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A Class of Quadrature Rules for Complex Cauchy Principal Value Integrals

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Abstract
This article is fully devoted to the numerical approximation of Cauchy-type integrals in the complex plane. A class of degree eight quadrature rules is formulated from a family of Gauss-type two-point rules based on the method of extrapolation. The basic rules are developed and then composite rules are constructed from the basic rules. The Error bounds for each rule are determined. The validity of the method has been demonstrated by the provision of numerical experiments and their results.

Keywords- Analytic function, Complex Cauchy principal value, Error bound, Quadrature rule.

1. Introduction

Cauchy principal value integrals are frequently encountered in research in applied mathematics, such as the theory of aerodynamics, scattering theory, the crack problem in plane elasticity, the singular eigen function method in neutron transport, and many other fields of science and engineering. They also occur in contour integration, which is considered an essential tool in mathematics, applied sciences, and engineering. The real CPV integral is given by,

$$K(f) = P\int_{\lambda-h}^{\lambda+h} \frac{f(x)}{x-\lambda}dx.$$

The numerical evaluation of the aforementioned integral has been the subject of extensive investigation. Some of the illustrious researchers who have formulated quadrature rules for numerical integration of real Cauchy principal value integrals are: Piessens (1970), Chawla and Jayarajan (1975), Elliott and Paget (1979), Orsi (1990), Diethelm (1995), Criscuolo (1997) and Li and Chen (2007).

Saff and Snider (2013) defined the complex Cauchy principal value integral in the following way.

$$I(g) = P\int_{z_0-h}^{z_0+h} \frac{g(z)}{z-z_0}dz \tag{1}$$

where, $g(z)$ is assumed to be an analytic in the disc,

$$D = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|, r > 1\}.$$

in the complex plane $\mathbb{C}$ containing the directed line segment L joining the points $z_0 - h$ to $z_0 + h$. According to this definition $I(g)$ is given by the limit,

$$\lim_{\epsilon \to 0} \int_{L'_\epsilon} \frac{g(z)}{z - z_0} dz.$$

provided the limit exists where L'_ϵ is the extended directed line segment with end points $Z_0 \pm h$ and semicircular indentation of radius ϵ and centre z_0. So far, very little work has been done to develop quadrature rules for complex CPV integrals. Milovanovic et al. (1984) have constructed an interpolatory type of rule for the approximate evaluation of the integral (1), which requires the evaluation of the first derivative of the function $g(z)$ at $z = z_0$. This rule is of precision at most eight and the precision, as well as the accuracy in approximation, diminishes if the $g'(z_0)$ is replaced by an approximation formula. Further Das and Hota (2012) have developed a derivative-free, one-parameter quadrature rule for the numerical evaluation of the integral (1). They assert that their rule is the one that accurately integrates the complex CPV integral compared to rules that have been published in the past. Moreover, Bej et al. (2012) have developed a few additional rules of the algebraic degree of precision eight using fewer nodes from the rules of Das and Hota (2012). Chen (2013) employed special Hermite interpolation polynomia, while Keller and Wrobel (2016) used a typical adaptive quadrature, and Legua and Sanchez-Ruiz (2017) applied residue theory techniques to calculate CPV integrals. Hasegawa and Sugiura (2019) presented an approximation method of Clenshaw-Curits type to solve the CPV integrals where they used Chebyshev polynomials for interpolating the smooth function of the integral. Yun (2020) used coordinate transformation techniques to evaluate CPV integrals. To enhance the accuracy of the numerical integration method through the coordinate transformation, he presented simple rational functions, including parameters. Xu et al. (2022) evaluated a particular type of CPV integrals with oscillatory integrands by transforming two line integrals and then computing them using the Gauss quadrature rule.
In their article, Saha et al. (2022) formulated non-classical quadrature schemes for the approximation of Cauchy-type oscillatory and singular integrals in the complex plane. Though these schemes have been developed for singular integrals with oscillatory kernels, Cauchy-type singular integrals can also be solved. However, to apply to an unknown integral, all these methods are not as simple and straightforward as standard quadrature rules meant for the numerical integration of definite integrals without having any kind of singularity. Some analytical methods are introduced by Gordon (2023) to solve complex CPV integrals. The Hilbert transform and Dirac delta function are used to encounter the problems. But the scope for applying these methods in the practical field is very limited. In this context, we have developed one and two parameter rules for the numerical approximations of complex CPV integrals of type (1). Here we have intended to formulate some progressive Gauss type and derivative-free quadrature rules for the numerical approximation of the complex CPV integral $I(g)$ involving all of their nodes on the line of integration. The proposed quadrature rules have many advantages over the other rules of this class.

(i) It does not require additional function evaluations in later stages.
(ii) There will be no further occurrence of any type of error like truncation errors, round-off errors, etc. due to the finite precision of the computing machine.
(iii) Real CPV integrals, real definite integrals, and complex line integrals of analytical functions over a line segment without encountering any singularities can all be numerically integrated by these rules.
(iv) Rules can be applied in compound form.

2. Formulation of Basic Rules

First, a two-point rule, denoted as $Q_g(\alpha)$ of precision of at least two, involving a parameter $\alpha(0 < \alpha \leq 1)$ is derived for the approximate evaluation of the complex CPV integral given in (1).On the basis of this rule, a class of (two parameters α_1 and α_2) four-point rules of precision at least four, have been constructed and, subsequently rules of degree six and eight have been derived.

2.1 Derivation of Two Point Rule

The following theorem serves as the foundation for the one-parameter family of two-point rules.

Theorem 1. *If $P(z)$ is a complex polynomial of degree less than or equal to two, then*

$$I_1 = P\int_{z_0-h}^{z_0+h} \frac{P(z)}{z-z_0}dz = w_0 P(z_0) + w_1[P(z_0+\alpha h) - P(z_0-\alpha h)] \quad (2)$$

where,

$$w_0 = 0 \; and \; w_1 = \frac{1}{\alpha} \quad (3)$$

Proof. Let

$$I_1 = P\int_{z_0-h}^{z_0+h} \frac{P(z)}{z-z_0}dz = w_0 P(z_0) + w_1[P(z_0+\alpha h) - P(z_0-\alpha h)] \quad (4)$$

If,

$$P(z) = (z-z_0)^k.$$

then the formula given in 4 exactly integrates I_1 for any even integer k and for any value of the constants w_0, w_1 and α. Therefore, to determine the constants w_0 and w_1 we make the assumption that I_1 is exact for,

$$P(z) = (z-z_0)^k; \; for \; k = 0,1.$$

This leads to the following set of equations,

$$w_0 = 0 \; and \; w_1 = \frac{1}{\alpha}.$$

From the linearity of the integrals, it follows that the formula (4) exactly integrates any complex polynomial of the form,

$$P(z) = \sum_{k=0}^{2} a_k (z-z_0)^k.$$

which is of degree two or less. This finishes the proof of the theorem.

