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# Approximation of the Generalized Lame Equations by the Strain Energy Functional

**Tatyana Maltseva**
Department of Structural Mechanics,
Industrial University of Tyumen, Tyumen, Russian Federation.
*Corresponding author*: maltsevatv@tyuiu.ru

**Yury Klochkov**
Department of Structural Mechanics,
Industrial University of Tyumen, Tyumen, Russian Federation.
E-mail: klochkovjs@tyuiu.ru

**Tatyana Saltanova**
School of Computer Science,
Tyumen State University, Tyumen, Russian Federation.
E-mail: tsaltanova@mail.ru

**Alexander Shusharin**
Department of Structural Mechanics,
Industrial University of Tyumen, Tyumen, Russian Federation.
E-mail: shusharinaa@tyuiu.ru

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**Abstract**
This paper considers a linear system of partial differential equations (PDEs) to describe the stress-strain state of a two-phase body under static load, such as water-saturated soil. It investigates the basic properties of a new general differential operator Lame. The equations differ from the classical Lame equations by including first derivatives, which account for the influence of pore water on soil mineral particles. The properties of the generalized Lamé operator are investigated for the application of variational methods to solve the problem. It also describes alternative of the Betti and Clapeyron formulas using strain energy results. The calculus of variations of the Galerkin method is used to solve the minimum functional problem. Properties of bilinear forms are established and a theorem on the existence and uniqueness of the solution of the two-phase equilibrium problem is proved. The finite element method is adapted for a kinematic model that considers excess residual pore pressures. A new stiffness matrix is obtained, which is the sum of two matrices: one for the soil skeleton and one for pore water. The adequacy of the mathematical model of a water-saturated foundation for a natural experiment is shown. The use of Korn's inequality implies limitations on elastic properties (homogeneity, anisotropy) and the geometry of the region (requiring regularity and smooth boundaries). The study illustrates that the methodology of mechanics of a deformable solid can be adapted with appropriate modifications to a two-phase body in a stabilized state. The finite element method is adapted for a kinematic model that considers excess residual pore pressures. A new stiffness matrix is obtained, which is the sum of two matrices: one for the soil skeleton and one for pore water. The finite element method is tested on the Flamand problem. The adequacy of the mathematical model of a water-saturated foundation for a full-scale experiment is shown. The problem of the action of distributed load on a water-saturated heterogeneous foundation was solved using the finite element method and the results were compared with experimental data. The effect of mesh partitioning on the accuracy of the numerical solution is also studied in the finite element method. The maximum discrepancy was no more than 26%.

**Keywords-** Generalized Lame system of equations, Asymmetric positive definite operator, Variational problem, Energy functional, Existence and uniqueness of solution.

**List of Symbols**

| | |
|---|---|
| Stress tensor, Pa | : $\sigma_{ij}$, $t_{ij}$, *i,j*=1,..3 |
| Deformation tensor | : $\varepsilon_{ij}$, *i,j*=1,..3 |
| Modulus of elastic deformation, MPa | : $E$ |
| Poisson's ratio | : $\nu$ |
| Shear modulus, MPa | : $G$ |
| Proportionality factor | : $\aleph$ |
| Sample height, m | : $h$ |
| Permanent Lame, MPa | : $\lambda$ |
| Symbol of Kronekera | :$\delta$ |
| Generalised Lamé operator | : $D_{ij}$, *i,j*=1,..3 |
| Volume force vector, H/m3 | : $K_i$, *i*=1,..3 |
| Normal vector | : $n_i$, *i*=1,..3 |
| Elastic potential | : $W$ |
| Partial derivative vector | : $\partial_i$, *i*=1,..3 |
| Volume deformation | : $\theta$ |
| Vector of load, H | : $q_i$, *i*=1,..3 |
| Displacement vector, m | : $u_i$, *i*=1,..3 |
| Space coordinates, m | : $x_i$, *i*=1,..3 |
| ***Superscript*** | |
| soil skeleton | : *s* |
| pore water (liquid) | : *l* |
| transpose operation | : T |
| ***Subscript*** | |
| coordinate indices | : *i*, *j* *i,j*=1,..3 |
| differentiation operation | : **,** |

## 1. Introduction

The expansion of urban infrastructure, construction and operation of various roads and engineering structures and development of oil and gas fields are in some cases are carried out on swampy areas consisting of soft soils. In these areas, Sediment runoff is hindered due to the flatness of the slightly rugged relief of the lowland. As a result, the groundwater level is high and the process of swamping is significant in the area. Previous studies have generally reduced soil consolidation to the compaction of saturated soil using heat conductivity equations. The use of heat conductivity equations is confirmed by experiments during the initial phase of consolidation but fundamentally diverges from experiments at the end of the consolidation process, since According to the theoretical models, pore water stresses diminish to zero. Field experiments by many authors indicate residual pore pressures in the stabilized state of the soils. The suggested models of filtration consolidation of soil are not applicable for analyzing of real soil settlement in a stabilized state, as the experiments show the presence of residual pore pressure in the soil after the filtration consolidation process is complete. The relevance of this study is associated with a new interdisciplinary approach to the mechanics of water-saturated soils. Loaded water-saturated soil is modeled from the standpoint of deformable solid mechanics as a unified whole (solid phase + liquid phase) at the end of the process of filtration consolidation. The soil is linearly deformable, with both phases contributing to load-bearing capacity. Air within water-saturated soil is not treated as a separate phase. The stress-strain state of the soil foundation is determined by elliptic Lame-type equations. This necessitates an analysis of the properties of the Lame-type operator and solutions of the Lame type equations to apply variational methods for solving the problem under consideration.

Energy or variational methods have an important place in solid mechanics both as an alternative to the more direct method of solving the governing partial differential equations (Barber, 2023). Variational methods are used to solve quasi-static processes in solid mechanics by replacing the integration of differential equations or systems of Lame differential equations under mixed boundary conditions. These methods reduce the solution of the variational problem to find the minimum of a quadratic functional, which is known for its high accuracy and wide applicability (Mikhlin, 1966). In the theory of elasticity, the quadratic functional of Galerkin form is represented by the potential energy of deformation (Galerkin, 1915; Perelman, 1967; Fletcher, 1984). Selvadurai (2007) reviewed the application of analytical methods in elasticity, poroelasticity, and plasticity to solve geomechanics boundary value problems. In works (Bukhartsev and Nguyen, 2014; Sarkar and Chakraborty, 2021; Sarkar and Chakraborty, 2022) the variational method which was developed within the limit equilibrium method determines the critical coefficient of the soil slope stability. The search for the collapse surface is carried out using the functional extremum dependent on the soil strength parameters. Onyelowe (2021) applied the variational method to optimize additive ratios for soil stabilization and improving its geotechnical properties. The authors (Wang, et al., 2022; Patterson et al., 2024) investigate variational (or energy) methods based on the principle of minimum deformation energy for static problems in active elastic solids. The variational method is used to solve problems of mechanics not only under conditions of small deformations, but also under conditions of large deformations using the example of a thin cylindrical shell (He et al., 2023). Additionally, a variational approach to fluid-structure interaction is used by Peschka et al. (2022). Numerical methods for solving integral and differential equations are widely presented in the book (Kumar and Ram, 2025).

The object of the study is the generalized Lamé operator, which is applicable in the mechanics of water-saturated soil. Unlike the classical model of linearly deformed soil, the generalized Lamé operator includes additional summands, specifically first partial derivatives. These additional terms reflect the influence of pore water on the stress-strain state of mineral soil particles (solid phase) under load.

Filtration consolidation models (Tsytovich et al., 1967; Mesri and Choi, 1985) are based on Darcy's filtration law, in both linear and nonlinear forms. Darcy's modified law governs the flow in porous media flow (Kapoor et al., 2024). After the filtration process ends, models based on Darcy's law cannot describe residual pore pressures in a stabilized state. The property of solutions of parabolic type equations is that in the absence of a water source, pore pressures are zero, causing the two-phase soil to transition to a single-phase soil. Field and laboratory tests (Yong et al., 2019; Lachinani et al., 2022; Xu et al., 2022; Zhou et al., 2022) have shown that there is excess residual pore pressure when all consolidation processes are complete. Therefore, filtration consolidation models are inapplicable for describing the stress-strain state of a two-phase (water-saturated) foundation.

