International Journal of Mathematical, Engineering and Management Sciences
Vol. 10, No. 5, 1447-1462, 2025
https://doi.org/10.33889/IJMEMS.2025.10.5.068

Ram Arti Publishers

Non-Polynomial Spline for Singularly Perturbed Differential-Difference Equation with Mixed Shifts and Layer Behavior

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(Received on November 30, 2024; Revised February 5, 2025 & March 7, 2025; Accepted on April 5, 2025)

Abstract
Various physical phenomena give rise to singularly perturbed differential equations with mixed shifts. Due to multiple parameters, singularly perturbed mixed delay boundary value problems are challenging to solve. This article considers a singularly perturbed differential-difference equation with delay and advance. To deal with the complexity of these equations, a non-polynomial spline numerical approach is adopted. For discretization, we have used a uniform mesh with equal spacing. Various theoretical results like stability and convergence are discussed. Numerical examples are solved to support the method and check the validity of the findings. The numerical order of convergence is determined and presented in the tables, along with the comparison of the results with the other existing methods. The comparison shows that the error is significantly less than the available solution in the literature. Also, the numerical order of convergence is determined to be equal to two. Graphs are drawn to observe the behavior of the solution for different values of parameters.

Keywords- Singular perturbation, Differential-difference equations, Boundary value problems, Mixed delay, Non-polynomial spline, Boundary layer.

1. Introduction

Singularly perturbed differential equations (SPDEs) involve a small parameter with the highest order derivative, and the singularly perturbed differential-difference equations (SPDDE) are the SPDEs with mixed shifts. SPDDE are complex boundary value problems (BVP) with a minimum of three parameters posing significant challenges in finding the solution. SPDDEs are prevalent in the mathematical modeling of many biological, engineering, and physical phenomena (Cahlon and Schmidt, 2005; Derstine et al., 1982; Segundo et al., 1968; Tuckwell, 1976, 1988; Tuckwell and Wan, 2005). Naidu (2002) presented an exhaustive review of the singular perturbation and time scale problems arising in control theory. Chi et al. (1986) have discussed a non-linear first-order SPDDE. The equation arises in the myelinated nerve axon. The potential change is observed from node to node as myelin insulates the membrane entirely. Stein (1967) was the first to represent the stochastic effect due to neuronal excitation as a differential-difference equation, specifically an SPDDE with mixed shifts of the following type:

$$\frac{\sigma^2}{2}y'' + (\mu - \upsilon)y'(\upsilon) + \lambda_E y(\upsilon + a_E) + \lambda_I y(\upsilon - a_I) - (\lambda_E + \lambda_I)y(\upsilon) = -1,$$
$$y(\upsilon) \equiv 0, \upsilon \in (\upsilon_1, \upsilon_2),$$

where, y is the expected first-exit time, σ and μ are variance, and the drift parameters, λ_E, and λ_I are mean rates of are excitatory and inhibitory synaptic inputs, modelled as Poisson process respectively, and a_E, a_I are small quantities that could depend on voltage. The occurrence of SPDDE in various areas is the motivation behind this study. Many researchers have used different numerical methods to deal with the complex structure of SPDDE. Traditional approaches often involve a variety of finite difference methods (FDM) on various types of mesh functions. Kadalbajoo and Sharma (2006) have employed exponentially fitted FDM on a uniform mesh. Authors Patidar and Sharma (2006) used non-standard FDM on uniform mesh and established uniform convergence. Duressa and Reddy (2015) have used the domain decomposition method. Many researchers have used polynomial based spline methods for SPDEs and SPDDEs. Ranjan (2023) applied the cubic spline method for SPDDE with a delay in the convection term with a fitting factor on the highest-order derivative. A quadratic B-spline method on the exponentially graded mesh is designed for fourth-order SPDE (Singh and Kumar, 2022). Mane and Lodhi (2023) solved a second-order SPDE with a discontinuous source term using the cubic B-spline technique. Lodhi and Mishra (2018) have approximated fourth-order SPDE via the quantic B-spline method. An FDM with two different non-uniform meshes is used for an SPDDE with large shifts (Elango and Unyong, 2022). A computational method based on FDM was used by Kiltu et al. (2021) SPDDE of reaction-diffusion type.