2.2 Two Point Formula

If $P(z)$ is supposed to be a polynomial that interpolates an analytic function $g(z)$ at the points,

$$z_2 = z_0 - \alpha h, z_0 \; and \; z_1 = z_0 + \alpha h.$$

then we have,

$$I(g) \approx w_0 g_0 + w_1[g_1 - g_2] = Q_g(\alpha) \quad (5)$$

where, $w_0 \; and \; w_1$ are given in Equation (3) and,

$$P(z_k) = g(z_k) = (z-z_0)^k.$$

for $k = 0,1,2$.

The rule given in (5) is the desired one-parameter family of two-point quadrature rules (actually the rule $Q_g(\alpha)$ is a three-point rule, but since the coefficient of g_0 in the rule $Q_g(\alpha)$ i.e. w_0 is zero, we say it as a family of two-point rules) of precision at least two for the approximate evaluation of the complex CPV integral $I(g)$ given in (1).

2.3 Asymptotic Error Estimate of Two Point Rule

The asymptotic error estimate of the one-parameter family of rules (Equation-(5)) for the numerical computation of a complex CPV integral of the type (1) is given in Theorem 2. Here it is supposed that the function $g(z)$ is analytic in the disc,

$$D = \{z \in C: |z - z_0| < \rho = r|h|;\ r > 1\}.$$

Theorem 2. *The truncation error* $E_{Q_g}(\alpha)$ *associated with the rule* $Q_g(\alpha)$ *is given by,*

$$|E_{Q_g}(\alpha)| \approx \frac{2|h|^3}{3(3!)}|1 - 3\alpha^2||g^{(3)}(z_0)| \tag{6}$$

for asymptotically small h.

Proof. Let us assume that, the function $g(z)$ is analytic in the disc,

$$D = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|;\ r > 1\}.$$

and the truncation error $E_{Q_g}(\alpha)$ associated with the rule $Q_g(\alpha)$ is given by,

$$E_{Q_g}(\alpha) = I(g) - Q_g(\alpha) \tag{7}$$

Then,

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{8}$$

where, $c_n = \frac{g^{(n)}(z_0)}{(n!)}$; are the Taylor's coefficients. As the above series is uniformly convergent in D, by integrating term by term of both sides of the series (8) we obtain,

$$\begin{aligned} I(g) &= 2hg'(z_0) + \frac{2h^3}{3(3!)}g^{(3)}(z_0) + \frac{2h^5}{5(5!)}g^{(5)}(z_0) \\ &+ \frac{2h^7}{7(7!)}g^{(7)}(z_0) + \frac{2h^9}{9(9!)}g^{(9)}(z_0) \\ &+ \frac{2h^{11}}{11(11!)}g^{(11)}(z_0) + \frac{2h^{13}}{13(13!)}g^{(13)}(z_0) + \ldots \end{aligned} \tag{9}$$

Again, by expanding each term of the rule $Q_g(\alpha)$ given in equation (5) about the point $z = z_0$ in the disc D by Taylor's expansion and then after simplification we obtain,

$$Q_g(\alpha) = 2hg'(z_0) + \frac{2\alpha^2 h^3}{3(3!)}g^{(3)}(z_0) + \frac{2\alpha^4 h^5}{5(5!)}g^{(5)}(z_0) + \frac{2\alpha^6 h^7}{7(7!)}g^{(7)}(z_0) + \ldots \tag{10}$$

Therefore, from Equation (7), we get,

$$E_{Q_g}(\alpha) = \frac{2h^3}{3(3!)}(1 - 3\alpha^2)g^{(3)}(z_0) + \frac{2h^5}{5(5!)}(1 - 5\alpha^4)g^{(5)}(z_0) + \ldots \tag{11}$$

Hence,

$$|E_{Q_g}(\alpha)| \approx \frac{2|h|^3}{3(3!)}|1 - 3\alpha^2||g^{(3)}(z_0)|.$$

for asymptotically small h.

The rules of higher precisions will be formulated from the above two point formula with the help of the following theorem.

Theorem 3. *If the rules*

$$S_1(g) = \sum_{i=0}^{m} A_i[g(z_0+\alpha_i h) - g(z_0-\alpha_i h)];\ 0<\alpha_i\le 1.$$

and

$$S_2(g) = \sum_{j=0}^{n} B_j\big[g(z_0+\beta_j h) - g(z_0-\beta_j h)\big];\ 0<\beta_j\le 1.$$

are of same precession $d(>0)$ and each of which numerically integrates the complex CPV integral $I(g)$ (given in Equation (1)) then, there exist a quadrature rule,

$$S(g) = \frac{1}{L+M}[LS_1(g) + MS_2(g)].$$

of precession $d+2$ numerically, also integrates the integral $I(g)$ for suitable L and M.

Proof. Let, $g(z)$ is analytic in

$$D = \{z\in\mathbb{C}: |z-z_0| < \rho = r|h|; r>1\}.$$

Then,

$$g(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k \tag{12}$$

where, $c_k = \frac{g^{(k)}(z_0)}{k!}$; for $k = 0,1,2$ ... are the Taylor's coefficients.

Since, the series in (12) is uniformly convergent, thus integrating both sides of the series term by term we obtain,

$$I(g) = \sum_{\mu=0}^{\infty} \frac{2c_{2\mu+1}}{2\mu+1} h^{2\mu+1}.$$

Further, expanding each term of the rules $Q(g)$ and $R(g)$ by Taylor's expansion about $z=z_0$; we get,

$$S_1(g) = \sum_{\mu=0}^{\infty}\sum_{i=0}^{m} 2U_i{\alpha_i}^{2\mu+1}c_{2\mu+1}h^{2\mu+1}.$$

and,

$$S_2(g) = \sum_{\mu=0}^{\infty}\sum_{j=0}^{n} 2V_j{\beta_j}^{2\mu+1}c_{2\mu+1}h^{2\mu+1}.$$

Denoting $E_{S_1}(g)$ and $E_{S_2}(g)$ are the truncation errors associated with the quadrature rules $S_1(g)$ and $S_2(g)$ meant for the approximate evaluation of complex CPV integral $I(g)$ respectively we obtain,

$$E_{S_1}(g) \approx O(|h|^{d+1}).$$

$$E_{S_2}(g) \approx O(|h|^{d+1}).$$

Since, $S_1(g)$ and $S_2(g)$ are of precision $d>0$(d is even).

