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Numerical Solution of Dual Fully Fuzzy Equations System Using Some Meta Heuristic Methods

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Abstract
Nowadays, there is more and more attention from researchers in solving various problems in the applied field of fuzzy equations, one of which is the fluid mechanics problem of reservoir piping. In this article, we investigate a solution to a dual fully fuzzy equations system involving triangular and trapezoidal fuzzy numbers using a numerical approach, especially meta-heuristic methods. We consider the equations system in crisp equations form as an optimization problem, and then its problems are solved. Here we compare the performance of four meta-heuristic methods, i.e., genetic algorithm (GA), chaos optimization algorithm (COA), particle swarm optimization (PSO), and grey wolf optimizer (GWO), to solve some cases in our problems. The results obtained show that PSO gives performance that is better than of other methods. Furthermore, the type of initial population number in GA affects how close the numerical solutions are to the exact solution.

Keywords- Fuzzy numbers, Dual fully fuzzy nonlinear equation systems, Optimization problem, Meta heuristic method.

1. Introduction
Fuzzy equations are one of the mathematical concepts that are widely used in science and engineering. This equation contains uncertain or vague numerical data which is considered to be more aspects of real problems. The concept of fuzzy was first introduced by Zadeh (1975). Fuzzy numbers are a generalization of crisp numbers which represent vague quantities and characterized by a membership function that assigns a degree of membership in the interval [0,1]. Based on the representation of the membership function, two

types of fuzzy numbers are most often used, namely, triangular and trapezoidal fuzzy numbers. One of the studies that attracts the attention of many researchers in fuzzy numbers is the dual fully fuzzy equation system. This system of equations has applied arithmetic operations on fuzzy numbers that are different from arithmetic operations on real numbers.

A fuzzy system of equations (FSE) is a system of equations in which all parameters, or some parameters, are fuzzy numbers. In the case of a linear system of equations, it can be written in the general form $A \otimes \mathrm{x} = \mathrm{b}$ with A being an $n \times n$ nonsingular crisp matrix, x and b are $n \times 1$ fuzzy vectors. In this case, the $n \times n$ fuzzy system is converted into a $2n \times 2n$ real equation system (Deswita and Mashadi, 2019). Many studies related to the solution of FSE have been carried out both analytically and numerically. In Friedman et al. (1998), the solution of FSE using an embedded method was studied about the uniqueness of the solution. Then Abbasbandy et al. (2005) used the conjugate gradient method for positive definite linear FSE problems. Dehghan et al. (2006) studied the iterative solution for linear FSE, and Miao et al. (2008) solved it using the Block SOR method.

In Dehghan et al. (2007), the authors introduced the system of fully fuzzy equations (SFFE) $A \otimes \mathrm{x} = \mathrm{b}$, where, b and A are fuzzy vector and fuzzy matrix, respectively. Especially for linear fully fuzzy systems of equations, Allahviranloo et al. (2011a) have studied the nonzero solutions of such systems and solved them using MOLP. Furthermore, Allahviranloo et al. (2011b) presented the maximal and minimal symmetry solutions of this type of system. Recently, Abbasi and Allahviranloo (2022) developed a new concept for the solution of fully fuzzy linear systems using fuzzy operation-based mean transmission.

In addition to using analytical methods, some researchers have also developed numerical or computational SFFE solutions. For example, Dehghan et al. (2006) discussed the solution of linear SFFE using computational methods. For the solution of the same system of equations, Dehghan and Hashemi (2006) obtained a solution using a decomposition procedure, then Dehghan et al. (2007) with their iterative technique.

The system of dual fully fuzzy equations (SDFFE) is a development of another form of SFFE. For the case of linear SDFFE, it has the general form $A \otimes \mathrm{x} \oplus \mathrm{b} = C \otimes \mathrm{x} \oplus \mathrm{d}$, and several studies have been conducted regarding the solution of the system. Among them, in Jafarian (2016), Jafarian solved SDFFE using ST decomposition method for trapezoidal fuzzy numbers and Safitri and Mashadi (2019) solved this system using LU factorization. Then Gemawati et al. (2018) used QR decomposition in solving the dual fully fuzzy linear equation system on trapezoidal numbers.