In this paper, the kinematic model of water-saturated soil is considered (Maltseva, 2022). This model for a two-phase (mineral particles of soil & pore water) soil mass considers the influence of pore water on the stress-strain state of the two-phase body. The two-phase of soil is confirmed by numerous field experiments (Bugrov et al., 1997; Maltseva et al., 2024). The model was tested on specific real scenarios in the reporting of Maltseva et al. (2020). In the article, within the kinematic model framework, solutions are constructed for the problems of loading the soil surface with typical loads describing the stress-strain state of each phase of a two-phase medium (soil skeleton + pore water), while considering the residual pore pressure. The deviation of the calculated residual pore pressures from the experimental data was no more than 5% (laboratory experiment) and 7% (full-scale experiment). According to the calculation method, a forecast of the deformation of the structure foundations made of weak water-saturated soils was developed. The need for this study is due to the inconsistency of filtration consolidation theory with full-scale and laboratory experiment results.

The kinematic soil model is based on the linear-deformable soil model but differs due to the two-phase nature of the soil. For the soil skeleton, the equations of state of the elastic medium are fulfilled. For pore water, hypotheses are introduced that take into account different properties of water in the soil pores compared to a solid body. The equation of state of pore water is formulated in the third hypothesis. The relationship between solid and liquid soil particles is established using equilibrium and kinematic interaction equations (hypothesis 4).

The kinematic model is based on the following hypotheses:
(i) Soil (peat, loam, clay) contains free incompressible water in the pores, which is hydraulically continuous.

(ii) The soil skeleton is linearly deformable. The load on the soil foundation is consistent with the limitation on the relative deformation of the soil skeleton $\varepsilon^s < 0.01$. Subsidence and swelling of the soil skeleton are not taken into account. Free and trapped air are combined with the soil skeleton and are not considered separately.

(iii) Unlike the theory of filtration consolidation, the expulsion of part of the water from the soil pores does not obey the filtration law and is described by a new physical equation: the relative deformation of the pore water $\varepsilon^l$ is caused by the pressure difference $d\sigma^l/dx_3$, not the pressure itself. The mathematical expression for the one-dimensional case is:

$$\frac{d\sigma^l}{dx_3} = \frac{E^l}{h}\varepsilon^l,$$

where, the superscript $l$ (liquid) refers to pore water, $E^l$ is the mechanical constant determined by the uniaxial compression test of a two-phase soil sample, $h$ is the height of the sample, $\varepsilon^l < 0.01$. The difference in pore pressure causes small relative movements of particles of the soil skeleton and pore water, rather than the speed of water movement.

(iv) Unlike the theory of filtration consolidation, the equation of conservation of mass of pore water is not used. Relative deformation of pore water $\varepsilon^l$ characterizes the change in relative porosity along the height of the sample. The relationship between solid and liquid soil particles is represented by relative linear deformations:

$$\varepsilon^s = -\aleph\varepsilon^l,$$

where, the parameter $\aleph > 0$ is determined experimentally and describes what part of the relative volume the liquid phase releases. The relationship between solid and liquid soil particles is represented through relative linear deformations.

Let us present a complete system of equations describing the stress-strain state of water-saturated soil for the spatial case *i*, *j*=1,2,3

Equilibrium equations,

$$\left(\sigma_{ij}^s - \sigma_{ij}^l\delta_{ij}\right)_{,j} = 0;$$

Equations of state for the soil skeleton

$$\sigma_{ii}^s = (2G + b_i)\varepsilon_{ii}^s + \lambda\theta, \qquad \theta = \varepsilon_{ii}^s;$$

$$\sigma_{ij}^s = G\varepsilon_{ij}^s, \qquad i \neq j,$$

$$G = \frac{E^s}{2(1+\nu)}, \quad \lambda = \frac{\nu E^s}{(1+\nu)(1-2\nu)}, \quad b_i = \frac{E_i^l}{\aleph_i^2};$$

Equations of state of pore water (hypothesis 3)

$$P_{ij}^{l} = E_i^l \varepsilon_{ij}^l \delta_{ij}, \qquad P_{ij}^{l} = h_i \sigma_{ij,j}^{l} \delta_{ij};$$

Phase interaction equations (hypothesis 4)

$$\varepsilon_{ii}^{s} = -\aleph_j \varepsilon_{ij}^{l} \delta_{ij};$$

Cauchy equations

$$\varepsilon_{ij}^{s} = \frac{1}{2}\left(u_{i,j}^{s} + u_{j,i}^{s}\right), \qquad \varepsilon_{ij}^{l} = u_{i,j}^{l} \delta_{ij}.$$

After transforming the system of equations describing the stabilized state of the two-phase soil after the end of the consolidation process, we obtained a solving system of linear differential equations of the Lamé type (1). The residual pore pressure is necessarily non-zero. Considering the above contributions, it is observed that none of the authors has explored this direction of work. In this paper, the properties of the generalized Lame operator were investigated. In a bounded simply connected three-dimensional domain with a piecewise smooth surface, an open internal domain was isolated by cutting out a layer along the boundary, in which a lineal of twice continuously differentiable functions was introduced. It was proved that the negative generalized differential Lame operator is positive definite under homogeneous mixed boundary conditions, but is not symmetric.

The application of variational methods to solving a mixed boundary value problem with the generalized Lame operator is based on the introduction of a variational equality with the Galerkin form. Using the projection theorem, the existence and uniqueness of a generalized solution to the equilibrium problem for a two-phase body were proved.

Analogues of three Betti formulas were obtained, and one of them was used to prove the asymmetry of the operator considered in this paper. In the theorem on the reciprocity of work for a two-phase body, an additional term appeared compared to a similar theorem for an elastic body; this term reflects the physical equations for the liquid phase. An analogue of the Clapeyron formula showed that in the representation of specific energy, the sum of the first two terms is a quadratic functional, i.e. a homogeneous function of the second degree, while the third term is a bilinear functional.

Thus, projection methods of mathematical physics, such as the Bubnov-Galerkin method, are applicable to find a solution to the system of Equations (1). The problem's formulation using a functional in the Galerkin form is generalized, allowing the solution to be determined in a wider class of functions compared to the solution based on Equations (1) and conditions (2). Variational formulations of elasticity theory problems are based on various variational principles, including those of Lagrange, Castigliano, Reissner and Hu-Washitz. It is known that the Lagrange principle on the work of external forces (both volume and surface) on possible displacements coincides with the principle of minimum potential energy. This article considers the prospects for extending the classical theory to non-Hookean elasticity law and finite displacements. The finite element method is closely related to variational formulations of elasticity theory problems. It is applied in solving both test and real problems of equilibrium of a two-phase body, as the problem's operator is limited and positively defined.

## 2. Problem Definition

The system of the kinematic model equations is a structure of generalized Lame equations and the unknown value is the vector of the solid phase displacements $u$ (real functions of the real argument). The system of second-order differential equations has the form (Maltseva, 2022):

$$D_{ij}u_j = K_i, \quad i, j=1,2,3 \tag{1}$$
$$D_{ij} = -\left(\left(G + \lambda + b_i\delta_{ij}\right)\partial_i\partial_j + G\delta_{ij}\partial_k\partial_k + c_i\delta_{ij}\partial_j\right),$$

where, $K$ - volume force vector;
$G, \lambda, b_i, c_i$ - positive constant coefficients;
$\delta_{ij}$ - symbol of Kronekera.

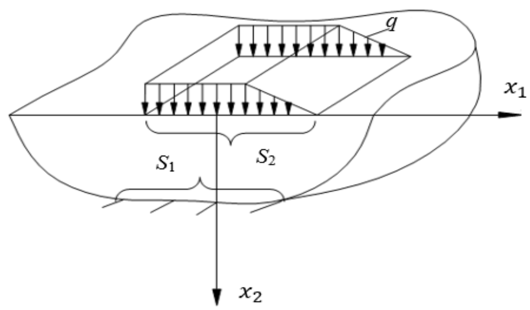


**Figure 1.** Schematic representation of the task.

The load case task of a two-phase massif was supplemented with boundary conditions of 3D part (Robin). There are no displacements on the part of the boundary $S_1$, and external loads are defined on another part of the boundary $S_2$ (**Figure 1**):

$$u_i|_{S_1} = 0, \quad t_{ij}u_j\big|_{S_2} = q_i, \quad i, j=1,2,3 \tag{2}$$
$$t_{ij} = \lambda n_i\partial_j + \left(G + b_i\delta_{ij}\right)n_j\partial_i + G\delta_{ij}n_k\partial_k,$$

where, $t_{ij}$ - stress tensor in the soil skeleton (II rang tensor).

