat{\xi}^{T}\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)}\right)^{T}\times\underset{qn\times qn}{\mathbf{V}^{-1}}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right)-\underset{qn\times1}{\left(\mathbf{I}_q\otimes\hat{\xi}\left(\hat{\xi}^{T}\hat{\xi}\right)^{-1}\hat{\xi}^{T}\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)}\right) \tag{19}$$

Equation (19) consisted of the last element in the form of $(\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})$, which produced a real number with a matrix of size 1×1, therefore the trace was the element itself. Therefore, Equation (19) was rewritten as Equation (20).

$$\ln\mathcal{L}\left(\boldsymbol{\Lambda}, vec\left(\hat{\mathbf{B}}\right)_{in}, \mathbf{V}; \hat{\mathbf{H}}\right) = -\frac{1}{2}\ln|\mathbf{V}| + \ln|\mathbf{A}| - \frac{1}{2}tr\left[\left(\mathbf{V}^{-1}\right)\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)^T \times \left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)\right] \tag{20}$$

The ln likelihood function in Equation (20) was then differentiated to **V** (Appendix) to obtain the estimated value. The derivative of the ln likelihood function in respect to **V** was set at zero. This was solved by equating the left-hand side of the equation with the right-hand side, resulting in the estimator of the parameter **V**, stated as follows:

$$\frac{\partial \ln\mathcal{L}\left(\boldsymbol{\Lambda}, \mathbf{V}; \hat{\mathbf{H}}\right)}{\partial \mathbf{V}} = -\frac{1}{2}tr\left(\mathbf{V}^{-1}\right) + \frac{1}{2}tr\left[\mathbf{V}^{-1}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right) \times \left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)^T \mathbf{V}^{-1}\right] = 0$$

$$tr\left(\underset{qn\times qn}{\mathbf{V}^{-1}}\right) = tr\left[\mathbf{V}^{-1}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)^T \mathbf{V}^{-1}\right].$$

Therefore, the estimator ***V*** was determined using the following Equation (21).

$$\underset{qn\times qn}{\hat{\mathbf{V}}} = \left(\left(\mathbf{I}_{qn} - \underset{qn\times qn}{\boldsymbol{\Lambda}^T \otimes \mathbf{W}}\right)\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \underset{qn\times qn}{\left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\left(\mathbf{I}_{qn} - \underset{qn\times qn}{\boldsymbol{\Lambda}^T \otimes \mathbf{W}}\right)\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}\right) \times \left(\left(\mathbf{I}_{qn} - \underset{qn\times qn}{\boldsymbol{\Lambda}^T \otimes \mathbf{W}}\right)\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \underset{qn\times qn}{\left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\left(\mathbf{I}_{qn} - \underset{qn\times qn}{\boldsymbol{\Lambda}^T \otimes \mathbf{W}}\right)\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}\right)^T \tag{21}$$

The estimators $vec(\mathbf{B})$ and **V** obtained using Equations (18) and (21), respectively were not in closed form because it contained parameter $\boldsymbol{\Lambda}$. Therefore, parameter $\boldsymbol{\Lambda}$ was estimated using a numerical approximation of the concentrated log-likelihood. The concentrated log-likelihood function for $\boldsymbol{\Lambda}$ was the ln-likelihood function substituted with estimators $vec(\mathbf{B})$ and **V**, depicting the ln-likelihood function only contained parameter $\boldsymbol{\Lambda}$. Before determining the concentrated log-likelihood function, estimator $vec(\mathbf{B})$ in Equation (18) was simplified as stated in Equation (22).

$$vec\left(\hat{\mathbf{B}}\right)_{initial} = \underset{(pq+q)\times qn}{\left(\mathbf{I}_q \otimes \left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\underset{qn\times qn}{\left(\boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)}\ \underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}$$

$$= \underset{(pq+q)\times qn}{\left(\mathbf{I}_q \otimes \left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \underset{(pq+q)\times qn}{\left(\mathbf{I}_q \otimes \left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\underset{qn\times qn}{\left(\boldsymbol{\Lambda}^T \otimes \mathbf{W}\right)}\ \underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} \tag{22}$$

Matrix $\underset{(pq+q)\times qn}{\left(\mathbf{I}_q \otimes \left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}$ represents an estimate of $vec(\mathbf{B})$ in the multivariate linear regression of $vec\left(\hat{\mathbf{H}}\right)$ to $\left(\mathbf{I}_q \otimes \hat{\xi}\right)$, while the other elements need to be modified, therefore Equation (22) becomes Equation (23).

$$\underset{(pq+q)\times 1}{vec\left(\hat{\mathbf{B}}\right)_{initial}} = \underset{(pq+q)\times 1}{vec\left(\hat{\mathbf{B}}\right)_{multi}} - \underset{(pq+q)\times 1}{vec\left(\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\mathbf{W}\hat{\mathbf{H}}\mathbf{\Lambda}\right)} \tag{23}$$

Let $\hat{\mathbf{B}}_{WH} = \underset{(p+1)\times n}{\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T}\mathbf{W}\hat{\mathbf{H}}$ the matrix of the regression coefficient estimator $\mathbf{W}\hat{\mathbf{H}}$ to $\hat{\xi}$, which is $(p+1)\times q$ in size, then Equation (23) becomes Equation (24).

$$\underset{(pq+q)\times 1}{vec\left(\hat{\mathbf{B}}\right)_{initial}} = \underset{(pq+q)\times 1}{vec\left(\hat{\mathbf{B}}\right)_{multi}} - \underset{(pq+q)\times qq}{\left(\mathbf{I}_q \otimes \hat{\mathbf{B}}_{WH}\right)}\underset{qq\times 1}{vec(\mathbf{\Lambda})} \tag{24}$$

Equations (24) and (21) were substituted into Equation (19) and the concentrated log-likelihood function was obtained using Equation (25).