In recent years, non-polynomial spline methods have emerged as promising alternatives due to their ability to provide more flexible and accurate approximations. Non-polynomial splines, also known as non-polynomial interpolates, are the piecewise functions defined over an interval. A variety of other functions are used in place of linear, quadratic, cubic, etc. polynomials, such as exponential, trigonometric, or rational functions. Hammad et al. (2022) have used ten such splines to approximate Fredholm integral equations. Debela and Duressa (2022) have used a non-polynomial spline with a fitting factor for an SPDDE with an integral boundary condition. They have also discussed the convergence and stability of the method. Rashidinia et al. (2008) have tackled second-order BVP using a non-polynomial cubic spline. The proposed method is illustrated by applying it to two numerical examples. Wakjira and Duressa (2020) used exponential spline for third-order SPDE; the method was proved to be sixth-order convergent. The trigonometric B-spline method for SPDE with a delay term is considered by Vaid and Arora (2019) and Ali et al. (2018).

It is observed that the numerical methods applied so far to handle the SPDDEs with mixed shifts are often very complex or need complex mesh structures (Kadalbajoo and Sharma, 2005; Mushahary et al., 2020). To overcome this limitation, a non-polynomial spline method with uniform mesh is proposed for SPDDE with mixed shifts. The advantage of applying the spline methods is that it gives the solution at every point of the domain. Also, the non-polynomial splines are easy to apply and computationally simple.

This research paper aims to investigate the effectiveness of non-polynomial spline methods in solving SPDDEs with mixed delay. The objective is to get a better approximate solution with an improved order of convergence. The article starts with a statement of the problem in Section 2, and the method is described in Section 3, along with convergence analysis and stability. The method is illustrated by applying it to three numerical examples in Section 4. Results and findings are given in the last conclusion section.

2. Problem Statement

Consider the following SPDDE with mixed shifts:

$$\varepsilon w_\varepsilon''(t) + a(t)w_\varepsilon'(t) + k_1(t)w_\varepsilon(t-\delta) + b(t)w_\varepsilon(t) + k_2(t)w_\varepsilon(t+\eta) = v(t), t \in (0,1) \tag{1}$$

with the boundary conditions

$w_\varepsilon(t) = \phi(t), -\delta \leq t \leq 0$ and $w_\varepsilon(t) = \psi(t), 1 \leq t \leq \eta$ (2)

where, ε is perturbation parameter, $0 < \varepsilon << 1$, $a(t)$, $b(t)$, $k_1(t)$, $k_2(t)$ and $v(t)$ are smooth over $(0, 1)$, δ and η are delay and advance parameters respectively. If δ and η tends to zero, then Equation (1) is an SPDE. The presence of ε causes layer behaviour in the solution of this BVP. The boundary region depends on whether coefficient of the convection term, i.e. $a(t)$ is positive or negative over (0,1). Using Taylor's series approximation in delay and advance terms in Equation (1), we get

$\varepsilon w_\varepsilon{}''(t) + p(t)w_\varepsilon{}'(t) + q(t)w_\varepsilon(t) = v(t)$ (3)

subject to the conditions

$$\begin{cases} w_\varepsilon(0) = \phi(0), \\ w_\varepsilon(1) = \psi(1) \end{cases} \quad (4)$$

where, $p(t) = a(t) - \delta k_1(t) + \eta k_2(t),\ q(t) = \lambda(t) + b(t) + k_2(t)$. The solution of the boundary value problem given by Equations (3)-(4) differ from the solution of Equations (1)-(2) by $O(\delta^2)$ and $O(\eta^2)$. As $0 < \delta << 1$, and $0 < \eta << 1$, solution of Equations (3)-(4) provides a good approximate solution to original Equations (1)-(2). Define the differential operator $L_{\varepsilon,\delta,\eta}$ corresponding to the Equations (3)-(4).

$$L_{\varepsilon,\delta,\eta} w_\varepsilon(t) \equiv \varepsilon w_\varepsilon{}''(t) + p(t)w_\varepsilon{}'(t) + q(t)w_\varepsilon(t) = v(t).$$

2.1 Properties of Continuous Problem

Continuous minimum principle: Let $\varphi(t)$ be a smooth function satisfying $\varphi(0) \geq 0,\ \varphi(1) \geq 0$ and $L_{\varepsilon,\delta,\eta} \leq \varphi(t),\ \forall t \in [0,1]$. Then $\varphi(t) \geq 0,\ \forall t \in [0,1]$.

Lemma 1: The solution of BVP given by Equations (3)-(4) is bounded and it is given as follows:

$$\|w_\varepsilon\| \leq \theta^{-1}\|f\| + max(|\phi|, \psi),$$

where, $\|\cdot\|$ is defined by $\|w_\varepsilon\| = \max\limits_{0 \leq t \leq 1} |w_\varepsilon|$.