The load vector ***q*** and the normal vector ***n*** to the surface are given on the surface of the body. Here vector of load and the normal vector is shown in bold. For brevity, all vectors, tensors, and operators will be shown in bold throughout the text. The variational formulation of the task reduces mathematical constraints on the desired solution as much as possible, enabling the construct schemes for numerical realization of the solution. The transition from the variational task to the goal of integration Equations (1) under boundary conditions (2) is reduced to the task of the functional minimum. A functional is a scalar multiplication (***Du***, ***u'***) where *u*' is some solution in a class of functions. This class of functions is broader because the functional reaches its lower bound on it. The functional is obtained using the methodology of elasticity theory for the classical Lamé equations. The additional summands in the generalized Lamé Equations (1) are extracted by representing the operator ***D*** as the sum of three operators ***A***, ***B***, and ***C***. Operator ***A*** represents the classical Lamé operator. Operator ***B*** is the second-order derivative. Operator ***C*** is a first order derivative. The sum of operators ***B***+***C*** distinguishes the generalized Lamé equations from the classical Lamé equations. Volume integrals were converted to surface integrals by Ostrogradsky's formula. The scalar product for the operator ***A*** (Lamé operator) is known by the formula (Mitrea, 2018):

$$(-\boldsymbol{Au}, \boldsymbol{u'}) = -\int_\Omega u'_i A_{ij} u_i d\Omega = 2\int_\Omega W^A(\boldsymbol{u}, \boldsymbol{u'})d\Omega - \int_S u'_i l_{ij} u_j dS\,,$$
$$W^A(\boldsymbol{u}, \boldsymbol{u'}) = W^A(\boldsymbol{u'}, \boldsymbol{u}), \quad l_{ij} = \lambda n_i\partial_j + Gn_j\partial_i + G\delta_{ij}n_k\partial_k\,,$$

where, $W^A(\boldsymbol{u})$ - elastic potential for isotropic soil at $\boldsymbol{u} = \boldsymbol{u}'$:

$$W^A(\boldsymbol{u}) = \frac{1}{2}\left(\lambda\theta^2 + 2G\varepsilon_{ij}\varepsilon_{ij}\right), \quad \theta = \varepsilon_{ij}\delta_{ij}.$$

The scalar multiplication for the new operators $\boldsymbol{B}$ and $\boldsymbol{C}$ have the formulas:

$$(-\boldsymbol{Bu}, \boldsymbol{u}') = -\int_{\Omega} u_i' B_{ij} u_j d\Omega == 2\int_{\Omega} W^B(\boldsymbol{u}, \boldsymbol{u}') d\Omega - \int_S b_i u_i' \delta_{ij} n_j \partial_i u_j dS\,,$$

$$(-\boldsymbol{Cu}, \boldsymbol{u}') = -\int_{\Omega} u_i' C_{ij} u_j d\Omega = \int_{\Omega} W^C(\boldsymbol{u}, \boldsymbol{u}') d\Omega = -\frac{1}{2}\int_S c_i u_i' \delta_{ij} n_i u_j dS \quad (3)$$

$$W^B(\boldsymbol{u}, \boldsymbol{u}') = \frac{1}{2} b_i \varepsilon_{ii} \varepsilon_{ii}'\,, \qquad W^B(\boldsymbol{u}, \boldsymbol{u}') = W^B(\boldsymbol{u}', \boldsymbol{u}),$$

$$W^C(\boldsymbol{u}, \boldsymbol{u}') = -c_i \varepsilon_{ii} u_i'\,, \qquad W^C(\boldsymbol{u}, \boldsymbol{u}') \neq W^C(\boldsymbol{u}', \boldsymbol{u}).$$

The bilinear form $W^C(\boldsymbol{u}, \boldsymbol{u}')$ is not commutative (permutative with respect to the elements $\boldsymbol{u}$ and $\boldsymbol{u}'$).

The piecewise smooth boundary $S=S_1+S_2$ of the finite region Ω represents a sphere of large radius and a half-plane. The directional cosines of the external normal are negative and at the same time do not turn to zero. At the volume integral of $W^C(\boldsymbol{u})$ in formula (3) is positive. Hence, the inequality $(\boldsymbol{Du}, \boldsymbol{u})>0$ is achieved for the operator $\boldsymbol{D}=-(\boldsymbol{A}+\boldsymbol{B}+\boldsymbol{C})$. It will be further shown that the operator $\boldsymbol{D}$ is positively defined, and Equation (1) has at most one solution. The proof of the existence of a single solution to the generalized Equations (1) is based on Betti's three formulas. Let $\boldsymbol{u}'$ and $u$ be two elastic displacement vectors, continuous and twice continuously differentiable functions in $\overline{\Omega}$. Let us compose the integral of the scalar multiplication $\int_{\Omega} \boldsymbol{u}' \cdot \boldsymbol{Du}\, d\Omega$.

Integrating by parts, we obtain the analog of Betti's formula I

$$\int_{\Omega} u_i' D_{ij} u_j d\Omega = 2\int_{\Omega} \left(W^A(\boldsymbol{u}, \boldsymbol{u}') + W^B(\boldsymbol{u}, \boldsymbol{u}') + \frac{1}{2} W^C(\boldsymbol{u}, \boldsymbol{u}')\right) d\Omega - \int_S u_i' t_{ij} u_j dS \quad (4)$$

The expression $2W^A(\boldsymbol{u}', \boldsymbol{u}) = \sum_{i,j=1}^{3}(2G+\lambda)\varepsilon_{ij}(\boldsymbol{u}')\varepsilon_{ij}(\boldsymbol{u})$ is the bilinear form of the strain components corresponding to the quadratic form when $\boldsymbol{u}'=\boldsymbol{u}$. The quadratic $2W^A(\boldsymbol{u}) = \sum_{i,j=1}^{3}(2G+\lambda)\varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u})$ is the doubled potential energy density of the elastic deformation. It is known from the elasticity theory that the form $W^A(\boldsymbol{u})$ is positive-definite.

In expression (4), we replace the displacement vector $\boldsymbol{u}'$ by the displacement vector $\boldsymbol{u}$ of the Betti formula:

$$\int_{\Omega} u_i D_{ij} u_j d\Omega = 2\int_{\Omega} \left(W^A(\boldsymbol{u}) + W^B(\boldsymbol{u}) + \frac{1}{2} W^C(\boldsymbol{u})\right) d\Omega - \int_S u_i t_{ij} u_j dS \quad (5)$$

The operator $\boldsymbol{D}$ is not symmetric. The non-symmetry is related to the operator $\boldsymbol{C}$. The summand $W^C(\boldsymbol{u})$ is absent in the theory of elasticity.

Subtracting from Equation (4) the expression in which the vectors $\boldsymbol{u}'$ and $\boldsymbol{u}$ swap places, we obtain the analog of Betti's formula III:

$$\int_{\Omega} \left(u_i' D_{ij} u_j - u_i D_{ij} u_j'\right) d\Omega = \int_{\Omega} \left(W^C(\boldsymbol{u}, \boldsymbol{u}') - W^C(\boldsymbol{u}', \boldsymbol{u})\right) d\Omega - - \int_S \left(u'_i t_{ij} u_j - u_i t_{ij} u'_j\right) dS \quad (6)$$

The sum $W^A(\boldsymbol{u}) + W^B(\boldsymbol{u})$ is a homogeneous function of degree two: $W^A(\boldsymbol{u}, \boldsymbol{u}') + W^B(\boldsymbol{u}, \boldsymbol{u}')$=$W^A(\boldsymbol{u}', \boldsymbol{u}) + W^B(\boldsymbol{u}', \boldsymbol{u})$.

Betti formula III shows that the generalized Lamé operator is not self-adjoint. The surface integral in (6) approaches zero in the case of a mixed boundary value problem with boundary conditions (2).

Using Equations (1), let us write expression (5) as an analog of the Clapeyron formula,

$$\int_\Omega u_i K_i d\Omega + \int_S u_i q_i dS = 2\int_\Omega \left(W^A(\boldsymbol{u}) + W^B(\boldsymbol{u}) + \frac{1}{2}W^C(\boldsymbol{u})\right) d\Omega.$$

The work of external volumetric and surface forces is used to transfer internal deformation energy to the two-phase body. When the load is removed from the body, the energy is converted into work, allowing the body to return to its initial state. According to formula (3), the volume integral of $W^C(\boldsymbol{u})$ is positive, and is a homogeneous function of the first degree with relation to linear deformations $\varepsilon_{ii}$.