$$\begin{aligned}
&\ln \mathcal{L}_{con}(\mathbf{\Lambda}) = \\
&-\frac{1}{2}\ln\left|\left(\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right)\right)\right. \\
&\left.\left(\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right)\right)^T\right| + \\
&+\ln\left|\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)\right| - \frac{1}{2}\left(\left(\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\hat{\mathbf{B}}\right)_{multi} - \left(\mathbf{I}_q \otimes \hat{\mathbf{B}}_{WH}\right)vec(\mathbf{\Lambda})\right)^T \times\right. \\
&\left(\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right)\right) \times \\
&\left(\left(\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right)\right)^T\right)^{-1} \\
&\left.\left(\left(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}\right)vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\hat{\mathbf{B}}\right)_{multi} - \left(\mathbf{I}_q \otimes \hat{\mathbf{B}}_{WH}\right)vec(\mathbf{\Lambda})\right)\right)
\end{aligned} \tag{25}$$

in an approximation of the linear coefficients $vec\left(\hat{\mathbf{B}}\right)_{multi}$. This was followed by carrying out a regression $\mathbf{W}\hat{\mathbf{H}}$ against $\hat{\xi}$ as well as determining the regression coefficients in a matrix $\hat{\mathbf{B}}_{WH}$ of size $(p+1)\times q$. The values of $vec\left(\hat{\mathbf{B}}\right)_{multi}$ and $\hat{\mathbf{B}}_{WH}$ were substituted into Equation (25). Subsequently, the eigenvalue of matrix $\mathbf{W}$ ($\psi_{\min}$) was calculated, forming a sequence $\boldsymbol{\lambda}$ with the initial and final values of $1/\psi_{\min}$ and 0.99 at a certain interval. Each value in the sequence $\boldsymbol{\lambda}$ was substituted with the diagonal of matrix $\mathbf{\Lambda}$, to produce $\ln L^{con}(\mathbf{\Lambda})$. The matrix $\mathbf{\Lambda}$ that produces the maximum value of $\ln L^{con}(\mathbf{\Lambda})$ was selected as $\hat{\mathbf{\Lambda}}$. The $vec\left(\hat{\mathbf{B}}\right)$ value was determined by substituting $\hat{\mathbf{\Lambda}}$ into the Equation (24).

3.4 The Spatial Dependency Test of MSAR-LVs

Spatial dependency test on the MSAR-LVs model was conducted using the LM test and using the following hypothesis basis:

$H_0 : \lambda_j = 0$

$H_1 : \lambda_j \neq 0; \quad j = 1, 2, ..., q$.

***Theorem* 2:**

If the MSAR-VLs model as in Equation (12), the error equation as in Equation (14), and the error distribution as in Equation (15), then the Lagrange Multiplier test is $\underset{1\times 1}{\text{LM}_\Lambda} = \dfrac{\left(\underset{1\times qn}{\tilde{\boldsymbol{\varepsilon}}^T} \underset{qn\times qn}{\boldsymbol{\Phi}_{\mathbf{y}}^{-1}} \underset{qn\times(pq+q)}{\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right)} \underset{(pq+q)\times 1}{vec\left(\mathbf{B}\right)} \right)^2}{\underset{1\times 1}{p}}$

where

$$\underset{1\times 1}{p} = \left(\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec\left(\mathbf{B}\right)\right)^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(vec\left(\boldsymbol{\eta}_n \mathbf{e}^T\right) - \left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec\left(\mathbf{B}\right)\right) + \left(\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec\left(\mathbf{B}\right) - vec\left(\boldsymbol{\eta}_n \mathbf{e}^T\right)\right)^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} vec\left(\boldsymbol{\eta}_n \mathbf{e}^T\right)$$

and $\tilde{\boldsymbol{\varepsilon}} = vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec\left(\mathbf{B}\right)$ and under H_0, the LM_Λ test statistic follows the $\chi^2_{(1)}$ distribution.

***Proof*:**

The LM test was based on estimation under the null hypothesis. The process of obtaining the LM statistic required the ln likelihood function of the MSAR-VLs model stated in Equations (16) and (17), the first and second derivatives of the ln likelihood function to $\boldsymbol{\Lambda}$ and $vec\left(\mathbf{B}\right)$, as well as the information matrix elements. The first and second derivatives of the ln likelihood function to $\boldsymbol{\Lambda}$ and $vec\left(\mathbf{B}\right)$ are found in Appendix.

Breusch & Pagan (1980), defined the LM test statistic as follows $\text{LM} = \hat{\mathbf{D}}_\lambda^T \hat{\boldsymbol{\psi}}^{-1} \hat{\mathbf{D}}_\lambda$, where $\hat{\mathbf{D}}_\lambda$ is the first derivative of the ln likelihood function to λ at $\lambda = 0$ and $\hat{\boldsymbol{\psi}}^{-1}$ is the (1,1) element of the inverse information matrix $\tilde{\boldsymbol{\psi}}_\theta$ of size 2×2. These elements are regarded as the second derivatives to each estimated parameter $\tilde{\boldsymbol{\psi}}_\theta = E\left[-\dfrac{\partial^2 L\left(\theta\right)}{\partial\theta\partial\theta'} \right]$.

The LM test statistic for the MSAR-LVs model is $\text{LM} = \hat{\mathbf{D}}_\Lambda^T \hat{\boldsymbol{\psi}}_{\Lambda\Lambda}^{-1} \hat{\mathbf{D}}_\Lambda$, where $\hat{\mathbf{D}}_\Lambda$ is the first derivative of the ln likelihood function to λ at $\lambda = 0$ and $\hat{\boldsymbol{\psi}}_{\Lambda\Lambda}^{-1}$ is the (1,1) element of the inverse information matrix $\tilde{\boldsymbol{\psi}}_\theta$. The information matrix for the MSAR-LVs model when $\lambda = 0$ is $\tilde{\boldsymbol{\psi}}_\theta = \begin{pmatrix} \underset{1\times 1}{\tilde{\psi}_{\Lambda\Lambda}} & \underset{1\times(pq+q)}{\tilde{\boldsymbol{\psi}}_{\Lambda\mathbf{B}}} \\ \underset{(pq+q)\times 1}{\tilde{\boldsymbol{\psi}}_{\mathbf{B}\Lambda}} & \underset{(pq+q)\times(pq+q)}{\tilde{\boldsymbol{\psi}}_{\mathbf{B}\mathbf{B}^T}} \end{pmatrix}$. Each element of the information matrix is found in Appendix.