Now, assuming the function $g(z)$ is analytic on D it can be shown that,

$$I(g) = S_1(g) + 2Uc_{d+1}h^{d+1} + \sum_{\mu=\frac{d}{2}+1}^{\infty} 2c_{2\mu+1}h^{2\mu+1}\left[\frac{1}{2\mu+1} - \sum_{i=0}^{m} U_i {\alpha_i}^{2\mu+1}\right] \tag{13}$$

and

$$I(g) = S_2(g) + 2Vc_{d+1}h^{d+1} + \sum_{\mu=\frac{d}{2}+1}^{\infty} 2c_{2\mu+1}h^{2\mu+1}\left[\frac{1}{2\mu+1} - \sum_{j=0}^{n} V_j {\beta_j}^{2\mu+1}\right] \tag{14}$$

by the help of Taylor's theorem where,

$$c_k = \frac{g^{(k)}(z_0)}{k!};\ fork = 0,1,2 \ldots$$

$$U = \left[\frac{1}{d+1} - \sum_{i=0}^{m} U_i {\alpha_i}^{d+1}\right].$$

and,

$$V = \left[\frac{1}{d+1} - \sum_{i=0}^{n} V_j {\beta_j}^{d+1}\right].$$

Multiplying V in equation (13) and $-U$ in Equation (14), then adding their results with subsequent simplifications, we obtain,

$$I(g) = \frac{1}{V-U}[VS_1(g) - US_2(g)] + \frac{c_{d+3}}{d+3}2h^{d+3} - \frac{1}{V-U}\left[V\sum_{i=0}^{m} U_i {\alpha_i}^{d+3} - U\sum_{j=0}^{n} V_j {\beta_j}^{d+3}\right] + \ldots \ldots \ldots$$

Substituting, $V = L$ and $U = -M$; the first term of above expression becomes,

$$S(g) = \frac{1}{L+M}[LS_1(g) + MS_2(g)].$$

It is the required generalized quadrature rule intended for the numerical integration of (1) and the corresponding truncation error is,

$$E_S(g) = \frac{1}{L+M}\left[LE_{S_1}(g) + ME_{S_2}(g)\right].$$

This proves the theorem.

Here, we say the rules $S_1(g)$ and $S_2(g)$ are ***Basic rules*** whereas the rule $S(g)$ constructed by following the process of extrapolation is the ***Composite rule***.

Based on the above theorem, we have constructed one rule of precision ten, three rules of precision eight and three more rules of precision six as composite rules from the basic two point rule as formulated below.

2.4 Derivation of Four Point Rules

Denoting $E_{Q_g}(\alpha_1)$ and $E_{Q_g}(\alpha_2)$ as the truncation errors in approximating the integral $I(g)$ by the rules

$Q_g(\alpha_1)$ and $Q_g(\alpha_2)$, we have,

$$I(g) = Q_g(\alpha_1) + E_{Q_g}(\alpha_1) \tag{15}$$

and,

$$I(g) = Q_g(\alpha_2) + E_{Q_g}(\alpha_2) \tag{16}$$

Now assuming the function $g(z)$ is analytic in the disc D the error terms $E_{Q_g}(\alpha_i)$; $for\ i = 1,2$ can be written as,

$$E_{Q_g}(\alpha_i) = \sum_{k=1}^{\infty} A_{2k+1} g^{(2k+1)}(z_0) \tag{17}$$

where,

$$\begin{cases} & A_{2k+1} = \eta_{2k+1}\xi(\alpha_i); \\ & \eta_{2k+1} = \dfrac{2h^{2k+1}}{(2k+1)!}; \\ and & \xi_{2k+1}(\alpha_i) = \dfrac{1}{2k+1} - \alpha_i^{2k}. \end{cases}$$

for $i = 1,2$ and $k = 1,2, \ldots$.

Now multiplying Equation (15) and (16) by $\xi_3(\alpha_2)$ and $-\xi_3(\alpha_1)$ respectively and then adding the results we obtain,

$$\begin{aligned} I(g) &= \tfrac{1}{3(\alpha_1^2-\alpha_2^2)}\left[(1-3\alpha_2^2)R_g(\alpha_1) - (1-3\alpha_2^2)R_g(\alpha_2)\right] \\ &+ \tfrac{1}{3(\alpha_1^2-\alpha_2^2)}\left[(1-3\alpha_2^2)E_g(\alpha_1) - (1-3\alpha_2^2)E_g(\alpha_2)\right] \end{aligned} \tag{18}$$

writing

$$k_1 = \tfrac{1-3\alpha_2^2}{3(\alpha_1^2-\alpha_2^2)} \text{ and } k_2 = \tfrac{3\alpha_1^2-1}{3(\alpha_1^2-\alpha_2^2)} \tag{19}$$

in Equation (18) we have,

$$I(g) \approx k_1 Q_g(\alpha_1) + k_2 Q_g(\alpha_2) = Q_g(\alpha_1, \alpha_2) \tag{20}$$

associated with the truncation error,

$$E_{Q_g}(\alpha_1, \alpha_2) = k_1 E_{Q_g}(\alpha_1) + k_2 E_{Q_g}(\alpha_2) \tag{21}$$

The rule given in Equation (20) is the desired family of two parameter four point rules.

2.5 Asymptotic Error Estimate of Four Point Rule

The asymptotic error estimates of the four point two parametric quadrature rule $Q_g(\alpha_1, \alpha_2)$ for the approximate evaluation of a complex CPV integral as formulated above are given in the following Theorem 4. Here, we assume that the function $g(z)$ is analytic in the disc,

$$D = \{z \in C: |z - z_0| < \rho = r|h|;\ r > 1\}.$$

Theorem 4. *The truncation error* $E_{Q_g}(\alpha_1, \alpha_2)$ *associated with the four point quadrature rule* $Q_g(\alpha_1, \alpha_2)$ *is given by,*

$$\left|E_{Q_g}(\alpha_1,\alpha_2)\right| \approx \frac{2|h|^5}{15(5!)}\left|(15\alpha_1^2-5)\alpha_2^2+3-5\alpha_1^2\right|\left|g^{(5)}(z_0)\right| \quad (22)$$

for asymptotically small h.