At absence of volumetric forces by Clapeyron's theorem for an elastic solid without considering pore water for comparison we have a homogeneous function $W^A(\boldsymbol{u})$ of the second degree

$$2\int_\Omega W^A(\boldsymbol{u}) d\Omega = \int_S u_i q_i dS.$$

## 3. Methodology

The operator $\boldsymbol{D}$ is defined on the set $M$ of continuous functions $\boldsymbol{u}$ together with their derivatives up to and including second-order in $\overline{\Omega}$. The set M is a lineal. The vector-functions $\boldsymbol{u}$ conforms homogeneous boundary conditions. The set $M$ is dense in the space $L_2(\Omega)$. The positive definiteness of the operator $\boldsymbol{D}$ is demonstrated relative to the norm of the Sobolev vector space $\boldsymbol{W}^{1,2}(\Omega)$. For the first two summands in the operator $\boldsymbol{D}$: $A_{ij} + B_{ij} = (G + \lambda + b_i\delta_{ij})\partial_i\partial_j + G\delta_{ij}\partial_k\partial_k$ (the negative Lame operator in the case of anisotropy) and the vector-function $\forall\, \boldsymbol{u} \in M$ complying with the homogeneous mixed conditions, Korn's inequality (Rectoris, 1985; Horgan, 1995) holds:

$$C_1^2 \int_\Omega c_{ijkl}\varepsilon_{kl}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u}')d\Omega \geq \int_\Omega \left(\sum_{i=1}^3 |u_i|^2 + \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_j}\right)^2\right) d\Omega,$$

where, $C_1^2$ is a positive constant independent of the choice of $\boldsymbol{u}$. The constant $C_1^2$ depends on the domain size and mechanical constants.

The use of Korn's inequality implies restrictions on the elastic properties (homogeneity, anisotropy) and the geometry of the region (requiring regularity and smooth boundaries). Specifically, in Flamand (Boussinesq) type problems, this region will be a semi-cylinder (hemisphere) of finite radius. This methodology will not be effective if these restrictions are not met. The article did not address the formulating the conditions to ensure the stability and accuracy of the variational method in a multidimensional irregular domain which requires additional research.

According to Korn's inequality, the operator $-(\boldsymbol{A} + \boldsymbol{B})$ is symmetric and positively defined in the Hilbert space $\boldsymbol{W}^{1,2}(\Omega)$:

$$(-(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u}, \boldsymbol{u}) \geq C^2 \|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}(\Omega)}, \quad C^2 = \frac{1}{C_1^2}.$$

For the third summand of the operator $\boldsymbol{D}$:$C_{ij} = c_i\delta_{ij}\partial_j$ the scalar product after integration by parts is written:

$$(-\boldsymbol{C}\boldsymbol{u}, \boldsymbol{u}') = \int_\Omega \sum_{i=1}^3 c_i \frac{\partial u'_i}{\partial x_i} u_i d\Omega - \int_S \sum_{i=1}^3 c_i u_i u'_i cos(\boldsymbol{n}, x_i) dS.$$

The operator (-$\boldsymbol{C}$) is asymmetric:

$$(-\boldsymbol{Cu},\boldsymbol{u}')-(\boldsymbol{u},-\boldsymbol{Cu}')=2\int_\Omega \sum_{i=1}^{3} c_i \frac{\partial u_i'}{\partial x_i} u_i d\Omega - \int_S \sum_{i=1}^{3} c_i u_i u'_i cos(\boldsymbol{n},x_i) dS,$$

because the volume integral is generally different from zero. The surface integral is zero.

At equality of elements $\boldsymbol{u}'=\boldsymbol{u}$, homogeneous boundary conditions and restrictions on the geometry of the region, the scalar product $(-\boldsymbol{Cu},\boldsymbol{u})=-\frac{1}{2}\int_S c_i u_i^2 cos(\boldsymbol{n},x_i)dS$ , $-1\le cos(\boldsymbol{n},x_i)\le 0$ is positive. For example, in the Flaman (Boussinesq) problems, the domain represents a half-cylinder (hemisphere) of finite radius. The axes $Ox_1$ and $Ox_2$ are in the day plane. An external load is applied to the day plane. The axis $Ox_3$ is directed inside the body, the directional cosine negative $cos(\boldsymbol{n},x_3)=-1$.

Therefore, the positive definiteness of the operator D for the case of homogeneous mixed boundary conditions with respect to the norm in the space $\boldsymbol{W}^{1,2}(\Omega)$ follows from the inequality:

$$(\boldsymbol{Du},\boldsymbol{u})\ge\gamma^2\|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}(\Omega)}.$$

For vector $\boldsymbol{u}$ we apply Friedrichs' inequality,

$$\|\boldsymbol{u}\|^2_{L_2(\Omega)}\le m_1\|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}(\Omega)}.$$

The operator $\boldsymbol{D}$ is positively defined in the space $L_2(\Omega)$:

$$(\boldsymbol{Du},\boldsymbol{u})\ge\gamma^2\|\boldsymbol{u}\|^2_{\boldsymbol{W}^{1,2}(\Omega)}\ge\frac{\gamma^2}{m_1}\|\boldsymbol{u}\|^2_{L_2(\Omega)}.$$

As a generalized (weak) solution of the considered mixed problem (1) - (2), we consider a function $\boldsymbol{v}\in V$ satisfying the variational problem:

$$(\boldsymbol{Du},\boldsymbol{v})=(\boldsymbol{K},\boldsymbol{v}),\forall\boldsymbol{v}\in V,\quad \boldsymbol{K}\in L_2(\Omega),\quad (\boldsymbol{K},\boldsymbol{v})\in V^*,\ \ V=\boldsymbol{W}^{1,2}(\Omega).$$

Condition $\boldsymbol{v}|_{S_1}=0$ let us denote by $\boldsymbol{v}^{\circ}\in\boldsymbol{W}^{\circ 1,2}(\Omega)$ $\boldsymbol{v}\in\mathring{\boldsymbol{W}}^{1,2}(\Omega)$. The space $V^*$ is conjugate to the space $V$. Let us apply the formula of integration by parts. The required smoothness of admissible functions will decrease. As a result, we obtain the Galerkin form

$$a(\boldsymbol{u},\boldsymbol{v})+c(\boldsymbol{u},\boldsymbol{v})=(\boldsymbol{K},\boldsymbol{v})+\int_{S_2}\boldsymbol{v}\cdot\boldsymbol{t}^{(\nu)}(\boldsymbol{u})dS \tag{7}$$

where,

$$a(\boldsymbol{u},\boldsymbol{v})=-\int_\Omega\left((G+\lambda)\theta\frac{\partial v_i}{\partial x_i}+G\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}+b_i\frac{\partial u_i}{\partial x_i}\frac{\partial v_i}{\partial x_i}\right)d\Omega\,,$$

$$c(\boldsymbol{u},\boldsymbol{v})=-\int_\Omega c_i\frac{\partial u_i}{\partial x_i}v_i d\Omega.$$

Let us formulate the properties of the form $d(\boldsymbol{u},\boldsymbol{v})=a(\boldsymbol{u},\boldsymbol{v})+c(\boldsymbol{u},\boldsymbol{v})$ and the form $c(\boldsymbol{u},\boldsymbol{v})$ as Lemma 1 and Lemma 2.

**Lemma 1:** *If the region* $\Omega$ *is bounded, then the form* $d(\boldsymbol{u},\boldsymbol{v})$ *is a bilinear continuous form on* $V\times V$.

***Proof:*** Let $\boldsymbol{u},\boldsymbol{v}\in V$, and the coefficients of the form are bounded

$$max(G,\lambda,b_i)\le m_1,\quad \partial_i u_i\in L_2(\Omega), v_i\in L_2(\Omega),\quad \partial_i=\frac{\partial}{\partial x_i},\quad i=1,2,3.$$

The estimation of the form $a(\boldsymbol{u},\boldsymbol{v})$ was done using the Cauchy-Bunyakovsky inequality

$$a(\boldsymbol{u},\boldsymbol{v}) \le m_1 \left|\int_\Omega \partial_i u_i \partial_j v_j d\Omega\right| \le m_1 \left(\int_\Omega |\partial_i u_i|^2 d\,\Omega\right)^{\frac{1}{2}} \left(\int_\Omega |\partial_i v|^2 d\,\Omega\right)^{\frac{1}{2}} \le m_1 \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V,$$

where, $\|\boldsymbol{u}\|_V = \left(\int_\Omega |\partial_i u_i|^2 d\,\Omega\right)^{\frac{1}{2}}$ is the norm in the Sobolev vector space $\boldsymbol{W}^{1,2}(\Omega)$. In the space $\boldsymbol{W}^{1,2}(\Omega)$ all partial derivatives of the first-order belong to the space $L_2(\Omega)$.

We also apply the Cauchy-Bunyakovsky inequality to the second summand of the form. The first derivatives of the product of function $\partial_i u_i v_i$ belong to $L_1(\Omega)$.

$$\left|\int_\Omega c_i \partial_i u_i v_i d\,\Omega\right| \le c|\partial_i u_i|_{L_2(\Omega)} |v_i|_{L_2(\Omega)}, \quad c = max(c_1, c_2, c_3).$$

By S. L. Sobolev's embedding theorem (Rectoris, 1985): $|\boldsymbol{u}|_{L_2(\Omega)} \le \mu(\Omega)\|\boldsymbol{u}\|_V$ form $c(\boldsymbol{u},\boldsymbol{v})$ is defined and satisfies the inequality:

$$|c(\boldsymbol{u},\boldsymbol{v})| \le m\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V, \qquad m = c\mu(\Omega).$$

The result is an inequality:

$$|d(\boldsymbol{u},\boldsymbol{v})| \le (m_1 + m)\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V.$$

The form $d(\boldsymbol{u},\boldsymbol{v})$ is bilinear and continuous, which was required to prove.