The element of the inverse information matrix (1,1) is $\underset{1\times1}{\tilde{\psi}_{\Lambda\Lambda}^{-1}} = \left(\underset{1\times1}{\tilde{\psi}_{\Lambda\Lambda}} - \underset{1\times(pq+q)}{\tilde{\boldsymbol{\psi}}_{\Lambda\mathbf{B}}} \left(\underset{(pq+q)\times(pq+q)}{\tilde{\boldsymbol{\psi}}_{\mathbf{BB}^T}} \right)^{-1} \underset{(pq+q)\times1}{\tilde{\boldsymbol{\psi}}_{\mathbf{B}\Lambda}} \right)^{-1}$ as stated in Equation (26).

$$\begin{aligned}\tilde{\psi}_{\Lambda\Lambda}^{-1} = &-\left(\left(\left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right) \right)^T \left(\mathbf{I}_q \otimes \mathbf{W} \right)^2 \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right) \right)^{-1} \\ &- \left(\left(vec\left(\boldsymbol{\eta}_n \mathbf{e}^T \right) \right)^T \left(\mathbf{I}_q \otimes \mathbf{W} \right)^2 \boldsymbol{\Phi}_{\mathbf{y}}^{-1} vec\left(\boldsymbol{\eta}_n \mathbf{e}^T \right) \right)^{-1} + \\ &+ \frac{1}{2} \left(\left(\left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right) \right)^T \left(\mathbf{I}_q \otimes \mathbf{W} \right)^2 \boldsymbol{\Phi}_{\mathbf{y}}^{-1} vec\left(\boldsymbol{\eta}_n \mathbf{e}^T \right) \right)\end{aligned} \tag{26}$$

The value of $\hat{\mathbf{D}}_\Lambda$ is the first derivative of the ln likelihood function to λ at $\lambda = 0$, namely

$$\hat{\mathbf{D}}_\Lambda = \left(vec\left(\hat{\mathbf{H}} \right) \right)^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(\mathbf{I}_q \otimes \mathbf{W} \right) \left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) - \left(\left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) \right)^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(\mathbf{I}_q \otimes \mathbf{W} \right) \left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) \tag{27}$$

Vector $vec\left(\hat{\mathbf{H}} \right) - \left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right)$ is the error of the OLS regression model, therefore $vec\left(\hat{\mathbf{H}} \right) - \left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right) = \tilde{\boldsymbol{\varepsilon}}$, the value of $\hat{\mathbf{D}}_\Lambda$ as stated in Equation (27) becomes Equation (28).

$$\hat{\mathbf{D}}_\Lambda = \tilde{\boldsymbol{\varepsilon}}^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(\mathbf{I}_q \otimes \mathbf{W} \right) \left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right) \tag{28}$$

The test statistic value $\text{LM} = \hat{\mathbf{D}}_\Lambda^T \hat{\boldsymbol{\psi}}_{\Lambda\Lambda}^{-1} \hat{\mathbf{D}}_\Lambda$ was determined from Equations (26) and (28)

$$\underset{1\times1}{\text{LM}_\Lambda} = \frac{\left(\underset{1\times qn}{\tilde{\boldsymbol{\varepsilon}}^T} \underset{qn\times qn}{\boldsymbol{\Phi}_{\mathbf{y}}^{-1}} \underset{qn\times(pq+q)}{\left(\mathbf{I}_q \otimes \hat{\xi} \right)} \underset{(pq+q)\times1}{vec(\mathbf{B})} \right)^2}{\underset{1\times1}{p}} \tag{29}$$

where,

$$\underset{1\times1}{p} = \left(\left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) \right)^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(vec\left(\boldsymbol{\eta}_n \mathbf{e}^T \right) - \left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) \right) + \left(\left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) - vec\left(\boldsymbol{\eta}_n \mathbf{e}^T \right) \right)^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} vec\left(\boldsymbol{\eta}_n \mathbf{e}^T \right)$$

and $\tilde{\boldsymbol{\varepsilon}} = vec\left(\hat{\mathbf{H}} \right) - \left(\mathbf{I}_q \otimes \hat{\xi} \right) \left(vec(\mathbf{B}) \right)$.

The LM test statistic was defined by Breusch & Pagan (1980), as $\text{LM} = \hat{\mathbf{D}}_\Lambda^T \tilde{\psi}_{\Lambda\Lambda}^{-1} \hat{\mathbf{D}}_\Lambda$. Assuming H_0: $\left[\Lambda, \rho, \boldsymbol{\alpha}' \right] = 0$ where there are p parameters from $\boldsymbol{\alpha}$ related to non-constraint, then according to Anselin (1988), $\text{LM} = \hat{\mathbf{D}}_\Lambda^T \tilde{\psi}_{\Lambda\Lambda}^{-1} \hat{\mathbf{D}}_\Lambda$ follows $\chi^2_{(2+p)}$. The evaluation of the MSAR-LVs model using the LM statistic test as stated in Equation (29) with H_0: $\lambda = 0$ then $\text{LM}_\Lambda \sim \chi^2_{(1)}$ or $\dfrac{\left(\tilde{\boldsymbol{\varepsilon}}^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1} \left(\mathbf{I}_q \otimes \hat{\xi} \right) vec(\mathbf{B}) \right)^2}{p} \sim \chi^2_{(1)}$.

The rejection region for H_0 focused on rejecting H_0 assuming $\mathrm{LM}_\Lambda \geq c \Leftrightarrow \frac{\left(\tilde{\boldsymbol{\varepsilon}}^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec(\mathbf{B})\right)^2}{p} \geq c$.

Testing carried out under H_0 and the distribution of $\frac{\left(\tilde{\boldsymbol{\varepsilon}}^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec(\mathbf{B})\right)^2}{p}$ is $\chi^2_{(1)}$, then $\alpha = P\left\{\frac{\left(\tilde{\boldsymbol{\varepsilon}}^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec(\mathbf{B})\right)^2}{p} \geq c\right\}$ with the corresponding value of c, which is $\chi^2_{(1;1-\alpha)}$. Therefore, the LM test for the MSAR-VLs model rejected H_0 if $\frac{\left(\tilde{\boldsymbol{\varepsilon}}^T \boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}\right) vec(\mathbf{B})\right)^2}{p} \geq \chi^2_{(1;1-\alpha)}$.

3.5 Partial Parameter Significance Test

Partial hypothesis testing of parameters aimed to determine which exogenous factor scores significantly affected the endogenous factor scores individually. Moreover, the partial testing of parameter $\boldsymbol{\lambda}$ was first conducted using the following hypothesis.

$H_0 : \lambda_j = 0$

$H_1 : \lambda_j \neq 0; \quad j = 1, 2, ..., q$

The test statistic used for the hypothesis testing is the Wald test, which requires the information of the Hessian matrix $\mathbf{H}_\Lambda(\hat{\boldsymbol{\Lambda}})$. This is the second derivative matrix of the concentrated log-likelihood function in Equation (25), stated in Equation (30) as follows.

$$\mathbf{H}_\Lambda(\hat{\boldsymbol{\Lambda}}) = \begin{bmatrix} \frac{\partial^2 \ln L^{con}}{\partial \lambda_1^2} & \frac{\partial^2 \ln L^{con}}{\partial \lambda_1 \partial \lambda_2} & \cdots & \frac{\partial^2 \ln L^{con}}{\partial \lambda_1 \partial \lambda_q} \\ \frac{\partial^2 \ln L^{con}}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ln L^{con}}{\partial \lambda_2^2} & \cdots & \frac{\partial^2 \ln L^{con}}{\partial \lambda_2 \partial \lambda_q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \ln L^{con}}{\partial \lambda_1 \partial \lambda_q} & \frac{\partial^2 \ln L^{con}}{\partial \lambda_2 \partial \lambda_q} & \cdots & \frac{\partial^2 \ln L^{con}}{\partial \lambda_q^2} \end{bmatrix} \tag{30}$$