In the case of non-linear SDFFE, it is generally difficult to solve analytically. Therefore, the numerical approach is one method that can be used. In general, in solving a system of nonlinear equations, the methods designed use many concepts in calculus and algebra, namely the concept of differentiability of a function and inverse function. These methods include Newton's method, gradient descent, and others. Kajani et al. (2005) used Newton method to solve dual fuzzy nonlinear equations in parametric form. With the same topic Waziri and Majid (2012) used Newton method for initial iteration and Broyden's method for the next iteration. Sedaghatfar et al. (2014) have transformed dual fully fuzzy linear equations system into two crisp equations form and then solve it. Then a combination of Newton's method and modified Adomian decomposition (Mosleh, 2013), a combination of Newton and Broyden methods (Waziri and Moyi, 2016) and through a multidimensional fuzzy arithmetic-based approach (Kołodziejczyk et al., 2020). Sulaiman et al. (2022) proposed a variant of Newton's method, namely the modified Newton method, for solving parameterized dual fuzzy nonlinear equations by introducing new parameters in determining the search direction of the Newton method.

However, there is a drawback to Newton's method, which is that in addition to having to choose an initial iteration value that is close enough to the solution to achieve convergence, it must also calculate the inverse of the Jacobian matrix at each iteration step. Some researchers have proposed several methods to overcome this problem. Among them, Sulaiman et al. (2018b) proposed the Samanskii method in solving dual fuzzy nonlinear equations to reduce the cost of calculating the Jacobian matrix in the Newton method. In another article, Sulaiman et al. (2018a) used the Levenberg-Marquardt method to overcome the weakness of Newton's method in solving nonlinear fuzzy equations. In Omesa et al. (2020), the authors have applied the quasi-Newton method to approximate the Jacobian matrix as the iteration progresses. Zakaria et al. (2023) in and Megarani and Zakaria (2024) have used Broyden's method to obtain the solution of non-linear SPDFF for the case of triangular fuzzy numbers.

In addition to calculus and algebra-based studies, several studies have examined the use of meta-heuristic methods in solving fuzzy number equations. One of the advantages of the meta-heuristic method is that it does not require the differentiability properties of the functions contained in the system of equations. In addition, the problem of the existence of the inverse of the Jacobian matrix, as in Newton's method, does not exist. Some meta-heuristic methods that have been widely used on various problems are the genetic algorithm (GA), Chaos Optimization Algorithm (COA), Grey Wolf Optimizer (GWO), Particle Swarm Optimization (PSO), and others. GA is a search algorithm based on the principles of natural selection and genetics, introduced by Holland in the 1970s and inspired by the biological evolution of living things (Holland, 1975). COA is a method to optimize possibly nonlinear complex functions of several variables by chaos search (Bing, 1997). The GWO is inspired by the grey wolf (Canis lupus), which mimics the grey wolf's leadership hierarchy and hunting mechanism in nature (Mirjalili et al., 2014). PSO was first introduced by Kennedy and Eberhart (1995), who modelled it on the social behaviour of flocks of birds or shoals of fish. PSO updates candidate solutions iteratively based on the movement of particles towards the best solution that has been found. PSO utilises swarm intelligence to find the optimal cost through collective movement.

Recently, the four meta-heuristic methods have been widely used in solving various problems, including fuzzy equations. In Mashinchi et al. (2007) used a GA approach in solving linear and quadratic fuzzy equations. Gerges et al. (2018) used GA to solve the Sudoku puzzle problems. Meanwhile, in Burkhart and Ruiz (2023), the authors have implemented GA for a hyperparameter optimization problem. Another example of GA applications can be seen in Abyan et al. (2022) and Fatimatuzzahra et al. (2025). In the meantime, COA and its various forms have been elaborated in Zhang et al. (2024). While GWO, Long et al. used it for solving large-scale function optimization problems (Long et al., 2018). In the application PSO, the cost optimization of the retrial queue model (Malik et al., 2021; Upadhyaya, 2020) has been elaborated. To improve the performance of the meta-heuristic methods in solving optimization problems, some researchers have combined two meta-heuristic methods in their algorithms. For example, in Bhandari et al. (2023, 2024), the authors used hybrid PSO and GWO for solving the allocation problem.