Theorem 1: The derivative of the solution of BVP given by Equations (3)-(4) satisfies the following inequalities:

$$\left\|w_\varepsilon{}^{(k)}\right\| \leq C\varepsilon^{-1}, k = 1,2,3,$$

where the constant $C > 0$ does not depend on ε .

Refer to Kadalbajoo and Sharma (2005), and Miller et al. (2012) for the proofs.

3. Method Description

This section describes the non-polynomial spline method, which interpolates the SPDDE with mixed shifts defined by Equations (3)-(4). Let π be the partition of the interval $(0, 1)$ obtained by dividing it into N equal parts $[t_i, t_{i+1}]$ each of width h, where $h = \frac{1}{N},\ t_0 = 0$ and $t_N = 1$. On each sub-interval of π define the non-polynomial spline function $S_N(t)$ as

$S_N(t) = \gamma_i + \lambda_i(t - t_i) + \mu_i \sin\tau(t - t_i) + \sigma_i \cos\tau(t - t_i)$ (5)

where, τ is a free parameter, $\gamma,\ \lambda,\ \mu$, and σ are constants. The non-polynomial spline $S_N(t)$ defined by Equation (5) interpolates $w_\varepsilon(t)$ at the nodal points, and is a class of functions from $C^2(0,1)$. Further, it is given by $span\{1, t, \cos\tau t, \sin\tau t\}$, and as $\tau \to 0$, it reduces to the cubic spline given by $span\{1,\ t,\ t^2, t^3\}$. This fact is visible from the relation

$$T = span\{1, t, cos\,\tau\, t, sin\,\tau\, t\} = span\left\{1, t, \frac{2}{\tau^2}(1 - cos\,\tau\, t), \frac{6}{\tau^3}(\tau t - sin\,\tau\, t)\right\}.$$

Thus $\lim_{\tau \to 0} T = \{1,\ t,\ t^2, t^3\}$. Let $w_\varepsilon(t_i)$ be the approximate solution of Equations (3)-(4) obtained by using non-polynomial spline function $S_N(t)$ on $[t_i, t_{i+1}]$ passing through the points $(t_i, w_\varepsilon(t_i))$ and $(t_{i+1}, w_\varepsilon(t_{i+1}))$. To determine the spline coefficients in Equation (5), we assume that $S_N(t)$ satisfies interpolation conditions at $t = t_i$, and $t = t_{i+1}$. We also assume that the first-order derivative is continuous at the common nodal points. We are using the following notations for the determination of constants in Equation (5).

$$S_N(t_i) = w_{\varepsilon i}, S_N(t_{i+1}) = w_{\varepsilon i+1}$$
$$S_N''(t_i) = M_i, S_N''(t_{i+1}) = M_{i+1} \quad (6)$$

Differentiating Equation (5), we obtain

$$S_N'(t) = \lambda_i + \mu_i \tau\, cos\,\tau\,(t - t_i) - \sigma_i \tau\, sin\,\tau\,(t - t_i) \quad (7)$$

$$S_N''(t) = -\mu_i \tau^2\, sin\,\tau\,(t - t_i) - \sigma_i \tau^2\, cos\,\tau\,(t - t_i) \quad (8)$$

Putting $t = t_i$ in Equation (8) and using Equation (6), we obtain

$$\sigma_i = -\frac{M_i}{\tau^2}, i = 0,1, \dots, N-1 \quad (9)$$

Putting $t = t_i$ in Equation (5) and using Equation (6), we obtain

$$\gamma_i = w_{\varepsilon i} + \frac{M_i}{\tau^2}, i = 0,1, \dots, N-1 \quad (10)$$

Putting $t = t_{i+1}$ in Equation (8) and using Equations (9) and (10), we have

$$\lambda_i = \frac{w_{\varepsilon i+1} - w_{\varepsilon i}}{h} + \frac{M_{i+1} - M_i}{\tau\theta}, i = 0,1, \dots, N-1 \quad (11)$$

and

$$\mu_i = \frac{M_i\, cos\,\theta - M_{i+1}}{\tau^2\, sin\,\theta}, i = 0,1, \dots, N-1 \quad (12)$$

where, $\theta = \tau h$.