The Wald test statistic stated in Equation (31), was characterized by a chi-square distribution.

$$Wald_{\lambda j} = \left(\frac{\hat{\lambda}_j}{\hat{se}\left(\hat{\lambda}_j\right)}\right)^2 \sim \chi^2_1 \tag{31}$$

where $\hat{se}\left(\hat{\lambda}_j\right)$ is the standard error of λ_j, namely $\sqrt{\mathrm{var}\left(\lambda_j\right)}$. Meanwhile, $\mathrm{var}\left(\lambda_j\right)$ is the main diagonal element of the negative inverse of the Hessian matrix or $-\left(\mathbf{H}_\Lambda(\hat{\boldsymbol{\Lambda}})\right)^{-1}$, corresponding in position to λ_j, and the Hessian matrix in the Equation (34). The decision-making criterion for the Wald test statistic in Equation (31) is H_0 rejected if the value of $Wald_{\lambda j} > \chi^2_{\alpha,1}$.

The partial significance test of β_{kj} was conducted to determine which parameters had a significant impact on the model. The hypothesis used to carry out partial testing on the parameters were stated as follows.

$H_0 : \beta_{kj} = 0$

$H_1 : \beta_{kj} \neq 0, \quad k = 1, 2, \ldots, p, \quad j = 1, 2, .., q$.

Partial hypothesis testing was conducted using the Wald test due to the capability in determining the significance of individual parameters in complex models, and consistent assumption of normal distribution. The test statistic in the Wald test was determined using Equation (32).

$$Wald_{\beta j} = \left(\frac{\hat{\beta}_{kj}}{\hat{se}\left(\hat{\beta}_{kj}\right)} \right)^2 \sim \chi^2_1 \quad (32)$$

where $\hat{se}\left(\hat{\beta}_{kj}\right)$ is the standard error of β_{kj}, namely $\sqrt{\text{var}\left(\beta_{kj}\right)}$. The value of $\text{var}\left(\beta_{kj}\right)$ is the main diagonal element of the variance-covariance $\hat{Var}\left(vec\left(\hat{\mathbf{B}}\right)\right)$. While the decision-making criterion of H_0 is H_0 rejected if the value of $Wald_{\beta j} > \chi^2_{\alpha,1}$.

4. Discussion Concerning the Application of MSAR-VLs on Economic Growth Modeling

The MSAR-LVs was used to depict economic growth and to illustrate its use. The section on methods described the selection of latent variables, respective indicators, conceptual framework, and economic growth model shown in **Figure 1**.

The outer model evaluation was conducted before obtaining the factor scores, and this was aimed to determine indicators reflecting the latent variables through loading and P-values. In addition, each loading factor and P-value were represented as shown in **Table 2**.

The indicators considered for removal from the model are those with a P-value greater than 0.1, unless it has a factor loading exceeding 0.6. Strong convergent validity mandated that every indicator under a latent variable must have high loadings greater than 0.6. (Dash and Paul, 2021). Therefore, the improved model in **Figure 2** was used for further analysis by excluding two social demographic indicators, namely labor force participation (SD2) and dependency ratio (SD4).

The revised economic growth framework was reassessed at the outer model level, ensuring that all indicators accurately reflected the corresponding latent variables. **Table 3** shows the results of the outer model evaluation of the revised framework.

Table 2. Loading factor and P-value for outer model.

Indicator <- latent variable	Loading factor	P -values	Updated model
ECO1 <- Eco_Growth	0.805	0.000	Maintained
ECO2 <- Eco_Growth	0.670	0.000	Maintained
HC1 <- Human_Cap	0.759	0.000	Maintained
HC2 <- Human_Cap	0.550	0.000	Maintained
HC3 <- Human_Cap	0.736	0.000	Maintained
HC4 <- Human_Cap	0.857	0.000	Maintained
HC5 <- Human_Cap	0.963	0.000	Maintained
HC6 <- Human_Cap	0.979	0.000	Maintained
HC7 <- Human_Cap	0.976	0.000	Maintained
SD1 <- Sos_Demog	0.849	0.146	Maintained
SD2 <- Sos_Demog	-0.517	0.230	Removed
SD3 <- Sos_Demog	-0.875	0.150	Maintained
SD4 <- Sos_Demog	0.432	0.193	Removed
SE1 <- Sos_Eco	0.414	0.069	Maintained
SE2 <- Sos_Eco	0.762	0.000	Maintained
SE3 <- Sos_Eco	0.516	0.001	Maintained
SE4 <- Sos_Eco	0.906	0.000	Maintained
SE5 <- Sos_Eco	0.919	0.000	Maintained

Table 3. Loading factor and P-value for revised outer model.

Indicator <- latent variable	Loading factor	T statistics	P –values
ECO1 <- Eco_Growth	0.795	7.280	0.000
ECO2 <- Eco_Growth	0.683	4.884	0.000
HC1 <- Human_Cap	0.763	11.965	0.000
HC2 <- Human_Cap	0.547	4.446	0.000
HC3 <- Human_Cap	0.740	8.200	0.000
HC4 <- Human_Cap	0.855	21.233	0.000
HC5 <- Human_Cap	0.963	123.857	0.000
HC6 <- Human_Cap	0.978	190.976	0.000
HC7 <- Human_Cap	0.976	190.847	0.000
SD1 <- Sos_Demog	0.930	1.988	0.047
SD3 <- Sos_Demog	-0.828	1.824	0.069
SE1 <- Sos_Eco	0.414	1.878	0.061
SE2 <- Sos_Eco	0.762	5.393	0.000
SE3 <- Sos_Eco	0.516	3.173	0.002
SE4 <- Sos_Eco	0.906	13.733	0.000
SE5 <- Sos_Eco	0.919	12.986	0.000

All indicators with a P-value less than 0.1, implied that the improved model accurately reflected the latent variables. The next stage focused on estimating the latent variables to obtain factor scores used as substitutes for the response and predictor variables, further modelled using Equation (11). The LM test was conducted using Theorem 2, and the model was based on **Figure 2**, with the results shown in **Table 4**.