As far as the literature search is concerned, there is no study on solving non-linear SPDFF using the meta-heuristic method. Therefore, in this article we will investigate whether meta-heuristic methods can be used to approximate the roots of SDFFE. Thus, SDFFE solving algorithm using a meta-heuristic method based on the mathematical model of optimization derived from SDFFE will be studied. Here we use GA, COA, GWO, and PSO to solve SDFFE. We selected the four methods by considering the advantages of each method. GA is a robust meta-heuristic method which provide optimization over large space state and can handling constraint optimization problems (Katoch et al., 2021). COA is one of the powerful global optimization techniques, which has outstanding capabilities in tackling complex optimization challenges. Recently, COA has been applied to many single-objective optimization problems, and simulation results have shown its effectiveness (Li et al., 2020). Meanwhile, the GWO algorithm has a simple structure, few control parameters, high convergence accuracy, and local minima escaping ability (Qiu et al., 2024). On the other hand, PSO has the major advantage of having fewer parameters to set. PSO obtains the best solution from particle interactions, but through the high-dimensional search space, PSO converges at a very slow speed towards the global optimum (Gad, 2022). In practice, this research will be studied and illustrated for non-linear SPDFF cases from related reference journals. Based on the results of the approximate solution obtained, the performance of the proposed method will be compared with the results using calculus or algebra-based methods, namely Broyden's method.

This paper is arranged as follows: some important basic theoretical concepts used in this study will be presented in Section 2. In Section 3, we describe the problem formulation of meta-heuristic method application to solve SDFFE and some examples of numerical results. Finally, the conclusion is given in Section 4.

2. Preliminary Notes

This section will present some important basic theoretical concepts used in this study.

2.1 Fuzzy Numbers

Definition: Let $F = \{(t, \tilde{u}_F(t)) | t \in \mathbb{R}, \tilde{u}_F : \mathbb{R} \to [0,1]\}$ fuzzy set. F is fuzzy number if $\tilde{u}_F$ has the following conditions (Mashinchi et al., 2007).

a) $\tilde{u}_F$ upper semicontinuous,
b) there exist real numbers p, q, r and s, with $p \leq q \leq r \leq s$, where,
   (i) $\tilde{u}_F(x) = 0$ outside the interval $[p, s]$,
   (ii) $\tilde{u}_F(x)$ monotonically increasing at $[p, q]$,
   (iii) $\tilde{u}_F(x)$ monotonically decreasing at $[r, s]$,
   (iv) $\tilde{u}_F(x) = 1$, on $[q, r]$.

Definition: A triangular fuzzy number $\tilde{t} = (p - a, p, p + b) = (t_m, t_l, t_u)$ is a fuzzy number with membership function i.e. (Zakaria et al., 2023).

$$u_{\tilde{t}} = \begin{cases} 1 - \frac{p-x}{a}, p - a \leq x \leq p \\ 1 - \frac{x-p}{b}, p \leq x \leq p + b \\ 0, \qquad other \end{cases} \tag{1}$$

Definition: A fuzzy number $\tilde{a} = (m, n, \alpha, \beta)$ is called a trapezoidal fuzzy number if its membership function is defined by Kumar et al. (2010):

$$u_{\tilde{a}} = \begin{cases} 1 - \frac{m-x}{\alpha}, \quad m - \alpha \leq x \leq m, \alpha > 0 \\ 1, \qquad m \leq x \leq n \\ 1 - \frac{x-n}{\beta}, \quad n \leq x \leq n + \beta, \beta > 0 \\ 0, \qquad other \end{cases} \tag{2}$$

Definition: Suppose given two triangular fuzzy numbers $\tilde{a} = (a_m, a_l, a_u)$ and $\tilde{b} = (b_m, b_l, b_u)$ then the calculation operation is as follows (Jafarian and Jafari, 2019).

a) $\tilde{a} \oplus \tilde{b} = (a_m + b_m, a_l + b_l, a_u + b_u)$,
b) $-\tilde{a} = (-a_m, -a_l, -a_u)$,

c) $\tilde{a} \ominus \tilde{b} = (a_m - b_u, a_l - b_l, a_u - b_m)$.

d) Multiplication on fuzzy numbers is denoted by $\hat{*}$, i.e.