**Lemma 2:** *For any open region* $\Omega$ *and* $S_1 \ne S$ *we have*

$$c(\boldsymbol{u},\boldsymbol{u}) = -\frac{1}{2}\int_{S_2} c_i u_i^2 cos(\boldsymbol{n},\boldsymbol{x})dS, \ \forall \boldsymbol{u} \in V \tag{8}$$

$$c(\boldsymbol{u},\boldsymbol{v}) = -c(\boldsymbol{v},\boldsymbol{u}) - \int_{S_2} c_i u_i v_i cos(\boldsymbol{n},\boldsymbol{x})dS, \qquad \forall \boldsymbol{u},\boldsymbol{v} \in V \tag{9}$$

***Proof:*** Formula (9) follows from formula (8) by replacing the function $\boldsymbol{u}$ by the expression $\boldsymbol{u}+\boldsymbol{v}$ in formula (9). Let us prove formula (8).

$$c(\boldsymbol{u},\boldsymbol{u}) = -\int_\Omega c_i \partial_i u_i u_i d\Omega = -\frac{1}{2}\int_\Omega c_i \partial_i (u_i)^2 d\Omega = -\frac{1}{2}\int_S c_i u_i^2 cos(v,x)dS,$$

which is exactly what we needed to prove.

**Theorem 1:** *Let* $\Omega$ *be a bounded region in* $R^3$ *u* $\boldsymbol{K}$ *- be a given element,* $\boldsymbol{K} \in L_2(\Omega)$*. Then problem* (1) *has a single solution* $\boldsymbol{u} \in V$.

***Proof:*** We substitute the equality $\boldsymbol{v} = \boldsymbol{u}$. into the Galerkin form (1). Based on the positive definiteness of the operator $\boldsymbol{D}$ with respect to the norm of the space $\boldsymbol{W}^{1,2}(\Omega)$, we have an estimate from below of the Galerkin form:

$$a(\boldsymbol{u},\boldsymbol{u}) + c(\boldsymbol{u},\boldsymbol{u}) \ge \gamma^2\|\boldsymbol{u}\|_V^2.$$

Let us give a projection theorem (Themam, 1981, p.28): «Let $\boldsymbol{W}$ - be a separable real Hilbert space (with norm $\|\cdot\|_{\boldsymbol{W}}$), and let $a(\boldsymbol{u},\boldsymbol{v})$ be a continuous bilinear form on $\boldsymbol{W}\times\boldsymbol{W}$ that is coercive, that is, there exists $\alpha > 0$ such that $a(\boldsymbol{u},\boldsymbol{u}) \ge \alpha\|\boldsymbol{u}\|_W^2, \forall \boldsymbol{u} \in \boldsymbol{W}$. Then for each $l$ of $\boldsymbol{W}^*$- space conjugate to $\boldsymbol{W}$, there exists one and only one element $\boldsymbol{u} \in \boldsymbol{W}$ such that $a(\boldsymbol{u},\boldsymbol{v}) =< l, \boldsymbol{v} >, \forall \boldsymbol{v} \in \boldsymbol{W}$».

Let us apply the projection theorem to equality (7). The space $\boldsymbol{W}$ in our case is the space $V$ with norm in the Sobolev space. Let us introduce the notations: $a(\boldsymbol{u},\boldsymbol{v}) = d(\boldsymbol{u},\boldsymbol{v})$и $< l,\boldsymbol{v} >= (\boldsymbol{K},\boldsymbol{v})$. The form $(\boldsymbol{K},\boldsymbol{v})$ is linear and continuous on $V$. The space $V$ is separable as a closed subspace of the separable space $\boldsymbol{W}^{1,2}(\Omega)$.

The variational method has some advantage over the grid method in the case of multidimensional problem, particularly when difference approximation of derivatives can lead to significant errors. However, the application of the variational method requires investigation the stability of the approximate solution and its convergence to the solution of problem (1) with boundary conditions (2). Ritz specified a direct method for solving the variational problem, which involves finding the minimum of the strain energy functional,

$$F(\boldsymbol{u}) = a(\boldsymbol{u},\boldsymbol{v})+c(\boldsymbol{u},\boldsymbol{v}) - (\boldsymbol{K},\boldsymbol{v}) - \int_{S_2} \boldsymbol{v}\cdot\boldsymbol{t}^{(\nu)}(\boldsymbol{u})dS$$

is carried out not in the space $H_D$, but in its n – dimensional subspace spanned by elements (linearly independent coordinates) $\varphi_1,\varphi_2,\dots,\varphi_n$. The solution of the variational problem is represented by a linear combination of these elements:

$$\boldsymbol{u}_n = \sum_{k=1}^{n} a_k\varphi_k,$$

where, $a_k$ are constant unknowns and are found from the minimum condition of the function of $n$ variables $F(\boldsymbol{u}_n)$.

The Ritz method has much in common with the displacement finite element method (FEM). It is most widely used in the mechanics of deformable solids. The difference between the traditional scheme of the Ritz method and the FEM lies in the choice of the system of approximating functions. In the Ritz method the approximation of displacements is performed in the entire domain of their definition, whereas in the FEM, the approximation is performed for each finite element (triangle or rectangle), allowing for the use of approximating functions of a simpler form. The FEM is applicable to the two-phase equilibrium problem considered in this article.

To demonstrate the convergence of the numerical solution to the analytical solution, we give a general convergence theorem (Themam, 1981).

Let $V$ be a Hilbert space, $b(\boldsymbol{u},\boldsymbol{v})$ be a coercive continuous bilinear form on $V$x$V$:

$$b(\boldsymbol{u},\boldsymbol{u}) \geq \alpha\|\boldsymbol{u}\|_V^2, \qquad \forall\boldsymbol{u}\in V \tag{10}$$

and $l$ is a linear continuous form on $V$. We denote by $\boldsymbol{u}$, the only solution in $V$ equation,

$$b(\boldsymbol{u},\boldsymbol{u}) = \langle l,\boldsymbol{u}\rangle, \qquad \forall\boldsymbol{u}\in V \tag{11}$$

To approximate an element $\boldsymbol{u}$, we define an arbitrary outer stable and convergent approximation of the space $V\{V_h p_h r_h\}_{h\in N}$. $V_h$ is an increasing sequence of finite-dimensional subspaces of $V$. $\cup V_h$ is dense in $V$; $\boldsymbol{p}_h$: $V_h \to L_2(\Omega)$ is a continuation operator; $\boldsymbol{r}_h$:$V \to V_h$ is a continuation operator. For $\forall h\in N$, we define a continuous bilinear form $b_h(\boldsymbol{u}_h,\boldsymbol{v}_h)$ on $V_h\times V_h$. The form $b_h(\boldsymbol{u}_h,\boldsymbol{v}_h)$ is coercive and satisfies the condition: $\exists\ \alpha_0 > 0$ and is independent of $h$, that

$$b_h(\boldsymbol{u}_h,\boldsymbol{u}_h) \geq \alpha_0\|\boldsymbol{u_h}\|_h^2, \ \ \forall\boldsymbol{u_h}\in V_h \tag{12}$$

Let there be a continuous linear form $l_h$ on the set of subspaces $V_h$ such that

$$\|l_h\|_{*h} \leq \beta \tag{13}$$

$\beta$ does not depend on $h$. For a fixed $h$ we construct a sequence of elements $\boldsymbol{u}_h \in V_h$ such that

$$b_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = \langle l_h, \boldsymbol{u}_h \rangle, \quad \forall \boldsymbol{u}_h \in V_h \tag{14}$$

Let's introduce the terms:

If the family $\boldsymbol{v}_h \xrightarrow[h\to 0]{weakly} \boldsymbol{v}$ and family $\boldsymbol{w}_h \xrightarrow[h\to 0]{strongly} \boldsymbol{w}$, then

$$\lim_{h\to 0} b_h(\boldsymbol{v}_h, \boldsymbol{w}_h) = b(\boldsymbol{v}, \boldsymbol{w}), \qquad \lim_{h\to 0} b_h(\boldsymbol{w}_h, \boldsymbol{v}_h) = b(\boldsymbol{w}, \boldsymbol{v}) \tag{15}$$

If the family $\boldsymbol{u}_h \xrightarrow[h\to 0]{weakly} \boldsymbol{u}$, then

$$\lim_{h\to 0} \langle l_h, \boldsymbol{u}_h \rangle = \langle l, \boldsymbol{u} \rangle \tag{16}$$

General convergence theorem (Themam, 1981) is: «**Theorem 2:** *If conditions* (10), (12), (13), (15), *and* (16) *are compiled the solution* $\boldsymbol{u}_h$ *of Equation* (14) *converges strongly to the solution* $\boldsymbol{u}$ *of Equation* (11) *with* $h \to 0$».