Based on **Table 4**, all results lead to the spatial autoregressive model, as represented in the following equation.

$$\hat{\boldsymbol{\eta}}_{Eco\_Growth} = \beta_0 + \lambda_1 \sum_{j=1;i\neq j}^{38} W_{ij}\hat{\eta}_{Eco\_Growth(j)} + \beta_1\hat{\boldsymbol{\xi}}_{Human\_Cap} + \beta_2\hat{\boldsymbol{\xi}}_{Sos\_Eco} + \beta_3\hat{\boldsymbol{\xi}}_{Sos\_Demog} \tag{33}$$

$$\hat{\boldsymbol{\eta}}_{Sos\_Demog} = \beta_0 + \lambda_2 \sum_{j=1;i\neq j}^{38} W_{ij}\hat{\eta}_{Sos\_Demog(j)} + \beta_1\hat{\boldsymbol{\xi}}_{Human\_Cap} \tag{34}$$

Table 5 summarized the results of parameter estimation and partial significance testing using the MLE and Wald test in sections 3.2 and 3.4, respectively with a significance threshold of $\alpha = 10\%$.

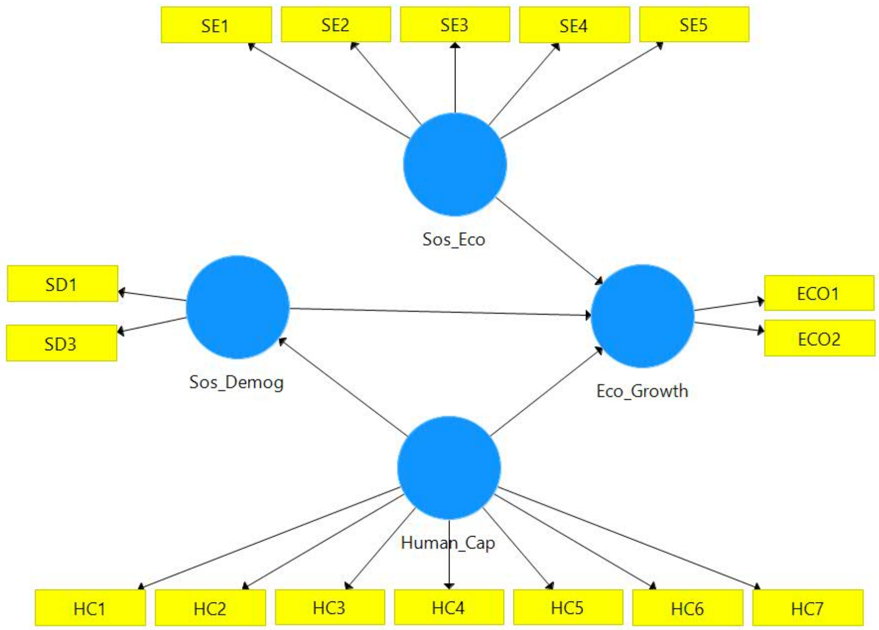

Figure 2. Revised model of economic growth.

Table 4. Lagrange multiplier test result.

Test	LM_{Λ}	**Result**
Economic growth	3.87×10^{6}	reject the null hypothesis
Social demograpic	5.75×10^{2}	reject the null hypothesis

Table 5. The estimation result of parameter model.

Economic growth		
Variable	**Coeficient**	**P-value**
λ_1 - spatial autoregressive coefficient	-0.094	0.071
β_0 - constant	-424.09	0.0098
β_1- human capital	13.32	0.0096
β_2 - social economic	3.23	0.4548
β_3 - social demograpic	-3.65	0.317
Social demograpic		
Variable	**Coeficient**	**P-value**
λ_2 - spatial autoregressive coefficient	0.19	0.141
β_0 - constant	-32.97	0.00
β_1- human capital	0.985	0.00

In the economic growth model, the spatial autoregressive coefficient, constant, and human capital had P-values less than 0.1. The result suggested that human capital was the most important factors influencing economic growth, characterized by a spill-over effect. As in Equation (33), the economic growth model was rewritten as in Equation (35).

$$\hat{\boldsymbol{\eta}}_{Eco\_Growth} = -424.09 - 0.094_1 \sum_{j=1;i\neq j}^{38} W_{ij}\hat{\eta}_{Eco\_Growth(j)} + 13.32\hat{\boldsymbol{\xi}}_{Human\_Cap} + 3.24\hat{\boldsymbol{\xi}}_{Sos\_Eco} - 3.65\hat{\boldsymbol{\xi}}_{Sos\_Demog} \tag{35}$$

The spatial autoregressive coefficient in the economic growth model was both significant and negative, indicating that spill-over effects from adjacent regions reduce economic growth in a given regency. A regency experienced a reduction in spill-over economic growth effects due to human capital support from neighbouring regencies.

The endogenous factor score is the economic growth of each regency, divided into four layers, as shown in **Figure 3**. Kediri City (code 30) is represented by the first layer of economic growth, using brown. Furthermore, Surabaya City (code 37) is the second layer with strong economic growth. The third layer, comprising several regencies with codes 15, 25, 35, etc characterized by medium economic growth, was depicted with light orange, while the fourth layer with the lowest economic growth was represented using the lightest color.

Surabaya City (code 37) is the capital of East Java Province, bordered by two neighbouring districts, namely Sidoarjo Regency (code 15) and Gresik (code 25). The economic growth model of Surabaya City can be expressed as stated in Equation (36).

$$\hat{\eta}_{Eco\_Growth[37]} = -424.09 - 0.047\hat{\eta}_{Eco\_Growth[15]} - 0.047\hat{\eta}_{Eco\_Growth[25]} + 13.32\hat{\xi}_{Human\_Cap} \tag{36}$$

The economic growth of Surabaya was significantly influenced by human capital, and this was proven by every one-point gain in human capital raises the economic growth by 13.32 points. However, it was negatively influenced by the economic growth of the neighboring regencies, Sidoarjo Regency (code 15) and Gresik (code 25). This implied that the economic growth of Surabaya tends to be decreasing due to the influence of human capital in both regencies.