Proof. The truncation error $E_{Q_g}(\alpha_1,\alpha_2)$ associated with the rule $Q_g(\alpha_1,\alpha_2)$ is given by,

$$E_{Q_g}(\alpha_1,\alpha_2) = I(g) - Q_g(\alpha_1,\alpha_2) \quad (23)$$

Now, by substituting $\alpha = \alpha_1, \alpha_2$ successively in Equation (8) and subsequently making simplifications with the help of Equations (20) and (19) we obtain,

$$\begin{aligned} Q_g(\alpha_1,\alpha_2) &= 2hg'(z_0) + \frac{2h^3}{3(3!)}g^{(3)}(z_0) + \frac{2h^5}{3(5!)}(\alpha_1^2+\alpha_2^2-3\alpha_1^2\alpha_2^2)g^{(5)}(z_0) \\ &+ \frac{2h^7}{3(7!)}(\alpha_1^4+\alpha_2^4+\alpha_1^2\alpha_2^2-3\alpha_1^4\alpha_2^2-3\alpha_1^2\alpha_2^4)g^{(7)}(z_0) + \dots \end{aligned} \quad (24)$$

Therefore, from Equation (9), we have,

$$\begin{aligned} E_{Q_g}(\alpha_1,\alpha_2) &= \frac{2h^5}{15(5!)}[(15\alpha_1^2-5)\alpha_2^2+3-5\alpha_1^2)]g^{(5)}(z_0) \\ &+ \frac{2h^7}{21(7!)}[3-7(\alpha_1^4+\alpha_2^4+\alpha_1^2\alpha_2^2)+21\alpha_1^2\alpha_2^2(\alpha_1^2+\alpha_2^2)]g^{(7)}(z_0) + \dots \end{aligned} \quad (25)$$

Hence,

$$|E_{Q_g}(\alpha_1,\alpha_2)| \approx \frac{2|h|^5}{15(5!)}|(15\alpha_1^2-5)\alpha_2^2+3-5\alpha_1^2||g^{(5)}(z_0)|.$$

for asymptotically small h. This establishes the theorem.

2.6 Rules of Precision Six

Now, by choosing suitable values of the parameters α_1 and α_2; we derive here some specific rules of precision six as special cases of (20). For instance, by taking $\alpha_1 = 1$ in (25) and then equating the coefficient of $g^{(5)}(z_0)$ to zero, we obtain the value of $\alpha_2 = \frac{1}{\sqrt{5}}$. Now, with these values of α_1 and α_2 the coefficients of the four point two parametric rule $Q_g(\alpha_1,\alpha_2)$ are obtained as,

$$k_1 = \frac{1}{6} \text{ and } k_2 = \frac{5}{6}.$$

Thus, the rule $Q_g(\alpha_1,\alpha_2)$ boils down to

$$Q_1(g) = \frac{1}{6}\left[Q_g(1) + 5Q_g\left(\frac{1}{\sqrt{5}}\right)\right] \quad (26)$$

which is the desired four point rule of precision six, meant for the numerical integration of complex CPV integral of the type (1) in the complex plane $\mathbb{C}$.

Now, without repeating the technique that we adopted in the formulation of the four point degree six rule $Q_1(g)$ here, we simply state few more rules belonging to this class of rules as,

$$Q_2(g) = \frac{1}{129}\left[49Q_g\left(\sqrt{\frac{5}{7}}\right) + 80Q_g\left(\frac{1}{\sqrt{10}}\right)\right] \quad (27)$$

and,

$$Q_3(g) = \frac{1}{14}\left[5Q_g\left(\sqrt{\frac{11}{15}}\right) + 9Q_g\left(\frac{1}{3}\right)\right] \tag{28}$$

The first leading term of the error expressions corresponding to each of the four point rules $Q_1(g), Q_2(g)$ and $Q_3(g)$ is obtained as,

(i) $E_{Q_1}(g) \approx 0.061\frac{|h|^7}{(7!)}|g^{(7)}(z_0)|$;

(ii) $E_{Q_2}(g) \approx 0.008\frac{|h|^7}{(7!)}|g^{(7)}(z_0)|$;

and,

(iii) $E_{Q_3}(g) \approx 0.002\frac{|h|^7}{(7!)}|g^{(7)}(z_0)|$.

The above expressions reveal that all the four point rules $Q_1(g), Q_2(g)$and $Q_3(g)$ are of degree of precision six. From their asymptotic error estimates, it is evident that the rule $Q_3(g)$ will provide more accurate result among them. The result of the numerical integration of the integrals shown in Table 3 to Table 9 clearly illustrates this fact.

2.7 Error Bounds of Four Point Degree Six Rules

The error bounds of the four point degree six quadrature rules $Q_1(g)$, $Q_2(g)$ and $Q_3(g)$ constructed in this section are obtained by following the technique by Lether (1971). Since the derivation of error bound is similar in each of the three rules, we have derived the error bound of the rule $Q_1(g)$ only in Theorem 5. Also, the error bounds of other two rules $Q_2(g)$ and $Q_3(g)$ are only stated in the same Theorem 5 without their detailed derivations, as their derivations will be merely a repetition.

Theorem 5. *If $g(z)$ is analytic in an open disc*
$D = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|;\ r > 1\}$.
then,
$(i)\ |E_{Q_1}(g)| \leq 2Me_{Q_1}(r)$;
$(ii)\ |E_{Q_2}(g)| \leq 2Me_{Q_2}(r)$;
$(iii)\ \left|E_{Q_3}(g)\right| \leq 2Me_{Q_3}(r)$.

where,

$$M = \underset{|z|=\rho}{Max}|g(z)|;$$

$$e_{Q_1}(r) = \left|ln\left(\frac{r+1}{r-1}\right) - \left\{\frac{30r^3-26r}{15r^4-18r^2+3}\right\}\right|;$$

$$e_{Q_2}(r) = \left|ln\left(\frac{r+1}{r-1}\right) - \left\{\frac{18060r^3-8686r}{9030r^4-7353r^2+645}\right\}\right|$$

and,

$$e_{Q_3}(r) = \left|ln\left(\frac{r+1}{r-1}\right) - \left\{\frac{1890r^3-966r}{945r^4-798r^2+77}\right\}\right|.$$

Each of which $\rightarrow 0$ as $r \rightarrow \infty$. The quantities $e_{Q_k}(r);\ k = 1, 2\ and\ 3$ are called error constants by Lether (1971).

Proof. Let, $E_{Q_1}(g)$ represents the truncation error of the degree six rule $Q_1(g)$. Then,

$$E_{Q_1}(g) = \sum_{\mu=3}^{\infty} c_{2\mu+1} h^{2\mu+1} E_{Q_1}[(z-z_0)^{2\mu+1})].$$

By the help of Taylor's theorem, where c_k is the Taylor's coefficient. Now, putting $z = z_0 + ht;\ t \in [-1,1]$, we get,

$$E_{Q_1}(g) = \sum_{\mu=3}^{\infty} c_{2\mu+1} h^{2\mu+1} E_{Q_1}(t^{2\mu+1}) \quad (29)$$

and subsequently it implies,

$$E_{Q_1}(g) = \sum_{\mu=3}^{\infty} 2c_{2\mu+1} h^{2\mu+1} \chi_{Q_1}(\mu) \quad (30)$$

where,

$$\chi_{Q_1}(\mu) \quad = \frac{1}{2\mu+1} - \frac{1}{6} - \left(\frac{5}{6}\right)\left(\frac{1}{5^\mu}\right) < \frac{1}{2\mu+1} - \frac{1}{6} < 0;\ for\ \mu > 3.$$

However, for $\mu = 3$;

$$\chi_{Q_1}(3) = \frac{-1}{42} - \frac{1}{150} < 0.$$

Therefore, using Cauchy-inequality we obtain,

$$\left|E_{Q_1}(g)\right| \leq 2M \sum_{\mu=3}^{\infty} \frac{1}{r^{2\mu+1}} \left|E_{Q_1}(t^{2\mu+1})\right| = 2M e_{Q_1}(r) \quad (31)$$

where,

$$e_{Q_1}(r) = \left|E_{Q_1}\left[\left(1-\frac{t}{r}\right)^{-1}\right]\right| = \left|ln\left(\frac{r+1}{r-1}\right) - \left\{\frac{30r^3-26r}{15r^4-18r^2+3}\right\}\right| \quad (32)$$

Therefore, from Equations (31) and (32) we obtain,

$$\left|E_{Q_1}(g)\right| \leq 2M e_{Q_1}(r).$$

This completes the proof of part (i) of Theorem 5. The rest part of the theorem can be established in the same way as it is done in case of Part (i). Hence, this proves the theorem.