$\tilde{a} \hat{*} \tilde{b} = (c_m, c_l, c_u)$.

where,

$$c_l = a_l \,.\, b_l,$$

$$c_m = \min(a_m \,.\, b_m, a_m \,.\, b_u, a_u \,.\, b_m, a_u \,.\, b_u),$$

$$c_u = \max(a_m \,.\, b_m, a_m \,.\, b_u, a_u \,.\, b_m, a_u \,.\, b_u).$$

If $\tilde{a}$ is any triangular fuzzy number and $\tilde{b}$ is non-negative, then:

$$\tilde{a} \hat{*} \tilde{b} = \begin{cases} (a_m \,.\, b_m, a_l \,.\, b_l, a_u \,.\, b_u), & a_m \geq 0 \\ (a_m \,.\, b_u, a_l \,.\, b_l, a_u \,.\, b_u), & a_m < 0, a_u \geq 0 \\ (a_m \,.\, b_m, a_l \,.\, b_l, a_u \,.\, b_m), & a_m < 0, a_u < 0 \end{cases}$$

***Definition*:** Suppose given two trapezoidal fuzzy numbers $\tilde{t} = (p, q, \alpha, \beta)$ and $\tilde{u} = (r, s, \gamma, \delta)$, then the algebra of trapezoidal fuzzy numbers (Gemawati et al., 2018).

a) $\tilde{t} \oplus \tilde{u} = (p + r, q + s, \alpha + \gamma, \beta + \delta)$,

b) $\tilde{t} \ominus \tilde{u} = (p - r, q - s, \alpha + \delta, \beta + \gamma)$,

c) For every $\tilde{t} = (p, q, \alpha, \beta)$, there exists $\bar{t} = (p, q, -\beta, -\alpha)$ such that $\tilde{t} \ominus \bar{t} = (0,0,0,0)$,

d) $\lambda \hat{*} \tilde{t} = \lambda \hat{*} (p, q, \alpha, \beta) = \begin{cases} (\lambda p, \lambda q, \lambda\alpha, \lambda\beta) & \lambda \geq 0 \\ (\lambda p, \lambda q, -\lambda\beta, -\lambda\alpha,) & \lambda < 0 \end{cases}$,

e) Multiplication $\tilde{t} \hat{*} \tilde{u}$

$$\tilde{t} \hat{*} \tilde{u} = \begin{cases} (pr, qs, p\gamma + r\alpha, q\delta + s\beta), & \tilde{t} > 0,\ \tilde{u} > 0 \\ (pr, qs, \alpha r - p\delta, \beta s - q\gamma), & \tilde{t} < 0,\ \tilde{u} > 0. \\ (pr, qs, -p\gamma - \alpha r, -q\delta - \beta s), & \tilde{t} < 0,\ \tilde{u} < 0 \end{cases}$$

3. Results and Discussion

In this study, we consider the SDFFE in its general form

$$\tilde{A}_1 \hat{*} \tilde{x} \oplus \tilde{A}_2 \hat{*} \tilde{x}^2 \oplus \ldots \tilde{A}_n \hat{*} \tilde{x}^n = \tilde{A}_{n+1} \hat{*} \tilde{x} \oplus \tilde{A}_{n+2} \hat{*} \tilde{x}^2 \oplus \ \ldots \oplus \tilde{A}_{2n} \hat{*} \tilde{x}^n \oplus \tilde{b} \tag{3}$$

where, $\tilde{A}_i (1 \leq i \leq 2n)$ is a matrix of fuzzy numbers, $\tilde{b}$ is vector of an unknown element, and $\mathbf{x}$ is a column vector of fuzzy. By using arithmetic operations, the SDFFE is converted to a crisp equation system (equation system in real numbers) in the form $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ with $\mathbf{f} = (f_1, f_2, \ldots, f_n, f_{n+1}, f_{n+2}, \ldots, f_{2n})^{\boldsymbol{T}}$ and $\mathbf{x} \in \mathbb{R}^{\mathbf{2n}}$**.** Based on the crisp equation system, the optimization problem is constructed as follows:

$$z = \min_{\mathbf{x}} F(\mathbf{x}) = \min_{\mathbf{x}} \sum_{i=1}^{2n} |f_i(\mathbf{x})| \tag{4}$$

For GA, we choose the fitness function as follows:

$$Fitness = \frac{1}{1+z} \tag{5}$$

The optimization problem (4) is then solved by meta-heuristic methods (GA, COA, GWO, and PSO). A solution of SDFFE is obtained if we get the minimum value of the optimization problem is or close to zero. The algorithm in pseudocode for every method is as follows.