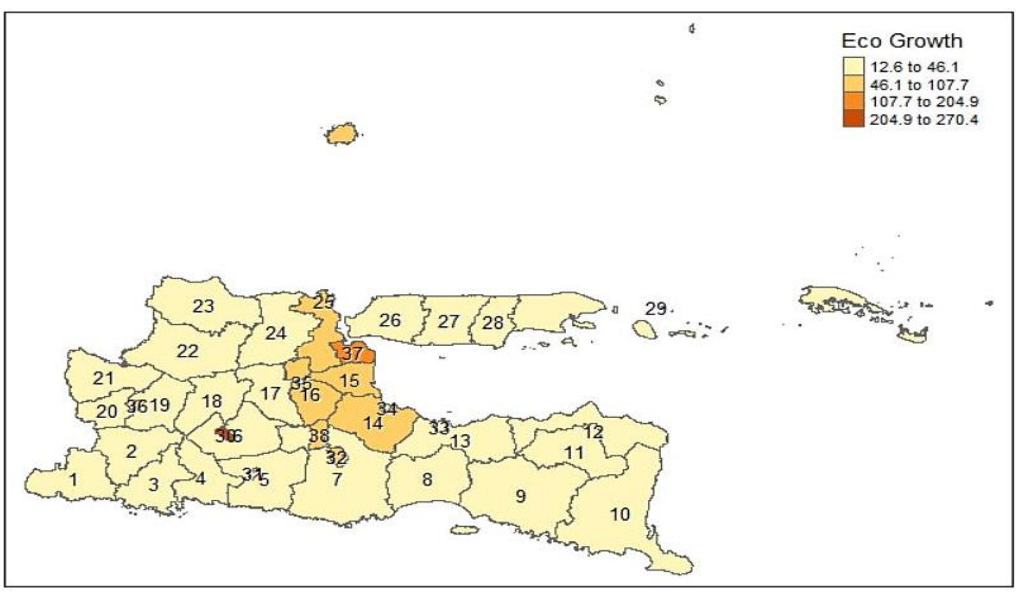

Figure 3. Distribution of economic growth factor scores.

In the social-demographic model, P-value of less than 0.1 depicted human capital, implying that this variable significantly influenced social demographics, without any spill-over effect. The social demographic model, stated in Equation (34), was expressed as in Equation (37).

$$\hat{\boldsymbol{\eta}}_{Sos\_Demog} = -32.97 + 0.19 \sum_{j=1;i\neq j}^{38} W_{ij}\hat{\eta}_{Sos\_Demog(j)} + 0.985\hat{\boldsymbol{\xi}}_{Human\_Cap} \tag{37}$$

The spatial autoregressive coefficient in the social demographic model is insignificant, depicting no spill-over spatial effect was obtained from the neighboring regencies.

The endogenous scores of social demographic factors for each regency/city were classified into four layers, as shown in **Figure 4**. The first layer represented the highest social demographic level, denoted with a dark green color, specifically the regencies with codes 15, 32, 37, and 38. The second layer was characterized by a high social demographic level with a light green color. The third and fourth layers had a medium and low social demographic level represented with bright yellow and red colors, respectively.

For example, Surabaya City (code 37), the capital of East Java Province, had the highest social demographic category, bordered by two neighboring regencies, namely Sidoarjo (code 15) and Gresik (code 25). In addition, the social demographic model was represented in Equation (38).

$$\hat{\eta}_{Sos\_Demog[37]} = -32.97 + 0.985\hat{\xi}_{Human\_Cap} \tag{38}$$

The social demographic modeling of Surabaya City was significantly influenced by human capital. Assuming human capital increases by 1 point, then the social demographic rises by 0.985 points.

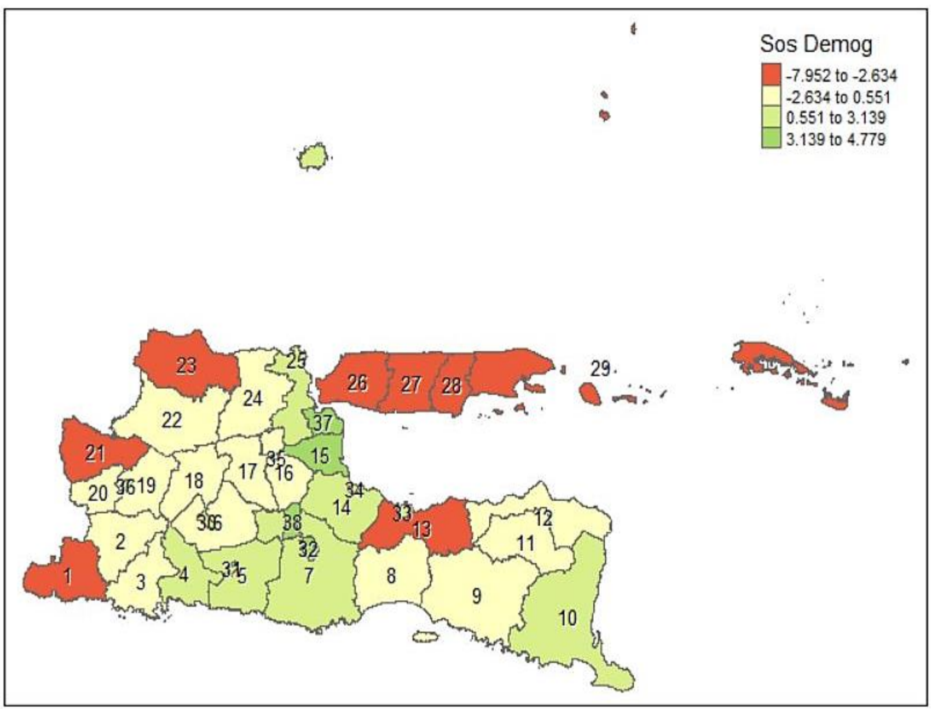

Figure 4. Distribution of of social_demographic factor scores.

The findings of the research indicate that human capital has a significant and positive influence on economic growth but also a negative spillover effect. This finding aligns with Jayadevan's research (Jayadevan, 2021), which stated that healthy human capital positively and significantly impacts economic growth. The research used SEM analysis without measuring spatial effects despite involving 181 countries as the research sample. In line with the research of Sarwar et al. (2021), which involved 83 developing countries, it was found that human capital positively impacts economic growth. The research used regression with the two-step GMM system method and did not involve spatial analysis. The similarity in the findings of this research is that the human capital variable positively influences economic development, while the difference is that it does not involve spatial effects. In fact, the economic growth of a country or region will influence its neighboring countries or regions. Thus, the economic growth of a region can be predicted through the coefficient value of its predictor variables and the spillover effects of the economic growth of its neighboring regions or countries. For example, the economic growth modeling in Surabaya City, as in Equation (36), is influenced by human capital (positive) and economic growth from the neighboring regencies (negative). This implies that the economic growth of Surabaya tends to decrease due to the influence of economic growth on both regencies. Meanwhile, the economic growth of both regencies is controlled by the human capital variable. This is suspected to be caused by the migration of young people with low education. Malamassam (2022) stated that rural areas were the main source of young, low-educated migrants to the capital city/province. Thus, the MSAR-VLs will provide more in-depth information and illustrate the phenomenon of the spillover effect.