Using the assumption that the $S_N'(t)$ is continuous at each nodal point, i.e., $S_{N_{i-1}}'(t_i) = S_{N_i}'(t_i)$, we have

$$\lambda_{i-1} + \mu_{i-1}\tau\, cos\,\theta - \sigma_{i-1}\tau\, sin\,\theta = \lambda_i + \mu_i \tau \quad (13)$$

Substituting the values of coefficients from Equations (9)-(12) in Equation (13), we get

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{1}{h^2}(w_{\varepsilon i+1} - 2w_{\varepsilon i} + w_{\varepsilon i-1}), i = 0,1, \dots, N-1 \quad (14)$$

where, $\alpha = \frac{1}{\theta^2}(\theta\, csc\,\theta - 1), \beta = \frac{1}{\theta^2}(1 - \theta\, cot\,\theta)$.

As $h \to 0, \theta = \tau h \to 0$. A simple application of L'Hospital's Rule gives us

$$\lim_{\theta \to 0} \alpha = \frac{1}{6} \text{ and } \lim_{\theta \to 0} \beta = \frac{1}{3}.$$

Using notations given by Equation (6) in Equation (3), we get

$$\varepsilon M_i + p_i w_{\varepsilon i}{}' + q_i w_{\varepsilon i} = v_i \quad (15)$$

where, $p_i = p(t_i),\ q_i(t_i),\ v_i = v(t_i)$.

From Jain (1984), w'_ε can be approximated as

$$\begin{cases} w'_{\varepsilon i} = \frac{w_{\varepsilon i+1} - w_{\varepsilon i-1}}{2h}, \\ w'_{\varepsilon i+1} = \frac{3w_{\varepsilon i+1} - 4w_{\varepsilon i} + w_{\varepsilon i-1}}{2h}, \\ w'_{\varepsilon i-1} = \frac{-w_{\varepsilon i+1} + 4w_{\varepsilon i} - 3w_{\varepsilon i-1}}{2h} \end{cases} \tag{16}$$

From Equation (15), we have

$$\begin{cases} M_i = v_i - p_i w'_{\varepsilon i} - q_i w_{\varepsilon i}, \\ M_{i-1} = v_{i-1} - p_{i-1} w'_{\varepsilon i-1} - q_{i-1} w_{\varepsilon i-1}, \\ M_{i+1} = v_{i+1} - p_{i+1} w'_{\varepsilon i+1} - q_{i+1} w_{\varepsilon i+1} \end{cases} \tag{17}$$

Using the values from Equations (16)-(17) in Equation (14), we obtain by rearranging the terms

$$w_{\varepsilon i-1}\left(\frac{3}{2\varepsilon}\alpha h p_{i-1} + \frac{1}{\varepsilon}\beta h p_i - \frac{1}{2\varepsilon}\alpha h p_{i+1} - \frac{1}{\varepsilon}\alpha h^2 q_{i-1} - 1\right) + w_{\varepsilon i}\left(-\frac{2}{\varepsilon}\alpha h p_{i-1} + \frac{2}{\varepsilon}\alpha h p_{i+1} - \frac{1}{\varepsilon}\beta h^2 q_i + 2\right) + w_{\varepsilon i+1}\left(\frac{1}{2\varepsilon}\alpha h p_{i-1} - \frac{1}{\varepsilon}\beta h p_i - \frac{3}{2\varepsilon}\alpha h p_{i+1} - \frac{1}{\varepsilon}\alpha h^2 q_{i+1} - 1\right) = -h^2\left(\frac{\alpha}{\varepsilon}v_{i-1} + \frac{2\beta}{\varepsilon}v_i + \frac{\alpha}{\varepsilon}v_{i+1}\right) \tag{18}$$

3.1 Stability

In this subsection, we have proved the following theorem to show the stability of the proposed method.

Theorem 2: For sufficiently small values of *h*, the tri-diagonal coefficient matrix *V* in the system given by Equations (18) is irreducible and diagonally dominant matrix; hence the proposed method is stable.

Proof: Let *V* denotes the tri-diagonal coefficient matrix in the system given by Equations (18), then

$$V(i,j) = \begin{cases} E_i^*, i - j = 1, \\ F_i^*, i = j, \\ G_i^*, j - i = 1, for i, j = 2,3, \dots, N-2, \end{cases}$$

where,

$$\begin{cases} E_i^* = \frac{3}{2}\alpha h p_{i-1} + \beta h p_i - \frac{1}{2}\alpha h p_{i+1} - \alpha h^2 q_{i-1} - \varepsilon, \\ F_i^* = -2\alpha h p_{i-1} + 2\alpha h p_{i+1} - \beta h^2 q_i + 2\varepsilon, \\ G_i^* = \frac{1}{2}\alpha h p_{i-1} - \beta h p_i - \frac{3}{2}\alpha h p_{i+1} - \alpha h^2 q_{i+1} - \varepsilon, \ for\ i = 2,\ 3,\ \dots, N-2. \end{cases}$$