The numerical evaluation of the error constants $e_{Q_n}(r);\ n = 1,2,3$ for $r > 1$ are shown in Table 1. It is established from the table that the numerical evaluation of the integral (1) can be done more accurately by the rule $Q_3(g)$ than the other rules in its class constructed in this article. The numerical values of the error constants $e_{Q_n}(r);\ n = 1,2,3$ provided in Table 1 and the graphs drawn in Figure 1, clearly support this claim.

Table 1. Error constants of rules Q_1, Q_2, Q_3.

	Rules with precision six		
r	$e_{Q_1}(r)$	$e_{Q_2}(r)$	$e_{Q_3}(r)$
1.1	0.5166908264	0.1296190617	0.1091564892
2.1	0.0005405799	0.0000891442	0.0000800958
3.1	0.0000271392	0.0000038743	0.0000015442
4.1	0.0000035069	0.0000004736	0.0000001691
5.1	0.0000007305	0.0000000960	0.0000000323
6.1	0.0000002040	0.0000000264	0.0000000085
7.1	0.0000000696	0.0000000089	0.0000000028
8.1	0.0000000274	0.0000000035	0.0000000011
9.1	0.0000000121	0.0000000015	0.0000000005

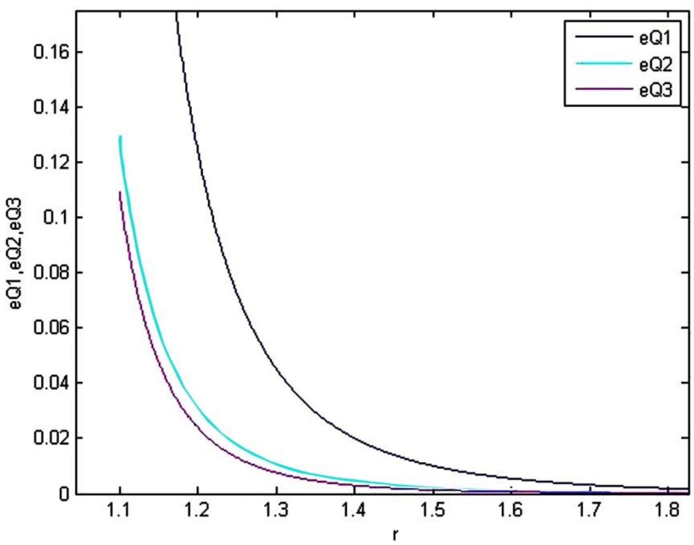

Figure 1. Graphs of error constants $e_{Q_1}(r)$, $e_{Q_2}(r)$ and $e_{Q_3}(r)$ for $r > 1$.

2.8 Rules of Precision Eight

In this subsection some quadrature rules have been formulated from the rules $Q_1(g), Q_2(g)$ and $Q_3(g)$ (given in Equations (26) to (28)) by following the method of extrapolation. Rules derived are of precision eight meant for the approximate evaluation of complex CPV integrals of the type (1). For the construction of rules by extrapolation, we have assumed here that the function $g(z)$ is sufficiently differentiable in,

$$D = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|; r > 1\}.$$

Further, since $E_{Q_n}(g)$ are the truncation errors associated with the quadrature rules $Q_n(g)$; i.e.,

$$E_{Q_n}(g) = I(g) - Q_n(g);\ for\ n = 1,2,3 \tag{33}$$

then it is easy to show that,

$$E_{Q_n}(g) = \begin{cases} \frac{2h^7}{7!}\left(\frac{-16}{525}\right) g^{(7)}(z_0) + \frac{2h^9}{9!}\left(\frac{-64}{1125}\right) g^{(9)}(z_0) + \ldots & for\ n = 1 \\ \frac{2h^7}{7!}\left(\frac{2}{525}\right) g^{(7)}(z_0) + \frac{2h^9}{9!}\left(\frac{671}{55125}\right) g^{(9)}(z_0) + \ldots & for\ n = 2 \\ \frac{2h^7}{7!}\left(\frac{16}{14175}\right) g^{(7)}(z_0) + \frac{2h^9}{9!}\left(\frac{704}{91125}\right) g^{(9)}(z_0) + \ldots & for\ n = 3. \end{cases} \tag{34}$$

and,

$$I(g) = Q_1(g) + \frac{2h^7}{7!}\left(\frac{-16}{525}\right) g^{(7)}(z_0) + \frac{2h^9}{9!}\left(\frac{-64}{1125}\right) g^{(9)}(z_0) + \cdots \tag{35}$$

$$I(g) = Q_2(g) + \frac{2h^7}{7!}\left(\frac{2}{525}\right) g^{(7)}(z_0) + \frac{2h^9}{9!}\left(\frac{671}{55125}\right) g^{(9)}(z_0) + \cdots \tag{36}$$

$$I(g) = Q_3(g) + \frac{2h^7}{7!}\left(\frac{16}{14175}\right) g^{(7)}(z_0) + \frac{2h^9}{9!}\left(\frac{704}{91125}\right) g^{(9)}(z_0) + \cdots \tag{37}$$

which are obtained by integrating the Taylor's series expansion of $g(z)$ given in Equation (12) and expansions of function at the nodes of the quadrature rule $Q_g(\alpha_1, \alpha_2)$ given in Equation (24) for,

$(\alpha_1, \alpha_2) = \left(1, \frac{1}{\sqrt{5}}\right); \left(\sqrt{\frac{5}{7}}, \frac{1}{\sqrt{10}}\right)$ and $\left(\sqrt{\frac{11}{15}}, \frac{1}{3}\right)$ respectively.