Other findings in this study indicate that human capital affects social demographics. The investment in human capital resulted in a productive demographic trend (Campbell and Okuwa, 2016), contributing more to poverty reduction in underserved areas (Hanisya et al., 2024; Wau, 2022). However, it does not have a significant spatial effect. This means that the social demographics of a region do not receive spillover effects of social demographics from neighbouring areas. It is in line with Jayadevan's research (Jayadevan, 2021), which states that healthy human capital reduces the poverty rate of residents in a region without any spillover effects from the social demographics aspects of another region.

5. Conclusions

In conclusion, the integrative development of the MSAR-LVs led to several findings. The MSAR-LVs model and error distribution were stated as $vec(\hat{\mathbf{H}}) = (\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W})^{-1} (\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}) vec(\mathbf{B}) + (\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W})^{-1} vec(\mathbf{\Xi})$ and $vec(\mathbf{\Xi}) \sim N_{qn}\left((\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W}) vec(\boldsymbol{\eta}_n \mathbf{e}^T), (\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W})\left((\mathbf{\Lambda}_{\mathbf{y}}^T \mathbf{\Theta}_{\varepsilon}^{-1} \mathbf{\Lambda}_{\mathbf{y}})^{-1} \otimes \mathbf{I}_n\right)(\mathbf{I}_{qn} - \mathbf{\Lambda}^T \otimes \mathbf{W})^T\right)$. The result of parameter estimation was obtained using the MLE method and means of concentrated log-likelihood for estimator $vec(\hat{\mathbf{B}})$ $vec(\hat{\mathbf{B}})_{initial} = \left(\mathbf{I}_q \otimes (\hat{\boldsymbol{\xi}}^T \hat{\boldsymbol{\xi}})^{-1} \hat{\boldsymbol{\xi}}^T\right)(\mathbf{I}_{qn} - \hat{\mathbf{\Lambda}}^T \otimes \mathbf{W}) vec(\hat{\mathbf{H}})$, where $\hat{\mathbf{\Lambda}}$ is the matrix of $\mathbf{\Lambda}$ that produced the maximum $\ln L^{con}(\mathbf{\Lambda})$ value, and $\ln L^{con}(\mathbf{\Lambda})$ is the $\mathbf{\Lambda}$ matrix whose diagonal contained the values in the $\boldsymbol{\lambda}$ sequence. Meanwhile, the Lagrange multiplier test statistic was stated as follows:

$$\mathrm{LM}_{\Lambda} = \frac{\left(\tilde{\boldsymbol{\varepsilon}}^T \mathbf{\Phi}_{\mathbf{y}}^{-1} (\mathbf{I}_q \otimes \hat{\boldsymbol{\xi}}) vec(\mathbf{B})\right)^2}{p},$$

where,

$$\underset{1\times 1}{p}=\left(\left(\mathbf{I}_q\otimes\hat{\boldsymbol{\xi}}\right)vec(\mathbf{B})\right)^T\boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(vec\left(\boldsymbol{\eta}_n\mathbf{e}^T\right)-\left(\mathbf{I}_q\otimes\hat{\boldsymbol{\xi}}\right)vec(\mathbf{B})\right)+\left(\left(\mathbf{I}_q\otimes\hat{\boldsymbol{\xi}}\right)vec(\mathbf{B})-vec\left(\boldsymbol{\eta}_n\mathbf{e}^T\right)\right)^T\boldsymbol{\Phi}_{\mathbf{y}}^{-1}vec\left(\boldsymbol{\eta}_n\mathbf{e}^T\right)$$

and $\tilde{\boldsymbol{\varepsilon}}=vec\left(\hat{\mathbf{H}}\right)-\left(\mathbf{I}_q\otimes\hat{\boldsymbol{\xi}}\right)vec(\mathbf{B})$ as well as under H_0, test statistic LM_{Λ} followed the distribution of $\chi^2_{(1)}$.

Wald test statistics for λ and β included $Wald_{\lambda j}=\left(\frac{\hat{\lambda}_j}{\hat{se}\left(\hat{\lambda}_j\right)}\right)^2\sim\chi^2_1$ and $Wald_{\beta j}=\left(\frac{\hat{\beta}_{kj}}{\hat{se}\left(\hat{\beta}_{kj}\right)}\right)^2\sim\chi^2_1$ with the decision-making criterion for the Wald test statistic was H_0 rejected if $Wald_{\lambda j}>\chi^2_{\alpha,1}$.

The findings of the research indicate that human capital has a significant and positive influence on economic growth but also a negative spillover effect. Several previous studies support this finding but did not involve spatial effects. It indicates that a regency's economic growth declines due to the economic growth patterns of neighboring regencies, which are, in turn, influenced by the human capital of these neighboring regencies. This effect may be attributed to the migration of low-educated young individuals to urban areas. Further research is needed to investigate the underlying causes of this finding.

Social demographic modeling is also found to be significantly influenced by human capital. This indicates that investment in human capital results in positive demographic trends. However, it does not have a significant spatial effect. It means that the social demographics of a region do not receive spillover effects of social demographics from neighboring areas. Although supported by several studies, additional in-depth investigation is warranted to substantiate these findings.

6. Limitation and Recommendation

The limitations of this study lie in the lack of development of simultaneous significance test and goodness-of-fit test of the MSAR-LVs. Furthermore, the developed model is limited to the multivariate-autoregressive spatial model in latent variables (MSAR-LVs). Recommendations for future research include developing simultaneous significance test and goodness-of-fit test to achieve a more comprehensive and complete model. Further studies are recommended to explore the multivariate-error spatial model in latent variables (MSEM-LVs) and the multivariate-spatial autoregressive moving average model in latent variables (SARMA-LVs). Not all datasets conform to the MSAR-LVs structure, necessitating alternative models that can adapt to varying data patterns.

Conflict of Interest

The authors declare no conflict of interest.

Acknowledgments

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AI Disclosure

The author(s) declare that no assistance is taken from generative AI to write this article.