Thus, for sufficiently small *h*, we have

$$\begin{aligned} \left|E_i^* + G_i^*\right| &= \left|2\alpha h p_{i-1} - 2\alpha h p_{i+1} - 2\varepsilon\right| \\ &= \left|2\alpha h\left(p_{i-1} - p_{i+1}\right) - 2\varepsilon\right| \\ &\le \left|F_i^*\right|,\ for\ i,\ j = 2,\ 3,\ \dots, N-2. \end{aligned}$$

Eliminate $w_{\varepsilon 0}$ and $w_{\varepsilon N}$ from Equation (18) using the boundary conditions given in Equations (4). Thus the first row of V is given as $[F_1^*\ G_1^*\ 0\ 0\ \cdots\ 0]$, where $F_1^* = -2\alpha h p_0 + 2\alpha h p_2 - \beta h^2 q_0 + 2\varepsilon$ and $G_1^* =$

$\frac{1}{2}\alpha h p_0 - \beta h p_1 - \frac{3}{2}\alpha h p_2 - \alpha h^2 q_2 - \varepsilon$. Further the last row of V is given as $[0 \ \cdots 0 \ \ E_{N-1}^* \ \ F_{N-1}^*]$, where $E_{N-1}^* = \frac{3}{2}\alpha h p_{N-2} + \beta h p_{N-1} - \frac{1}{2}\alpha h p_N - \alpha h^2 q_{N-2} - \varepsilon$ and $F_{N-1}^* = -2\alpha h p_{N-2} + 2\alpha h p_N - \beta h^2 q_{N-1} + 2\varepsilon$. Clearly, for small values of h, $|G_1^*| \le |F_1^*|$ and $|E_{N-1}^*| \le |F_{N-1}^*|$.

Hence V is diagonally dominant. Further even for $h \to 0, E_i{}^* \neq 0$, and $G_i{}^* \neq 0, i = 1,2,\dots,N-1$. Thus, V is an irreducible matrix. Hence proved.

3.2 Convergence Analysis

In this subsection, convergence analysis of the proposed method is described.

Writing Equation (18) in the matrix form as:

$$Aw_\varepsilon + h^2 BF = U \tag{19}$$

where, A is a diagonally dominant tri-diagonal matrix given by $A = P + hQR$, where $P = P(i,j)$ is given by

$$P(i,j) = \begin{cases} 2, i = j = 1,2,\dots,N-1 \\ -1, |i-j| = 1 \\ 0, \text{otherwise}, \end{cases}$$

$$QR = Z(i,j) = \begin{cases} \frac{2\alpha}{\varepsilon}(p_2 - p_0) - \frac{2\beta}{\varepsilon} h q_1, i = j = 1 \\ \frac{3}{2\varepsilon}\alpha p_{i-1} + \frac{1}{\varepsilon}\beta p_i - \frac{1}{2\varepsilon}\alpha p_{i+1} - \frac{1}{\varepsilon}\alpha h q_{i-1}, i > j \\ \frac{2}{\varepsilon}\alpha(p_{i+1} - p_{i-1}) - \frac{2}{\varepsilon}\beta h q_i, i = j \\ \frac{1}{2\varepsilon}\alpha p_{i-1} - \frac{1}{\varepsilon}\beta p_i - \frac{3}{2\varepsilon}\alpha p_{i+1} - \frac{1}{\varepsilon}\alpha h q_{i+1}, i < j \\ \frac{2\alpha}{\varepsilon}(p_N - p_{N-1}) - \frac{2\beta}{\varepsilon} h q_{N-1}, i = j = N-1, \end{cases}$$

$F = (v_1, v_2, \dots, v_{N-1})^T, w_\varepsilon = (w_{\varepsilon 1}, w_{\varepsilon 2}, \dots, w_{\varepsilon N-1})^T$,

$$B = \begin{bmatrix} 2\beta & \alpha & & & & \\ \alpha & 2\beta & \alpha & & & \\ & \alpha & 2\beta & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 2\beta & \alpha \\ & & & & \alpha & 2\beta \end{bmatrix},$$

and $U = (u_1, 0, \dots, 0, u_{N-1})^T$, where $u_1 = -h^2\alpha v_0, u_{N-1} = -h^2 \alpha v_N$.

Let $W = \left(W(t_1), W(t_2), \dots, W(t_{N-1})\right)^T$ be the exact solution of Equations (1)-(2) at nodal points. Using these in Equation (19), we obtain

$$AW + h^2 BF = T(h) + U \tag{20}$$

where, $T(h) = \left(T(t_1), T(t_2), \dots, T(t_{N-1})\right)^T$ is the truncation error obtained by numerical approximation.