Now, by suitably combining any two of the three quadrature rules $Q_1(g)$, $Q_2(g)$ and $Q_3(g)$ by the help of Equations (35), (36) and (37) we get,

$$I(g) = \frac{1}{19}[27Q_3(g) - 8Q_2(g)] + \left(\frac{968}{165375}\right)\left(\frac{2h^9}{9!}\right) g^{(9)}(z_0) + \ldots \tag{38}$$

$$I(g) = \frac{1}{28}[Q_1(g) + 27Q_3(g)] + \left(\frac{128}{23625}\right)\left(\frac{2h^9}{9!}\right) g^{(9)}(z_0) + \ldots \tag{39}$$

and

$$I(g) = \frac{1}{9}[Q_1(g) + 8Q_2(g)] + \left(\frac{248}{55125}\right)\left(\frac{2h^9}{9!}\right) g^{(9)}(z_0) + \ldots \tag{40}$$

We denote the first term on the right hand side of the Equation (38) by $Q_{32}(g)$, i.e.,

$$Q_{32}(g) = \frac{1}{19}[27Q_3(g) - 8Q_2(g)] \tag{41}$$

and claim it as the required quadrature rule of precision eight obtained from the rules $Q_2(g)$ and $Q_3(g)$ by extrapolation for numerical integration of complex CPV integrals of type (1). Again, the corresponding truncation error (denoted by $E_{Q_{32}}(g)$) associated with the rule $Q_{32}(g)$ is obtained as,

$$\begin{aligned} E_{Q_{32}}(g) &= I(g) - Q_{32}(g) \\ &= \frac{1}{19}\left[27E_{Q_3}(g) - 8E_{Q_2}(g)\right] \end{aligned} \tag{42}$$

Similarly, two more quadrature rules,

$$Q_{13}(g) = \frac{1}{28}[Q_1(g) + 27Q_3(g)] \tag{43}$$

and,

$$Q_{12}(g) = \frac{1}{9}[Q_1(g) + 8Q_2(g)] \tag{44}$$

Each of precision eight are obtained from rules $Q_1(g)$,$Q_2(g)$. The truncation errors associated with these two rules are given by,

$$\begin{aligned} E_{Q_{13}}(g) &= I(g) - Q_{13}(g) \\ &= \frac{1}{28}\left[E_{Q_1}(g) + 27E_{Q_3}(g)\right] \end{aligned} \tag{45}$$

and,

$$\begin{aligned} E_{Q_{12}}(g) &= I(g) - Q_{12}(g) \\ &= \frac{1}{9}\left[E_{Q_1}(g) + 8E_{Q_2}(g)\right] \end{aligned} \tag{46}$$

respectively, for the numerical computation of complex CPV integrals of type (1). This method may be continued for the formulation of quite a large number of quadrature rules of a higher degree of precision involving more points. Again, it is observed from Equations (41), (43) and (44) that, no additional evaluation of functions at any of the nodes is required when a complex CPV integral (1) is numerically approximated by each of these rules Q_{32} ,Q_{13} and Q_{12}. This is due to the fact that each of these rules are a weighted mean of two rules from the set of three quadrature rules Q_1 ,Q_2 and Q_3. As a result, the numerical approximations of an integral obtained by the application of each of the rules Q_{32} ,Q_{13} and Q_{12} or any rule, belonging to this class of rules are not affected by any type of additional errors like truncation error, round-off error or machine error, usually occuring due to the finite precision of digital computing machine. It is also worth mentioning that all the quadrature rules Q_{32} ,Q_{13} and Q_{12} given in Equations (41), (43) and (44) being of precision eight, approximate the integrals of type (1) more accurately than the rules Q_1 ,Q_2 and Q_3. This fact has been vividly seen in the approximate values obtained by numerical approximations of some standard test integrals whose exact values are otherwise known. Also, these quadrature rules have been successfully applied for the numerical computation of line integrals of analytic functions in the complex plane, real CPV integrals as well as real definite integrals without having any kind of singularities. The results of numerical integrations are depicted in Section-3. It is appropriate to state here that the rules Q_1, Q_2 and Q_3 may be termed as ***Basic rules*** and the rules derived from them (i.e., Q_{32}, Q_{13} and Q_{12}) by extrapolation be termed as ***Composite rules***.

2.9 Error Bound of Degree Eight Rules

We have established the error bounds of the degree eight quadrature rules Q_{32} and Q_{13} in Theorem 6 using the method credited to Lether(1971). However, by following the same procedure the error bound of another degree eight quadrature rule Q_{12} can't be obtained as it is done for both of the rules Q_{32} and Q_{13} for the reasons explained below. Since,

$$I(g) = Q_{12}(g) + E_{Q_{12}}(g) \tag{47}$$

and $E_{Q_{12}}(g)$ is a linear operator thus, by using the transformation,

$$z = z_0 + ht; for\ t \in [-1,1].$$

we obtain

$$E_{Q_{12}}(g) = \sum_{\nu=4}^{\infty} 2a_{2\nu+1}h^{2\nu+1}E_{Q_{12}}(t^{2\nu+1}).$$

which in turn boils down to

$$E_{Q_{12}}(g) = \sum_{\nu=4}^{\infty} 2a_{2\nu+1}h^{2\nu+1}\psi_{Q_{12}}(\nu) \tag{48}$$

where,

$$\psi_{Q_{12}}(\nu) = \frac{1}{2\nu+1} - \frac{1}{54}\left\{1 + \frac{1}{5^{\nu-1}}\right\} - \frac{8}{1161}\left\{49\left(\frac{5}{7}\right)^{\nu} + \frac{8}{10^{\nu-1}}\right\}.$$

is not of one sign for $\nu \geq 4$.

However, the asymptotic error estimate of the rule Q_{12} has been given in Table 1. Next, we determine the error bounds of other two-degree eight quadrature rules Q_{32} and Q_{13}, as given in the following theorem.

Theorem 6. *If $g(z)$ is analytic in a open disc*

$$D = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|;\ r > 1\}.$$

then,

$$\left|E_{Q_{ij}}(g)\right| \leq 2Me_{Q_{ij}}(r);\ ij = 32,13$$

where,

$$M = \underset{|z|=\rho}{Max}|g(z)|.$$

$$e_{Q_{32}}(r) = \left|ln\left(\frac{r+1}{r-1}\right) - \left\{\frac{56700r^7 - 75150r^5 + 27648r^3 - 2006r}{28350r^8 - 47025r^6 + 23829r^4 - 3591r^2 + 165}\right\}\right|.$$

and,

$$e_{Q_{13}}(r) = \left| ln\left(\frac{r+1}{r-1}\right) - \left\{\frac{28350r^7 - 48510r^5 + 23058r^3 - 2866r}{14175r^8 - 28980r^6 + 18354r^4 - 3780r^2 + 231}\right\}\right|.$$

Each of which $\to 0$ as $r \to \infty$. The quantities $e_{Q_{ij}}(r)$, for $ij = 32,13$ are defined as error constants due to Lether (1971).