Algorithm 1: Pseudo code for GA (Mangla et al., 2019)

INPUT: Input function $F(\mathbf{x})$
OUTPUT: vector **x** denotes the solution of SDFFE
Initialize population with random individuals
Evaluate the fitness of each individual in the population
while termination condition not met **do**
    Select parents from the population based on fitness
    Perform crossover on parents to create offspring
    Perform mutation on offspring
    Evaluate fitness of offspring
    Select individuals to form the new population
end while
Return the best individual from the population

Algorithm 2: Pseudo code for GWO (Mirjalili et al., 2014)

INPUT: Input function $F(\mathbf{x})$, a, B, D
OUTPUT: vector **x** denote the solution of SDFFE
Initiate the grey wolf population $y_i, i = 1,2,3, \dots ,2n$
Initiate a, B, D
Evaluate fitness of every search factor
$y_\alpha =$ the best search factor
$y_\beta =$ the second best search factor
$y_\delta =$ the third best search factor
while t < maximum number of iteration **do**
    For every search factor do
        Arbitrarily initiate c1 and c2
        Update the place of the present search factor
        Update a, B, D
    End for
    Estimate the fitness of every search factors
    Update $y_\alpha, y_\beta, y_\delta$
    $t = t + 1$
end while
Return y_α

Algorithm 3: Pseudo code for COA (Luo et al., 2008)

INPUT: Input function $F(\mathbf{x})$
OUTPUT: vector **x** denotes the solution of SDFFE
Step 1. Initialization
Initialize the chaos variable's iterative, the fine search flag, the maximum chaos variable's iterative K_{max}, the maximum fine search flag r_{max}
Produce $\mathbf{t}^0$ by random in [0,1] and $\mathbf{t}^0 \neq (0,0.25,0.5,0.75,1.0)$
Let $\mathbf{t}^k = \mathbf{t}^0$, the current best chaos variable $\mathbf{t}^* = \mathbf{t}^0$, $a_i^r = a_i$, $b_i^r = b_i$, produce $\mathbf{x}^*$ by random in the search space, calculate $F^* = F(\mathbf{x}^*)$

Step 2. Mapping variable
Calculate $x_i^k = a_i^r + t_i^k(b_i^r - a_i^r)$
Step 3. Best so far
If $F(\mathbf{x}^k) < F^*$, update $F^* = F(\mathbf{x}^k)$, $\mathbf{x}^* = \mathbf{x}^k, \mathbf{t}^* = \mathbf{t}^k$ end if
Step 4. Chaos variable iteration
$k = k + 1$, calculate $t_i^{k+1} = 4t_i^k(1.0 - t_i^k)$
Step 5.
If $k < k_{max}$, return step 2; else $r = r + 1$, go to step 6
Step 6. Change search range
$a_i^{r+1} = x_i^* - \gamma(b_i^r - a_i^r)$, $b_i^{r+1} = x_i^* + \gamma(b_i^r - a_i^r)$
If $a_i^{r+1} < a_i^r$ then $a_i^{r+1} = a_i^r$ end if
If $b_i^{r+1} > b_i^r$ then $b_i^{r+1} = b_i^r$ end if
Step 7.
If $r < r_{max}$ Produce $\mathbf{t}^0$ by random $k = 0$, $\mathbf{t}^k = \mathbf{t}^0$ go to step 2;
Else the COA is terminated, $\mathbf{x}^*$ is a solution

Algorithm 4: Pseudocode for PSO (Kennedy and Eberhart, 1995)

INPUT: Input function $F(\mathbf{x})$
OUTPUT: vector $\mathbf{x}$ denotes the solution of SDFFE
Initialize population of particles with random positions and velocities
Initialize global best positions and fitness to infinity
while the termination condition is not met, **do**
    For each particle, do
        Update velocity based on current position, personal best, and global best
        Update position based on velocity
        Evaluate the fitness of the current position
        if the current position is better than the personal best, then
            Update personal best position and fitness
        else if the current position is better than the global best, then
            Update the global best position and fitness
        end if
    End for
end while
Return the global best position and fitness

For illustration of our method, we give two examples of problems. Based on the general form of SDFFEs, here we use nonlinear SDFFEs in the polynomial form.

Problem 1. Consider the SDFFE in triangular fuzzy number as follow (Zakaria et al., 2023).

$$\begin{cases} (2,3,5) \circledast \tilde{\boldsymbol{x}} \oplus (2,4,5) \circledast \tilde{\boldsymbol{y}} \oplus (1,2,3) \circledast \tilde{\boldsymbol{x}}^2 \oplus (3,5,6) \circledast \tilde{\boldsymbol{y}}^2 = \\ \qquad (2,4,5) \circledast \tilde{\boldsymbol{x}}^2 \oplus (3,4,5) \circledast \tilde{\boldsymbol{y}}^2 \oplus (5,12,42) \\ \\ (1,2,3) \circledast \tilde{\boldsymbol{x}} \oplus (3,4,6) \circledast \tilde{\boldsymbol{y}} \oplus (3,4,5) \circledast \tilde{\boldsymbol{x}}^2 \oplus (1,3,4) \circledast \tilde{\boldsymbol{y}}^2 = \\ \qquad (2,3,4) \circledast \tilde{\boldsymbol{x}}^2 \oplus (2,4,5) \circledast \tilde{\boldsymbol{y}}^2 \oplus (2,24,47) \end{cases} \tag{6}$$