Let us apply this theorem and show the convergence of the numerical solution obtained by FEM.

Let $\Omega$ - be an open bounded region in the space $R_2$. By $\mathfrak{I}_h$ we denote a regular triangulation $\Omega$, a collection of two-dimensional simplexes that meet the condition

$$\sigma(h) \le \alpha, \;\; \rho(h) \to 0 \tag{17}$$

where,

$$\rho(h) = \underbrace{sup \rho_J}_{J \in \mathfrak{I}_h}, \qquad \rho'(h) = \underbrace{inf \rho'_J}_{J \in \mathfrak{I}_h}, \qquad \sigma(h) = \underbrace{sup(\rho_J/\rho'_J)}_{J \in \mathfrak{I}_h},$$

where, $\rho = \rho_J$ is the diameter of the smallest ball containing J (two-dimensional simplex, in particular triangle, rectangle); $\rho' = \rho'_J$ is the diameter of the largest ball contained in J.

If J is a triangle, then the inequality it is known (Themam, 1981):

$$\frac{1}{tg\frac{\theta}{2}} \le \frac{\rho_J}{\rho'_J} \le \frac{2}{sin\theta},$$

where, $\theta$ is the smallest angle of the two-dimensional simplex J. Condition (17) means that the smallest angle for all triangles $J \in \mathfrak{I}_h$ s bounded from below: $\theta \ge \theta_0 > 0$.

**Proposition:** If $\rho(h) \to 0$ and $\sigma(h) < \alpha$, then the solution $\boldsymbol{u}_h$ of task (14) converges to the solution $\boldsymbol{u}$ of task (11).

The proof follows from positive definiteness of the continuous bilinear form $d(\boldsymbol{u}, \boldsymbol{v})$ and fulfillment of the conditions of the convergence theorem (10), (12), (13), (15), and (16).

The variational problem is solved by methods of finding functions that give the minimum value to a given functional. Some boundary conditions (kinematic) must be considered precisely when choosing the coordinate functions of the numerical solution. Natural (static) boundary conditions are approximated automatically when solving the variational problem. It is not necessary to take into account the natural boundary conditions when selecting the coordinate functions. Thus, variational methods (Ritz method), projection methods of Bubnov-Galerkin type and the finite element method (Ern and Guermond, 2004) based on them are applicable for finding the solution of the generalized Lamé Equations (1) with boundary conditions (2).

## 4. Results and Discussion

According to FEM, the matrix form for Equation (7) was obtained. The flat region occupied by the two-phase medium was divided into triangular finite elements (the case of plane deformation). The displacements of the $ijm$ triangle vertices were expressed by the displacement vector $\delta = \{u_1^i\ u_2^i\ u_1^j\ u_2^j\ u_1^m\ u_2^m\}$, which is unknown.

In the triangular element $ijm$ we define as coordinate functions:

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2\ , \qquad u_2 = \alpha_4 + \alpha_5 x_1 + \alpha_6 x_2,$$

where, $\alpha_1, \dots, \alpha_6$ are unknown constant coefficients. The coefficients $\alpha_1, \dots, \alpha_6$ are determined from the system of linear algebraic equations. Let us give the final formulas for displacements:

$$u_1 = \frac{1}{2\Delta}\left[(p_i + d_i x_1 + n_i x_2)u_1^i + (p_j + d_j x_1 + n_j x_2)u_1^j + (p_m + d_m x_1 + n_m x_2)u_1^m\right],$$

$$u_2 = \frac{1}{2\Delta}\left[(p_i + d_i x_1 + n_i x_2)u_2^i + (p_j + d_j x_1 + n_j x_2)u_2^j + (p_m + d_m x_1 + n_m x_2)u_2^m\right],$$

$$p_i = x_1^j x_2^m - x_1^m x_2^j,\ \ n_i = x_1^m x_1^j, \qquad d_i = x_2^j - x_2^m\ , \qquad 2\Delta = \begin{vmatrix} 1 & x_1^i & x_2^i \\ 1 & x_1^j & x_2^j \\ 1 & x_1^m & x_2^m \end{vmatrix}.$$

The coefficients $p, n, d$ with other indices are obtained by cyclic permutation. From the vector of unknown displacements by means of the Cauchy equations we pass to the vector of relative deformations:

$$\{\varepsilon\} = [N]\{\delta\} \tag{18}$$

where,

$$[N] = \frac{1}{2\Delta}\begin{pmatrix} d_i & 0 & d_j & 0 & d_m & 0 \\ 0 & n_i & 0 & n_j & 0 & n_m \\ n_i & d_i & n_j & d_j & n_m & d_m \end{pmatrix}.$$

Then, based on Hooke's law and the physical equation of state for pore water (hypothesis 2) and using Lagrange's principle, the final formulas were written down:

$$([N]^T[D][N] + [M]^T[D_l][N])\{\delta\} = \{K\} \tag{19}$$

The matrices $[D]$ and $[D_l]$ describe the mechanical properties of the soil skeleton and pore water, respectively. Matrices $[D_l]$, $[M]$ are new and correspond to operator $\boldsymbol{C}$ in Equation (1):

$$[D_l] = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [M] = \frac{1}{2\Delta}\begin{pmatrix} \xi_i^* & 0 & \xi_j^* & 0 & \xi_m^* & 0 \\ 0 & \xi_i^* & 0 & \xi_j^* & 0 & \xi_m^* \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where, $\xi_k^* = p_k + d_k x_1 + n_k x_2,\ (k = i, j, m)$.

The upper index $T$ denotes the transpose operation of the matrix. The expression in brackets of formula (19) $[N]^T[D][N] + [M]^T[D_l][N]$ is the stiffness matrix for a triangular two-phase element. It is independent of the loads acting on the element and can be calculated for each element separately. Physically, the components of this matrix represent the coefficients of the displacement method equations for calculating a single element. A global stiffness matrix is obtained for a set of elements. To increase the order of approximation, rectangular elements were also considered. These results are not presented in the article.

The Flaman test problem of loading a half-plane by a vertical load $Q$ is considered. An analytical solution is known for this problem. A two-phase body of unit thickness is bounded from above by a semi-cylinder of small radius $\rho$ and a day surface, and from below by a semi-cylinder of large radius $R$. The region is divided into triangular elements. Volumetric forces are absent. The parameters of the problem are taken from laboratory experiments of A.V. Nabokov (Maltseva et al., 2024): $E_s = 8.1\ MPa$, $E_l = 3.3\ MPa$, $\nu = 0.3$, $\aleph = 0.52$, $h = 1\ m$, $Q = 0.077\ MN/m$.

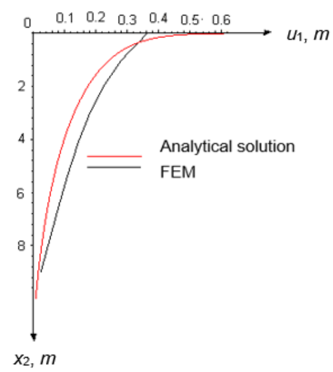


**Figure 2.** Comparison of horizontal displacements, $x_1 = 0$ (grid $10 \times 10$).

**Figure 6** shows the comparison of vertical displacements of the day surface of the half-space on different grids of the domain partitioning. The grid dimensionality affects the error of the numerical solution. The best result is obtained with 25x25 cm triangulation. **Figure 7** shows the variation of vertical displacements $u_2$ along the depth of the compressible layer in the section located at a distance of 1.5m from the vertical axis of symmetry of the half-space.

The calculations showed that the numerical solution agrees quite well with the analytical one. The maximum discrepancy was no more than 22 %.