Proof. Proceeding in the same vein as it is done in the case of the Theorem 5, it can be shown here that,

$$E_{Q_{ij}}(g) = \sum_{\mu=4}^{\infty} 2a_{2\mu+1}h^{2\mu+1}\psi_{ij}(\mu).$$

where for $\mu \geq 4$,

$$\psi_{ij}(\mu) = \begin{cases} \frac{1}{2\mu+1} - \frac{135}{266}\left(\frac{11}{15}\right)^{\mu} + \frac{1}{14}\left(\frac{1}{9^{\mu-1}}\right) + \frac{392}{2451}\left(\frac{5}{7}\right)^{\mu} + \frac{3920}{16641}\left(\frac{1}{10}\right)^{\mu} \geq 0; & for \quad ij = 32 \\ \frac{1}{2\mu+1} - \frac{1}{168}\left\{1 + \left(\frac{1}{5^{\mu-1}}\right)\right\} - \frac{27}{392}\left\{5\left(\frac{11}{15}\right)^{\mu} + \left(\frac{1}{9^{\mu-1}}\right)\right\} \geq 0; & for \quad ij = 13. \end{cases} \tag{49}$$

As a result, $E_{Q_{ij}}(g)$; for $ij = 32,13$ are of one sign. Therefore these error bounds can be obtained in the same way as it is done in Theorem 5; for which we have omitted the proof here.

2.10 Comparative Analysis

Here, we have made a modest attempt to make a comparative analysis as well as to discuss some common features of all the rules (Q_1, Q_2, Q_3, Q_{32}, and Q_{12}) derived in this section. The important characteristics of these rules are described in the following points.

(i) The rules Q_1, Q_2 and Q_3 are four point rules where the number of functional evaluation required in case of rules Q_{32}, Q_{13}, and Q_{12} is eight.
(ii) All these rules referred here have their nodes lying on the line of integration. This preserves a basic characteristic of integration of a function on a line segment in $\mathbb{R}$ or $\mathbb{C}$ analytically.
(iii) All the weights associated with these rules are real numbers. So the rules can be employed for the real defnite integrals.
(iv) The first leading term of the error expressions associated with each of these six rules are given in the following table in order of their increasing accuracy.

Table 2. Leading term of the error expressions of the rules.

Rules	Leading term of the error expressions
Q_1	$-0.061\frac{h^7}{7!}g^{(7)}(z_0)$
Q_2	$0.0076\frac{h^7}{7!}g^{(7)}(z_0)$
Q_3	$0.002\frac{h^7}{7!}g^{(7)}(z_0)$
Q_{32}	$0.0059\frac{h^9}{9!}g^{(9)}(z_0)$
Q_{13}	$0.0054\frac{h^9}{9!}g^{(9)}(z_0)$
Q_{12}	$0.0045\frac{h^9}{9!}g^{(9)}(z_0)$

From the asymptotic error estimates, as tabulated in the second column of Table 2, it is self-explanatory that the rule Q_3 shall provide better results in numerical integration as compared to the rules Q_1 and Q_2. However, the rule Q_{12} shall provide better accuracy among all of these rules considered. This fact is very much observed during the numerical approximation of the integrals considered in Section 3.

3. Numerical Experiments

In this section, we present the result of numerical experiments obtained using the proposed schemes. The Quadrature rules (Q_1, Q_2, Q_3, Q_{32}, Q_{13}, Q_{12}) developed in Section 2 are meant for the numerical evaluation of complex CPV integrals of type (1). However, these rules are also applicable to evaluate other types of integrals.

3.1 Evaluation of Complex CPV Integrals

Here, we consider the following integrals:

$$I_1 = P\int_{-i}^{i} \frac{e^z}{z} dz, \qquad I_2 = P\int_{-i}^{i} \frac{(1+z)e^z}{z} dz,$$

$$I_3 = P\int_{-i}^{i} \frac{(1+z\cos z)}{z} dz, \qquad I_4 = P\int_{\frac{1-i}{4}}^{\frac{-1+i}{4}} \frac{\tan^{-1} z}{z} dz,$$

$$I_5 = P\int_{\frac{1+i}{2}}^{\frac{3(1+i)}{2}} \frac{\sin z}{z-(1+i)} dz.$$

The exact values of these integrals correct to fifteen decimal places, computed by using the data available in Abramowitz and Stegun (1964) are found to be the following:

$$I_1 = 1.892166140734366i, \quad I_2 = 3.575108110350159i,$$
$$I_3 = 2.350402387287603i, \quad I_4 = -0.506613635510659 + 0.492764356203114i,$$
$$I_5 = 1.817558673962320 - 0.205725120888008i.$$

The complex CPV integrals I_1, I_2, I_3, I_4 and I_5 have been evaluated by the rules developed in section 2. The computed values of the five integrals and the absolute errors have been appended in the following five tables.

Table 3. Numerical evaluation of complex principal value integrals.

Rules	Approximate Value of I_1	Abs. Error	Approximate Value of I_2	Abs. Error
$Q_1(g)$	$1.892154356768595i$	1.2×10^{-5}	$3.575014450641384i$	9.4×10^{-5}
$Q_2(g)$	$1.892167586370264i$	1.4×10^{-6}	$3.575119545275700i$	1.1×10^{-5}
$Q_3(g)$	$1.892166546822965i$	4.1×10^{-7}	$3.575111276895897i$	3.2×10^{-6}
$Q_{32}(g)$	$1.892166109118838i$	3.2×10^{-8}	$3.575107795472823i$	3.1×10^{-7}
$Q_{13}(g)$	$1.892166111463880i$	2.9×10^{-8}	$3.575107818815379i$	2.9×10^{-7}
$Q_{12}(g)$	$1.892166116414523i$	2.4×10^{-8}	$3.575107868094110i$	2.4×10^{-7}
Exact Value	$\mathbf{1.892166140734366}i$		$\mathbf{3.575108110350159}i$	

Table 4. Numerical evaluation of complex principal value integrals.

Rules	Approximate Value of I_3	Abs. Error
$Q_1(g)$	$2.350489907519472i$	8.7×10^{-5}
$Q_2(g)$	$2.350391190203891i$	1.1×10^{-5}
$Q_3(g)$	$2.350398860218264i$	3.5×10^{-6}
$Q_{32}(g)$	$2.350402089697999i$	3.0×10^{-7}
$Q_{13}(g)$	$2.350402111907592i$	2.7×10^{-7}
$Q_{12}(g)$	$2.350402158794511i$	2.3×10^{-7}
Exact Value	$\mathbf{2.350402387287603}\boldsymbol{i}$	

Table 5. Numerical evaluation of complex principal value integrals.