Applying Taylor's series in Equation (18) about $t = t_i$, and using Equation (15), truncation error $T(h)$ can be given by

$$T_i(h) = (-1 + 2(\alpha + \beta))\varepsilon h^2 w_\varepsilon''(\zeta_i) + \frac{1}{3}(\alpha p_{i+1} - \beta p_i + \alpha p_{i+1})h^4 w_\varepsilon'''(\zeta_i) + \frac{1}{12}\left(\varepsilon(-12\alpha + 1) + \alpha h(p_{i+1} - p_{i+1})\right)h^4 w_\varepsilon^{(4)}(\zeta_i) + O\left(h^5\right), t_{i-1} < \zeta_i < t_i \tag{21}$$

From Equations (20) and (21), we have

$$A(W - w_\varepsilon) = AE = T(h) \tag{22}$$

with $E = W - w_\varepsilon = (e_1, e_2, \dots, e_{N-1})^T$. We will now prove that $\|E\|$ is bounded.

Lemma 2: If D is a square matrix, and $\|D\| < 1$. Then $(I + D)^{-1}$ exists, and $\left\|(I + D)^{-1}\right\| < \dfrac{1}{1 - \|D\|}$.

Proof: Refer Rashidinia et al. (2008).

From Equations (20) and (22), we get

$E = A^{-1}T = (P + hQR)^{-1}T = (I + hP^{-1}QR)^{-1}P^{-1}T.$

This implies, $\|E\| \le \|(I + hP^{-1}QR)^{-1}\| \|P^{-1}\| \|T\| \le \frac{\|P^{-1}\| \|T\|}{\|(I + hP^{-1}QR)\|}$.

Then

$$\|E\| \le \frac{\|P^{-1}\| \|T\|}{1 - h\|P^{-1}\| \|QR\|} \tag{23}$$

provided $h\|P^{-1}\| \|QR\| \le 1$.

From Henrici (1962) we have

$$\|P^{-1}\| \le \frac{1}{8h^2} \tag{24}$$

this implies

$$\|QR\| \le 8\alpha p + 6\beta q + 2\alpha q \tag{25}$$

where, $p = max|p(t_i)|$ and $q = max|q(t_i)|$, $0 < t_i < 1$. Consider the following two cases, according to the values of α and β.

Case I: If $\alpha + \beta = \frac{1}{2} and\ \alpha \ne \frac{1}{12}$

From Equation (21), we have $\|T\| \le \lambda_1 h^4 W_4$, where $W_4 = \max_{0<\xi<1} \left|w^{(4)}(\xi)\right|$. Using this in Equation (23), we get

$$\|E\| \leq \frac{\lambda_1 h^2 W_4}{1-\omega pq} \equiv O(h^2) \tag{26}$$

Hence, the method is second-order method.

Case II: If $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$

In this case, from Equation (21), we have

$$T_i(h) = \frac{1}{36}(p_{i+1} - 5p_i + p_{i+1})h^4 w_\varepsilon'''(\zeta_i) + \frac{1}{144}\left(h(p_{i+1} - p_{i+1})\right)h^4 w_\varepsilon^{(4)}(\zeta_i) + O(h^5), t_{i-1} < \xi_{i-1} < t_{i+1}.$$

For this choice of α and β, our method is an optimal second-order method.

4. Numerical Results and Discussion

This section illustrates three numerical instances to verify the validity of the proposed method. Maximum absolute error (MAE) E_N is calculated by using the formula:

$$E_N = \max_{0\leq i\leq N} |w_\varepsilon(t_i) - S_N(t_i)|,$$

where, $w_\varepsilon(t_i)$ is the actual solution and $S_N(t_i)$ is the approximate solution obtained by non-polynomial spline method at the nodal points $t_i, i = 0,1, \dots, N$.

The rate of convergence (ROC) is determined using the formula:

$$R = log_2(E_N/E_{2N}).$$

In the first example, the advance term is absent i.e., $k_2(t) = 0$; in the second one, only the advance term is present, i.e., $k_1(t) = 0$, whereas in the third example, both the parameters are present. MATLAB is used for numerical computations.