Let $\tilde{\mathbf{x}} = (x_1, x_2, x_3)$ and $\tilde{\mathbf{y}} = (y_1, y_2, y_3)$ are non-negative triangular fuzzy numbers. By the arithmetic operation of triangular fuzzy numbers, Equation (6) becomes

$$\begin{cases} 2x_1 + 2y_1 - x_1^2 - 5 = 0 \\ 3x_2 + 4y_2 - 2x_2^2 + y_2^2 - 12 = 0 \\ 5x_3 + 5y_3 - 2x_3^2 + y_3^2 - 42 = 0 \\ x_1 + 3y_1 + x_1^2 - y_1^2 - 4 = 0 \\ 2x_2 + 4y_2 + x_2^2 - y_2^2 - 24 = 0 \\ 3x_3 + 6y_3 + x_3^2 - y_3^2 - 47 = 0 \end{cases} \tag{7}$$

So that in the optimization problem (4) we have

$$F(\mathbf{x}, \mathbf{y}) = \sum_1^6 |f_i(\mathbf{x}, \mathbf{y})| \tag{8}$$

where,

$$\begin{cases} f_1(\mathbf{x}, \mathbf{y}) = 2x_1 + 2y_1 - x_1^2 - 5 \\ f_2(\mathbf{x}, \mathbf{y}) = 3x_2 + 4y_2 - 2x_2^2 + y_2^2 - 12 \\ f_3(\mathbf{x}, \mathbf{y}) = 5x_3 + 5y_3 - 2x_3^2 + y_3^2 - 42 \\ f_4(\mathbf{x}, \mathbf{y}) = x_1 + 3y_1 + x_1^2 - y_1^2 - 4 \\ f_5(\mathbf{x}, \mathbf{y}) = 2x_2 + 4y_2 + x_2^2 - y_2^2 - 24 \\ f_6(\mathbf{x}, \mathbf{y}) = 3x_3 + 6y_3 + x_3^2 - y_3^2 - 47 = 0 \end{cases}$$

The approximation solution of optimization problem using each method can be seen in **Table 1**.

Table 1. The approximation solution of problem 1.

Variable	Approximation solution using method				
	GA	**COA**	**GWO**	**PSO**	**Broyden (Zakaria et al., 2023)**
x_1	0.9974	1.0035	1.0029	1	1
x_2	3.9975	4.0002	4.0001	4	4
x_3	5.9968	6.0000	6.0001	6	6
y_1	2.0168	2.0026	2.0007	2	2
y_2	3.9976	4.0002	4.0002	4	4
y_3	6.9959	7.0001	7.0000	7	7
Elapsed time	1205 sec	25 sec	333 sec	11 sec	2.81 sec

Note that for the GA method, if the initial population is integer numbers, then the solution obtained is the same as the solution using the Broyden method. Furthermore, **Figure 1** and **Figure 2** present a triangular fuzzy number for our solution.

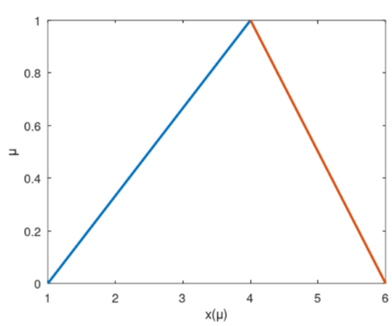

Figure 1. Triangular fuzzy number representation graph of **x** in Problem 1.