Rules	Approx. Value of I_4	Abs. Error
$Q_1(g)$	$-0.506610246316862 + 0.492769262629850i$	6.0×10^{-6}
$Q_2(g)$	$-0.506613978055818 + 0.492763691116544i$	7.5×10^{-7}
$Q_3(g)$	$-0.506613670666537 + 0.492764117647976i$	2.4×10^{-7}
$Q_{32}(g)$	$-0.506613541239472 + 0.492764297240158i$	1.1×10^{-7}
$Q_{13}(g)$	$-0.506613548368334 + 0.492764301397329i$	1.0×10^{-7}
$Q_{12}(g)$	$-0.506613563418156 + 0.492764310173578i$	8.5×10^{-8}
Exact Value	$\mathbf{-0.506613635510659 + 0.492764356203114}\boldsymbol{i}$	

Table 6. Numerical evaluation of complex principal value integrals.

Rules	Approx. Value of I_5	Abs. Error
$Q_1(g)$	$1.817558809095785 - 0.205723744869242i$	1.3×10^{-6}
$Q_2(g)$	$1.817558655483211 - 0.205725292739725i$	1.7×10^{-7}
$Q_3(g)$	$1.817558667195166 - 0.205725171683909i$	5.1×10^{-8}
$Q_{32}(g)$	$1.817558672126516 - 0.205725120713039i$	1.8×10^{-9}
$Q_{13}(g)$	$1.817558672263045 - 0.205725120726242i$	1.7×10^{-9}
$Q_{12}(g)$	$1.817558672551274 - 0.205725120754116i$	1.4×10^{-9}
Exact Value	$\mathbf{1.817558673962320 - 0.205725120888008}\boldsymbol{i}$	

We have evaluated the integrals using a series of quadrature rules: Q_1, Q_2, Q_3, Q_{32}, Q_{13}, Q_{12}. The numerical results in the tables show that the values produced by applying the series of increasingly precise rules, converge to a value that is equal to the exact value of the corresponding integrals, which is accurate up to eight decimal point.

3.2 Evaluation of Line Integrals of Analytic Functions

In this subsection, we have considered the following line integrals.

$$I_6 = P\int_{-i}^{i} e^z dz,\ I_1 = P\int_{-i/2}^{i/2} \cos z\ dz.$$

Table 7. Numerical evaluation of complex line integrals.

Rules	Approx. Value of I_1	Abs. Error	Approx. Value of I_2	Abs. Error
$Q_1(g)$	1.682860093872789i	8.2×10^{-5}	$1.042191277895125i$	6.7×10^{-7}
$Q_2(g)$	1.682951958905436i	10.0×10^{-6}	$1.042190527130796i$	8.4×10^{-8}
$Q_3(g)$	1.682944730072933i	2.8×10^{-6}	$1.042190585739540i$	2.5×10^{-8}
$Q_{32}(g)$	$1.682941686353984i$	2.8×10^{-7}	$1.042190610416907i$	5.7×10^{-10}
$Q_{13}(g)$	$1.682941707351499i$	2.6×10^{-7}	$1.042190610459383i$	5.2×10^{-10}
$Q_{12}(g)$	1.682941751679586i	2.2×10^{-8}	$1.042190610549055i$	4.4×10^{-10}
Exact Value	**1.682941969615793i**		$\mathbf{1.042190610987495}\boldsymbol{i}$	

3.3 Evaluation of Real CPV Integrals

The following real CPV integrals are numerically approximated in this subsection. I8 = P R 1 −1 1+xcosx x dx, I9 = P R 3/2 1/2 sinx x−1 dx. The approximate values with their absolute errors are reflected in Table 8.

$$I_8 = P\int_{-1}^{1}\frac{1+x\cos x}{x}dx,\ \ I_9 = P\int_{1/2}^{3/2}\frac{\sin x}{x-1}dx.$$

Table 8. Numerical evaluation of real principal value integrals.

Rules	Approx. Value of I_8	Abs. Error	Approx. Value of I_9	Abs. Error
$Q_1(g)$	1.682860093872789	8.2×10^{-5}	0.532854099299606	5.1×10^{-8}
$Q_2(g)$	1.682951958905436i	10.0×10^{-6}	0.532854156329300	6.3×10^{-9}
$Q_3(g)$	1.682944730072933	2.8×10^{-6}	0.532854151864677	1.8×10^{-9}
$Q_{32}(g)$	1.682941686353984	2.8×10^{-7}	0.532854149984835	3.4×10^{-11}
$Q_{13}(g)$	1.682941707351499	2.6×10^{-7}	0.532854149987353	3.1×10^{-11}
$Q_{12}(g)$	1.682941751679586	2.2×10^{-8}	0.532854149992668	2.6×10^{-11}
Exact Value	**1.682941969615793**		**1.042190610987495*i***	

3.4 Evaluation of Real Definite Integrals

The following real definite integrals are numerically approximated in this subsection.

$$I_{10} = \int_{-1}^{1} e^x dx,\ \ I_{11} = \int_{-1/2}^{1/2}\cos x dx.$$

Table 9. Numerical evaluation of real definite integrals.

Rules	Approx. Value of I_{10}	Abs Error	Approx. Value of I_{11}	Abs. Error
$Q_1(g)$	2.350489907519472	8.7×10^{-5}	0.958850421323795	6.6×10^{-7}
$Q_2(g)$	2.350391190203891	1.1×10^{-5}	0.958851158706536	8.1×10^{-8}
$Q_3(g)$	2.350398860218264	3.5×10^{-6}	0.958851100959398	2.3×10^{-8}
$Q_{32}(g)$	2.350402089697999	3.0×10^{-7}	0.958851076644813	5.6×10^{-10}
$Q_{13}(g)$	2.350402111907592	2.7×10^{-7}	0.958851076686698	5.2×10^{-10}
$Q_{12}(g)$	2.350402158794511	2.3×10^{-7}	0. 958851076775121	4.3×10^{-10}
Exact Value	**2.350402387287603**		**0.958851077208406**	

4. Conclusion

The present study aimed to investigate a quadrature rule to obtain a numerical approximation of CPV integrals in the complex plane. Furthermore, we demonstrated the presence of such rules theoretically and experimentally. We also obtained the asymptotic error estimate and the error bound for each rule. The numerical results in the tables in Section 3 show that, the values produced by applying a series of increasingly precise rules, converge to a value that is equal to the exact value of the corresponding integrals, which is accurate up to eight to eleven figures after the decimal point. So, when a series of such rules of increasing accuracy are applied to an integral for its numerical evaluation, one can confidently accept the value of the integral (whose value is not feasible to acquire analytically) obtained corresponding to the quadrature rule of highest precision. Once more, these rules are also applicable to the numerical integration of:

(i) complex definite integrals on the line segment having end points $z_0 - h$ and $z_0 + h$.
(ii) Real CPV and real definite integrals.

In the future, we want to extend these rules to evaluate complex CPV integrals with oscillatory kernels and also to compute multidimensional complex integrals.

Conflict of Interest
The authors confirm that there is no conflict of interest to declare for this publication.

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