Example 1: Consider the following singularly perturbed delay differential equation

$$\varepsilon w_\varepsilon'' + w_\varepsilon'(t) + 2w_\varepsilon(t-\delta) - 3w_\varepsilon(t) = 0,$$

with the boundary conditions

$$w_\varepsilon(t) = 1, -\delta \leq t \leq 0, \text{ and } w_\varepsilon(t) = 1.$$

The exact solution is $w_\varepsilon(t) = a_1 \, exp(r_1 t) + a_2 \, exp(r_2 t)$,

where,

$r_1 = \frac{-(1-2\delta)+\sqrt{(1-2\delta)^2+4\varepsilon}}{2\varepsilon}$, $r_2 = \frac{-(1-2\delta)-\sqrt{(1-2\delta)^2+4\varepsilon}}{2\varepsilon}$, $a_1 = \frac{1-exp(r_2)}{exp(r_1)-exp(r_2)}$, and $a_2 = 1 - a_1$.

Table 1 gives MAE, and **Table 2** gives ROC for Example 1, for different α, β, N and δ values. **Table 3** gives the comparison of the values obtained by non-polynomial spline with existing solution in the literature. **Figure 1** provides a comparison of exact and approximate solution for Example 1 with $\varepsilon = 10^{-2}, \delta = 2\varepsilon$ and $N = 50$. It can be seen that the proposed method provides a good approximation to the actual solution than the existing solution in the literature. **Figure 2** represents the behaviour of the solution for different values of δ for Example 1. Very less variation is observed in the solution for different values of δ.

Table 1. MAE for Example 1 with $\varepsilon = 0.1$ and different values of δ, α, and β.

α and β	δ	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
1/4, 1/4	0.01	2.4630E-02	2.1575E-04	2.1564E-06	2.1762E-08
1/6, 1/3		2.2361E-02	1.9714E-04	1.9702E-06	1.9898E-08
1/14, 3/7		1.9800E-02	1.7588E-04	1.7575E-06	1.7455E-08
1/18, 4/9		1.9376E-02	1.7233E-04	1.7220E-06	1.7418E-08
1/24, 11/24		1.9006E-02	1.6923E-04	1.6910E-06	1.7107E-08
1/12, 5/12		2.0118E-02	1.7854E-04	1.7841E-06	1.8037E-08
1/4, 1/4	0.05	2.2036E-02	1.9586E-04	1.9564E-06	1.9444E-08
1/6, 1/3		1.9735E-02	1.7647E-04	1.7629E-06	1.7505E-08
1/14, 3/7		1.7138E-02	1.5431E-04	1.5417E-06	1.5295E-08
1/18, 4/9		1.6709E-02	1.5062E-04	1.5048E-06	1.5244E-08
1/24, 11/24		1.6334E-02	1.4739E-04	1.4726e-06	1.4922E-08
1/12, 5/12		1.7461E-02	1.5708E-04	1.5693E-06	1.5571E-08
1/4, 1/4	0.09	1.9486E-02	1.7638E-04	1.7623E-06	1.7501E-08
1/6, 1/3		1.7164E-02	1.5621E-04	1.5611E-06	1.5486E-08
1/14, 3/7		1.4544E-02	1.3315E-04	1.3311E-06	1.3189E-08
1/18, 4/9		1.4110E-02	1.2932E-04	1.2928E-06	1.3451E-08
1/24, 11/24		1.3732E-02	1.2596E-04	1.2593E-06	1.2792E-08
1/12, 5/12		1.4869E-02	1.3604E-04	1.3598E-06	1.3475E-08

Table 2. ROC for Example 1 with $\varepsilon = 0.1$ and different values of δ, α, and β.

α and β	δ	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
1/4, 1/4	0.01	2.1503E+00	2.0005E+00	2.0000E+00	2.0448E+00
1/6, 1/3		2.1433E+00	2.0007E+00	2.0000E+00	2.0515E+00
1/14, 3/7		2.1354E+00	2.0010E+00	2.0000E+00	2.0294E+00
1/18, 4/9		2.1340E+00	2.0011E+00	2.0000E+00	2.5397E+00
1/24, 11/24		2.1328E+00	2.0012E+00	2.0000E+00	2.0601E+00
1/12, 5/12		2.1364E+00	2.0011E+00	2.0000E+00	1.6768E+00
1/4, 1/4	0.05	2.1318E+00	2.0012E+00	1.9999E+00	2.0270E+00
1/6, 1/3		2.1249E+00	2.0011E+00	2.0000E+00	2.0282E+00
1/14, 3/7		2.1169E+00	2.0010E+00	2.0000E+00	2.0316E+00
1/18, 4/9		2.1156E+00	2.0010E+00	1.9999E+00	2.0667E+00
1/24, 11/24		2.1143E+00	2.0010E+00	2.0000E+00	2.0695E+00
1/12, 5/12		2.1178E+00	2.0010E+00	1.9999E+00	2.0355E+00
1/4, 1/4	0.09	2.1145E+00	2.0010E+00	2.0000E+00	2.0305E+00
1/6, 1/3		2.1077E+00	2.0009E+00	2.0001E+00	2.0371E+00
1/14, 3/7		2.0999E+00	2.0003E+00	2.0001E+00	2.0392E+00
1/18, 4/9		2.0985E+00	2.0003E+00	2.0001E+00	2.1135E+00
1/24, 11/24		2.0974E+00	2.0002E+00	2.0001E+00	2.0785E+00
1/12, 5/12		2.1008E+00	2.0005E+00	1.9999E+00	2.0406E+00