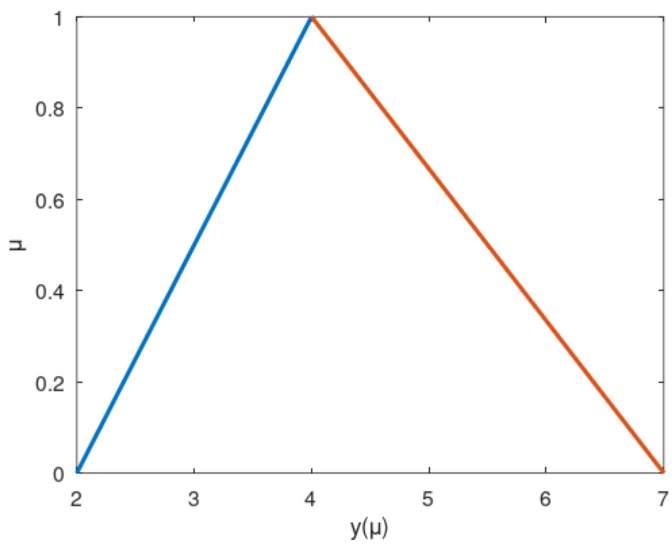

Figure 2. Triangular fuzzy number representation graph of **y** in Problem 1.

Problem 2. Consider the SDFFE in trapezoidal fuzzy number as follows (Zakaria et al., 2023).

$$\begin{cases} (5,7,3,2) \circledast \tilde{\boldsymbol{x}}^2 \oplus (3,5,1,2) \circledast \tilde{\boldsymbol{y}}^2 \oplus (6,7,16,3) = \\ \quad (5,6,1,2) \circledast \tilde{\boldsymbol{x}}^2 \oplus (4,5,3,3) \circledast \tilde{\boldsymbol{y}}^2 \oplus (2,9,4,5) \\ (3,6,2,2) \circledast \tilde{\boldsymbol{x}}^2 \oplus (4,6,1,3) \circledast \tilde{\boldsymbol{y}}^2 \oplus (6,10,11,7) = \\ \quad (4,5,1,1) \circledast \tilde{\boldsymbol{x}}^2 \oplus (5,8,1,2) \circledast \tilde{\boldsymbol{y}}^2 \oplus (1,2,3,4) \end{cases} \tag{9}$$

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ are non-negative trapezoidal fuzzy numbers. By the arithmetic operation of trapezoidal fuzzy numbers, Equations (6) become

$$\begin{cases} y_1^2 - 4 = 0 \\ x_2^2 - 2 = 0 \\ 2x_1^2 - y_3^2 - 2y_1^2 + 12 = 0 \\ x_4^2 - y_2^2 + 1 = 0 \\ -x_1^2 - y_1^2 + 5 = 0 \\ x_2^2 - 2y_2^2 + 8 = 0 \\ -x_3^2 + x_1^2 - y_3^2 + 8 = 0 \\ x_4^2 + x_2^2 - 2y_4^2 + y_2^2 + 3 = 0 \end{cases} \tag{10}$$

So that in the optimization problem (4) we have

$$F(\mathbf{x}, \mathbf{y}) = \sum_1^8 |f_i(\mathbf{x}, \mathbf{y})| \tag{11}$$

where,

$$\begin{cases} f_1(\mathbf{x},\mathbf{y}) = y_1^2 - 4 \\ f_2(\mathbf{x},\mathbf{y}) = x_2^2 - 2 \\ f_3(\mathbf{x},\mathbf{y}) = 2x_1^2 - y_3^2 - 2y_1^2 + 12 \\ f_4(\mathbf{x},\mathbf{y}) = x_4^2 - y_2^2 + 1 \\ f_5(\mathbf{x},\mathbf{y}) = -x_1^2 - y_1^2 + 5 \\ f_6(\mathbf{x},\mathbf{y}) = x_2^2 - 2y_2^2 + 8 \\ f_7(\mathbf{x},\mathbf{y}) = -x_3^2 + x_1^2 - y_3^2 + 8 \\ f_8(\mathbf{x},\mathbf{y}) = x_4^2 + x_2^2 - 2y_4^2 + y_2^2 + 3 \end{cases}.$$

The approximation solution of the optimization problem using each method can be seen in **Table 2**.

Table 2. The approximation solution of Problem 1.