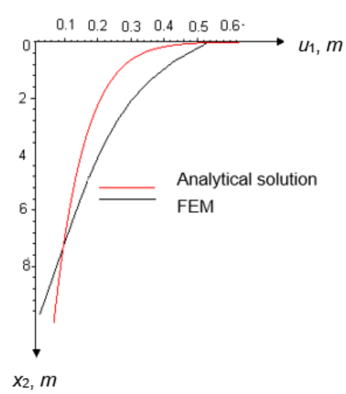


**Figure 3.** Comparison of horizontal displacements, $x_1 = 0$ (grid $15 \times 15$).

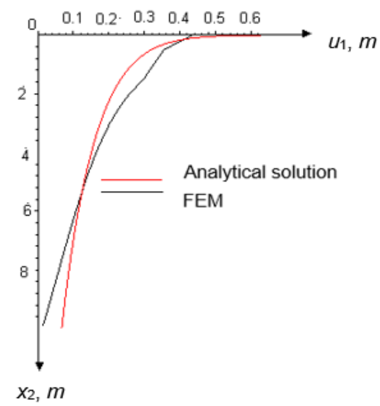


**Figure 4.** Comparison of horizontal displacements, $x_1 = 0$ (grid $20 \times 2.0$).

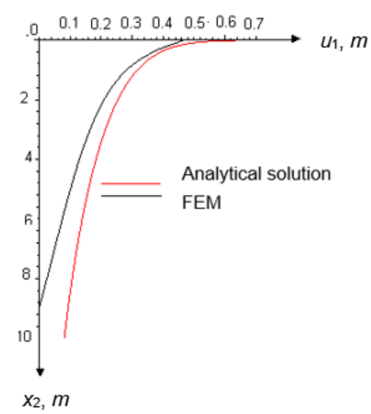


**Figure 5.** Comparison of horizontal displacements, $x_1 = 0$ (grid $25 \times 25$).

As can be seen from the graphs shown in **Figures 2-5**, the analytical solution agrees quite well with the solution obtained by FEM, especially on a 0.2 m grid 20x20.

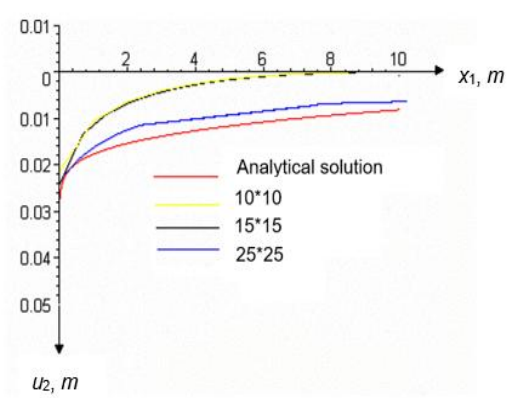


**Figure 6.** Comparison of vertical displacements on different grids, $x_2 = 0$.

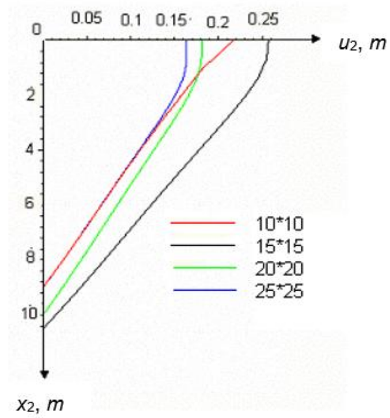


**Figure 7.** Vertical displacements on different grids, $x_1 = 1.5$ m.

To test this methodology, a real problem of loading a heterogeneous water-saturated soil base was solved. To compare the solution of this problem with the results of the natural experiment by Bugrov et al. (1997), all input parameters were set as in the experiment. The soil base was multilayered (**Figure 8**) and subjected to a uniformly distributed load of $q = 0.054\ MPa$ at a distance of $a = 5\ m$ from the symmetry axis.

The stress-strain state of each layer is described using a system of differential Equations (1) with different constant coefficients $G, \lambda, b_i, c_i$ for each soil layer and boundary conditions (2). The mechanical parameters of the problem for each layer are summarized in **Table 1**. The Poisson's ratio $\nu = 0.3$ was one averaged value for all layers.

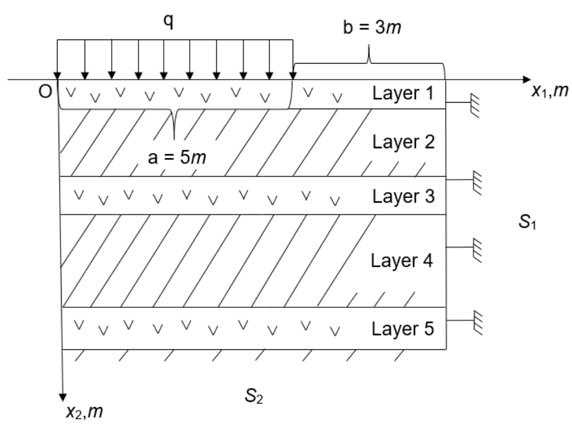


**Figure 8.** Schematic diagram of the soil foundation.

**Table 1.** Mechanical characteristics of soil layers.

| Layer number | Layer thickness, *m* | $E_l$, MPa | $E_s$, MPa | ℵ |
|---|---|---|---|---|
| layer 1 | 0.7 | 0.035 | 0.956 | 0.38 |
| layer 2 | 1.7 | 0.468 | 1.955 | 0.5 |
| layer 3 | 0.9 | 0.065 | 5.662 | 0.09 |
| layer 4 | 4 | 0.294 | 2.941 | 0.5 |
| layer 5 | 1.8 | 0.192 | 3.425 | 0.43 |

On the symmetry axis $Ox_2$ and boundary $S_1$ there are no horizontal displacements of soil particles, while on boundary $S_2$ there are no displacements at all. The finite element method (FEM) was applied to solve the problem. The novelty of this approach is in the construction of a new stiffness matrix corresponding to the generalized Lame operator for each layer of the soil foundation. The domain was discretized using rectangular elements.

**Figure 9** and **Figure 10** show the results of numerical solutions of vertical displacements of the soil skeleton and pore water at the layer boundaries. The displacements of pore water particles decrease with increasing the depth as the pore water is clamped by mineral soil particles.

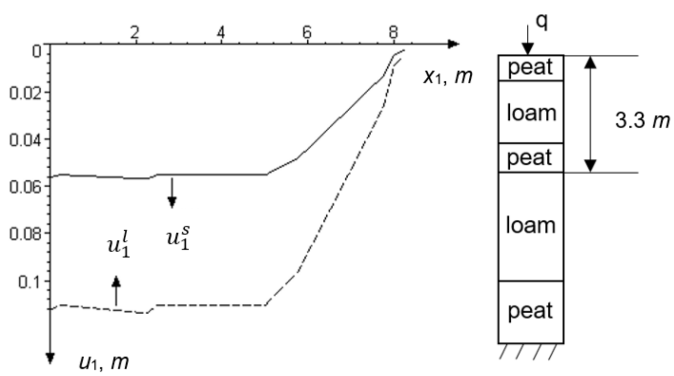


**Figure 9.** Vertical displacements of soil particles at the depth 3.3 m.

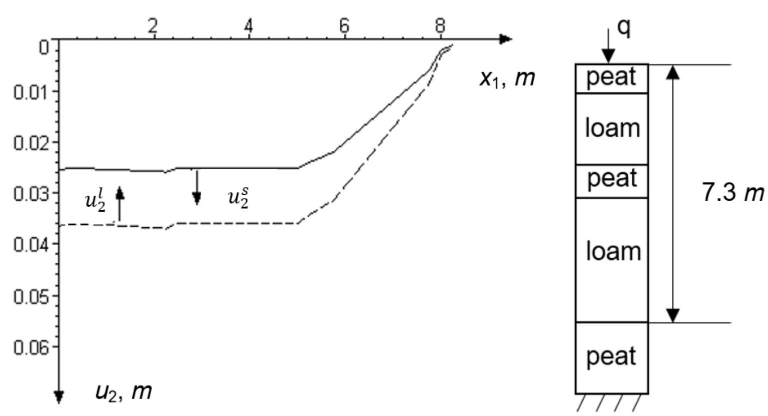


**Figure 10.** Vertical displacements of soil particles at the depth 7.3 m.

At a distance of 3.3 m from the day surface, the vertical displacements were 0.001 m. The farther the layer is from the day surface, the smaller the displacements of soil particles and pore water.

Closer to the day surface (at a distance of 2.5 m) the maximum displacements of the soil skeleton amounted to 0.06 m. For the layer located at a distance of 7.3 m from the day surface, vertical displacements of the soil skeleton are equal to 0.001 m.

**Figures 11-13** show the horizontal displacements of soil skeleton particles and pore water for different vertical cross sections.

The vertical displacements decrease by an order of magnitude when moving away from the line of action of the external load for different layers.