Appendix

First and Second Derivatives of the Ln Likelihood Function and Elements of the Information Matrix

(i) The first derivative of the ln likelihood function to $\mathbf{\Lambda}$ is

$$\frac{\partial \mathcal{L}\left(\mathbf{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial \mathbf{\Lambda}}_{1\times 1} = \left(vec\left(\hat{\mathbf{H}}\right)\right)^T \mathbf{V}^{-1}\left(\mathbf{I}_q \otimes \mathbf{W}\right)\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right) + \\ -\left(\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^T \mathbf{A}^{-1}\mathbf{V}^{-1}\left(\mathbf{I}_q \otimes \mathbf{W}\right)\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)$$

(ii) The first derivative of the ln likelihood function to $vec\left(\mathbf{B}\right)$ is

$$\frac{\partial \mathcal{L}\left(\mathbf{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial\left(vec\left(\mathbf{B}\right)\right)}_{(pq+q)\times 1} = -2\left(\mathbf{I}_q \otimes \hat{\xi}\right)^T \mathbf{V}^{-1} vec\left(\mathbf{\Xi}\right)$$

(iii) The second derivative of the ln likelihood function to $\mathbf{\Lambda}$ is

$$\frac{\partial^2 \mathcal{L}\left(\mathbf{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial^2 \mathbf{\Lambda}} = 2\left(vec\left(\hat{\mathbf{H}}\right)\right)^T \mathbf{V}^{-2}\left(\mathbf{I}_q \otimes \mathbf{W}\right)^2 \mathbf{\Phi}_{\mathbf{y}}\mathbf{A}^T\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right) \\ -2\left(\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^T \mathbf{V}^{-2}\left(\mathbf{I}_q \otimes \mathbf{W}\right)^2 \mathbf{\Phi}_{\mathbf{y}}\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right) \\ -\left(\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^T \mathbf{V}^{-1}\left(\mathbf{I}_q \otimes \mathbf{W}\right)^2 \mathbf{A}^{-2}\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)$$

(iv) The second derivative of the ln likelihood function to $vec\left(\mathbf{B}\right)$ is

$$\frac{\partial \mathcal{L}\left(\mathbf{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial \mathbf{\Lambda}\partial\left(vec\left(\mathbf{B}\right)\right)}_{1\times(pq+q)} = \left(vec\left(\hat{\mathbf{H}}\right) - 2\mathbf{A}^{-1}\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^T \mathbf{V}^{-1}\left(\mathbf{I}_q \otimes \mathbf{W}\right)\left(\mathbf{I}_q \otimes \hat{\xi}\right)$$

(v) The second derivative of the ln likelihood function to $\mathbf{\Lambda}$ is

$$\frac{\partial \mathcal{L}\left(\mathbf{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial\left(vec\left(\mathbf{B}\right)\right)\partial \mathbf{\Lambda}}_{(pq+q)\times 1} = 2\left(\mathbf{I}_q \otimes \hat{\xi}\right)^T \left(\mathbf{I}_q \otimes \mathbf{W}\right)\mathbf{V}^{-1}\left(-vec\left(\hat{\mathbf{H}}\right) + 2\mathbf{A}^{-1}\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)$$

(vi) The second derivative of the ln likelihood function to $vec\left(\mathbf{B}\right)$ is

$$\frac{\partial \mathcal{L}\left(\mathbf{\Lambda},\mathbf{B},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial^2\left(vec\left(\mathbf{B}\right)\right)}_{(pq+q)\times(pq+q)} = 2\left(\mathbf{I}_q \otimes \hat{\xi}\right)^T \mathbf{V}^{-1}\left(\mathbf{I}_q \otimes \hat{\xi}\right)$$

(vii) The first derivative of the ln likelihood function as Equation (24) to $\mathbf{V}$ is

$$\frac{\partial \ln \mathcal{L}\left(\boldsymbol{\Lambda},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial \mathbf{V}} = \frac{\partial}{\partial \mathbf{V}}\left[-\frac{1}{2}\ln|\mathbf{V}| + \ln|\mathbf{A}| - \frac{1}{2}tr\left[\mathbf{V}^{-1}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)^T \times \left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)\right]\right]$$

$$\frac{\partial \ln \mathcal{L}\left(\boldsymbol{\Lambda},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial \mathbf{V}} = -\frac{1}{2}\frac{\partial\left(\ln|\mathbf{V}|\right)}{\partial \mathbf{V}} - \frac{1}{2}\frac{\partial}{\partial \mathbf{V}}tr\left[\mathbf{V}^{-1}\left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)^T \times \left(\mathbf{A}vec\left(\hat{\mathbf{H}}\right) - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\mathbf{A}vec\left(\hat{\mathbf{H}}\right)\right)\right]$$

$$\frac{\partial \ln \mathcal{L}\left(\boldsymbol{\Lambda},\mathbf{V};\hat{\mathbf{H}}\right)}{\partial \mathbf{V}} = -\frac{1}{2}tr\left(\underset{qn\times qn}{\mathbf{V}^{-1}}\right) + \frac{1}{2}tr\left[\underset{qn\times qn}{\mathbf{V}^{-1}}\left(\underset{qn\times qn}{\mathbf{A}}\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \underset{qn\times qn}{\left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)}\underset{qn\times qn}{\mathbf{A}}\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}\right) \times \left(\underset{qn\times qn}{\mathbf{A}}\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)} - \left(\mathbf{I}_q \otimes \hat{\xi}\left(\hat{\xi}^T\hat{\xi}\right)^{-1}\hat{\xi}^T\right)\underset{qn\times qn}{\mathbf{A}}\underset{qn\times 1}{vec\left(\hat{\mathbf{H}}\right)}\right)^T\underset{qn\times qn}{\mathbf{V}^{-1}}\right]$$

(viii) Information matrix elements (1,1)

$$\underset{1\times 1}{\tilde{\psi}_{\Lambda\Lambda}} = -2\left(vec\left(\boldsymbol{\eta}_n\mathbf{e}^T\right)\right)^T\boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \mathbf{W}\right)^2\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right) + 3\left(\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^T\boldsymbol{\Phi}_{\mathbf{y}}^{-1} \times \left(\mathbf{I}_q \otimes \mathbf{W}\right)^2\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)$$

(ix) Information matrix elements (2,2)

$$\underset{(pq+q)\times(pq+q)}{\tilde{\boldsymbol{\psi}}_{\mathbf{BB}^T}} = -2\left(\mathbf{I}_q \otimes \hat{\xi}\right)^T\boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \hat{\xi}\right)$$

(x) Information matrix elements (1,2)

$$\underset{1\times(pq+q)}{\tilde{\boldsymbol{\psi}}_{\Lambda\mathbf{B}}} = -\left(vec\left(\boldsymbol{\eta}_n\mathbf{e}^T\right) - 2\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)^T\boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(\mathbf{I}_q \otimes \mathbf{W}\right)\left(\mathbf{I}_q \otimes \hat{\xi}\right)$$

(xi) Information matrix elements (2,1)

$$\underset{(pq+q)\times 1}{\tilde{\boldsymbol{\psi}}_{\mathbf{B}\Lambda}} = 2\left(\mathbf{I}_q \otimes \hat{\xi}\right)^T\left(\mathbf{I}_q \otimes \mathbf{W}\right)\boldsymbol{\Phi}_{\mathbf{y}}^{-1}\left(vec\left(\boldsymbol{\eta}_n\mathbf{e}^T\right) - 2\left(\mathbf{I}_q \otimes \hat{\xi}\right)vec\left(\mathbf{B}\right)\right)$$

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