Variable	Approximation solution using method				
	GA	COA	GWO	PSO	Broyden (Zakaria et al., 2023)
x_1	1.0006	1.0000	1.0000	1.0000	1.0000
x_2	1.4129	1.4140	1.4147	1.4142	1.4142
x_3	1.7319	1.7336	1.7332	1.7321	1.7321
x_4	2.0007	2.0002	2.0006	2.0000	2.0000
y_1	2.0008	2.0002	2.0002	2.0000	2.0000
y_2	2.2367	2.2360	2.2361	2.2361	2.2361
y_3	2.4493	2.4487	2.4488	2.4495	2.4495
y_4	2.6459	2.6458	2.6461	2.6458	2.6458
Elapsed time	37 min	25.84 sec	393.8 sec	11.33 sec	2.95 sec

The representation of trapezoidal fuzzy number for our solution can be seen in **Figure 3** and **Figure 4.**

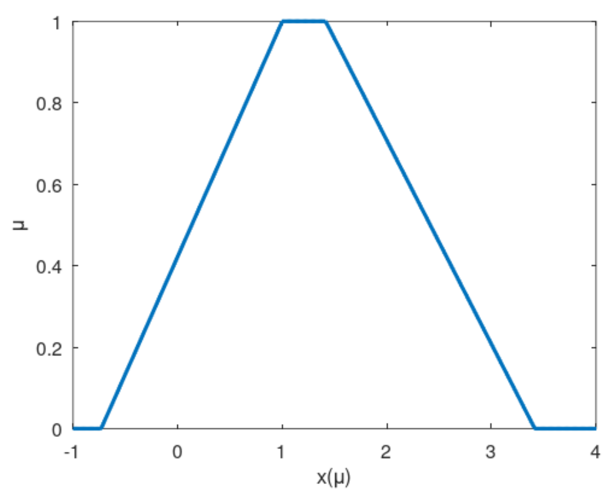

Figure 3. Trapezoidal fuzzy number representation graph of **x** in Problem 2.

Based on the results obtained in solving Problem 1 and Problem 2, as presented in **Table 1** and **Table 2**, it shows that the proposed meta-heuristic method can be used as an alternative method in determining the approximate solution of SDFFE, as expected. The results of the four meta-heuristic methods show performance that is not inferior to calculus or algebra-based methods, namely Broyden's method, although in terms of computation time Broyden's method is still superior. PSO provides the best performance where the solution obtained is the same as the Broyden method, only losing in terms of computing time. Meanwhile, GA in this case requires the longest computation time. The accuracy in determining computational parameters such as the number and type of initial population, the number of iterations, and the initial population interval greatly affects the root results obtained as well as computation time. When the SDFFE solution is an integer, then determining the type of population in the form of an integer is highly recommended because the solution obtained will be the same, and the computation time is much shorter, almost close to the computation time of PSO and COA. Conversely, if the solution is a real number, then the initial population is recommended to be a real number as well.

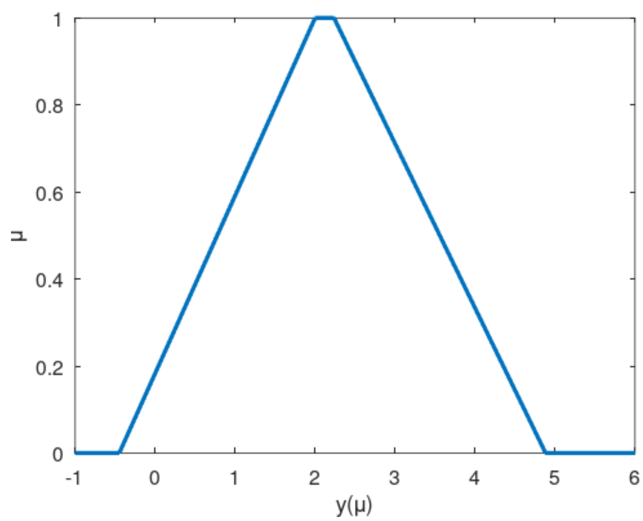

Figure 4. Trapezoidal fuzzy number representation graph of **y** in Problem 2.

4. Conclusion

In this article, we have investigated and compared the performance of four meta-heuristic methods, i.e., genetic algorithm (GA), chaos optimization algorithm (COA), grey wolf optimizer (GWO), and particle swarm optimization (PSO), to solve some cases in dual fully fuzzy equation system problems. The results show that PSO gives an approximation solution that is closer to an approximation solution using the calculus-based method, namely the Broyden method. Furthermore, the type of initial population number in GA affects how close the numerical solutions are to the exact solution. In this study, the performance of the meta-heuristic method in determining the solution of non-linear SDFFE is limited to polynomial equations, it has not been tested on other types of non-linear equations, such as transcendental equations. This is because the theory on the concept of fuzzy numbers in the type of transcendental functions has not been studied. The study of this concept can be a topic for future research.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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The author(s) declare that no assistance is taken from generative AI to write this article.

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