**Figures 14** and **15** show a comparison of the obtained solution to the problem with the data of a natural experiment. **Figure 14** shows graphs of vertical displacements of points on the daytime surface of a water-saturated soil base, obtained by FEM and from experimental data in the section $x_2 = 0\ m$. The discrepancy ranged from 5% to 10% under load at a distance of more than 7 m from the axis of symmetry. The maximum settlement according to the experimental data was 0.46 m, while according to FEM the settlement was 0.44 m at a distance of 3.8 m from the axis of symmetry. **Figure 15** shows the graphs of horizontal displacements obtained by FEM and experimentally in the section $x_1 = 1\ m$. The maximum discrepancy on the daylight surface was 16.2%. At the boundary of the first layer, the displacements coincided with the experimental ones, at the boundary of the second layer, the discrepancy was 11%, at the boundary of the third layer, the discrepancy was 7%, and at the boundary of the fourth layer, the maximum discrepancy was 26%. The numerical solution of the problem agrees quite well with the experimental data. The computational efficiency is shown on a finer 25x25 grid. The maximum discrepancies between the experimental data and the numerical solution for horizontal displacements of soil particles were 10%, for pore pressures there are 6% (**Figures 15-16**). The approach considered will be applicable for large-scale projects. In case of a shortage of available computing resources, cloud computing can be used.

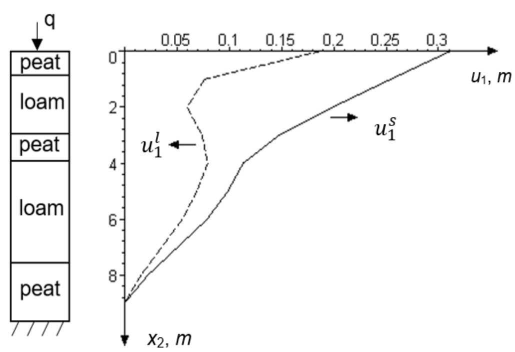


**Figure 11.** Horizontal displacements ($x_1 = 2\ m$).

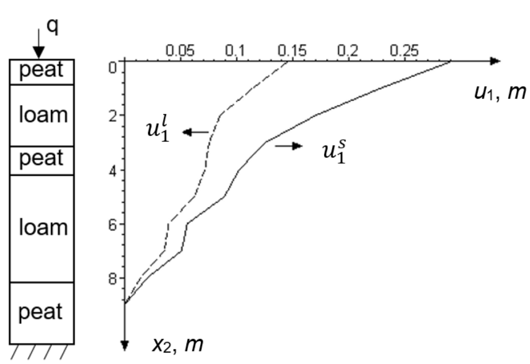


**Figure 12.** Horizontal displacements ($x_1 = 5\ m$).

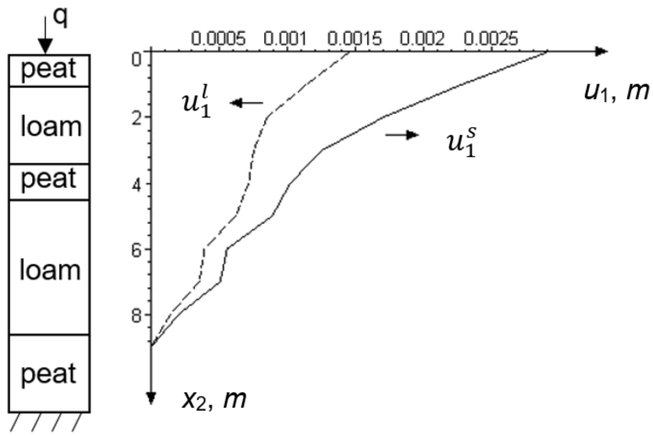


**Figure 13.** Horizontal movements ($x_1 = 8\ m$).

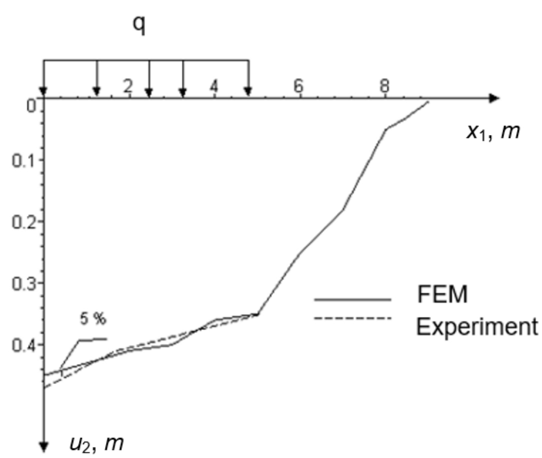


**Figure 14.** Vertical displacements of the soil skeleton on the daytime surface ($x_2 = 0\ m$).

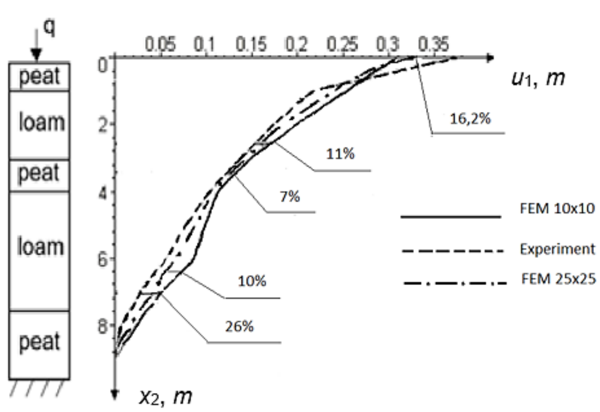


**Figure 15.** Horizontal displacements of the soil skeleton for a vertical section ($x_1 = 1\ m$).

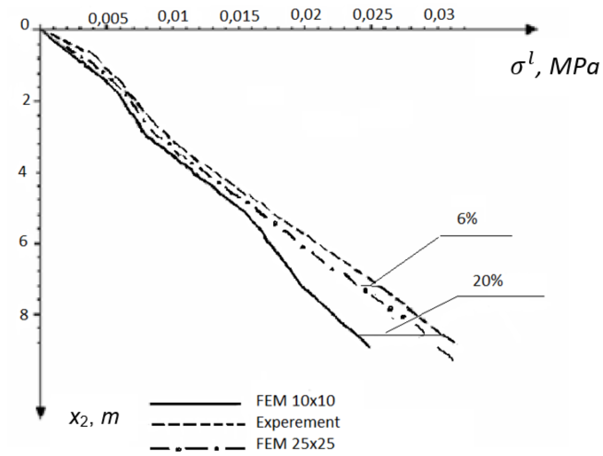


**Figure 16.** Residual pore pressure.

**Figure 16** shows the change in residual pore pressure with depth. The maximum discrepancy between the numerical solution and the experimental data was 20% (10x10 grid) and there are 6% (25x25 grid).

## 5. Conclusion

The paper considers the generalized differential Lamé operator. It is shown that the operator is positively definite on the set of twice continuously differentiable functions and is not symmetric like the Lamé operator. The applicability of variational methods to the solution of the mixed boundary value problem is demonstrated. The existence and uniqueness of a generalized solution of the problem of equilibrium of a two-phase body are proved.

The analogs of three Betti formulas are obtained, and one of them shows the asymmetry of the operator. To approximate the system of generalized Lame equations, an energy functional (analog of the Clapeyron formula) is constructed. The deformation energy is represented by three components: the sum of the first two summands is a quadratic functional, and the third summand is a bilinear functional that describes the effect of pore water on the soil skeleton.

The model was calibrated on a Flamand-type test problem. The finite element method was used to solve a real problem of loading a non-uniform soil foundation with a distributed load from a structure. The numerical solution agrees quite well with the experimental data. The maximum discrepancy was no more than 10% on a 25x25 grid.

The results of this study have implications for soil mechanics and applied mathematics. They will lead to additional studies and applications to various applied problems within the framework of the generalized Lame operator model.

The model presented here is based on small-deformation assumptions and excludes soil creep to first describe soil behavior through elasticity theory, with subsequent extension to viscoelasticity theory. The viscoelastic formulation accounts for soil creep, though this extension is not addressed in the current article.

One of the causes of soil deformation is soil creep (Nguyen, 2012). The influence of creep deformation of soft soil on engineering construction cannot be ignored (Yuan et al., 2023). The change in the modulus of linear deformation of the soil over time within the framework of the model considered by the authors of the article will enable solving problems that account for the simultaneous influence of pore water on the soil skeleton and the viscoelastic properties of the soil skeleton on the stress-strain state of the soil foundation (Maltseva et al., 2024). In the work of Yin et al. (2022) & Chen et al. (2024) a general simple method is considered that is suitable for calculating settlements during compaction of layered viscous clay soils without vertical drainage holes or with them under complex loading conditions with good accuracy. This is the first aspect. Secondly, the further prospects of this study include justifying the use of design solutions such as vertical reinforcement in road construction, the use of sand-reinforced pads and piles in low-rise construction to enhance the bearing capacity of viscoelastic water-saturated soil foundations. Studying the influence of the viscoelastic properties of the skeleton of water-saturated soil will significantly expand the field of geotechnical design, using more accurate and efficient methods for calculating soil foundations.

### Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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