Table 3. Comparison of MAE for Example 1 with Melesse et al. (2019) with $\varepsilon = 0.1, \alpha = 1/24, \beta = 11/24$ and $N = 100$.

δ	Non-polynomial Spline	Melesse et al. (2019)
$\delta = 0.00\varepsilon$	1.75E-04	2.46E-02
$\delta = 0.20\varepsilon$	1.64E-04	2.56E-02
$\delta = 0.50\varepsilon$	1.47E-04	2.96E-02
$\delta = 0.80\varepsilon$	1.32E-04	3.33E-02
$\delta = 1.00\varepsilon$	7.15E-05	5.38E-02

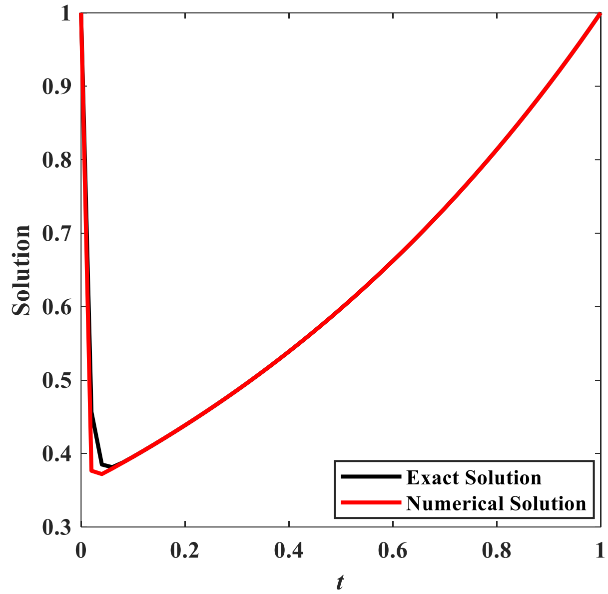

Figure 1. Exact and approximate solution of example 1 for $\varepsilon = 10^{-2}, \delta = 2\varepsilon$ and $N = 50$.

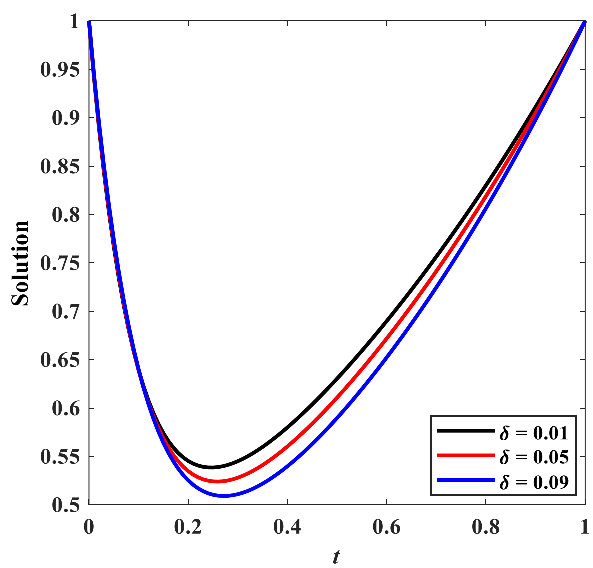

Figure 2. Solution of Example 1 for $\varepsilon = 10^{-1}, N = 100$ and different values of δ.

Example 2: Consider the following singularly perturbed differential equation with advance parameter

$$\varepsilon w_\varepsilon''(t) + w_\varepsilon'(t) - 3y(t) + 2w_\varepsilon(t+\eta) = 0,$$

with the boundary conditions

$$w_\varepsilon(0) = 1 \text{ and } w_\varepsilon(t) = 1, 1 \leq t \leq \eta.$$

The exact solution is $w_\varepsilon(t) = a_1\, exp(r_1 t) + a_2\, exp(r_2 t)$,

where, $r_1 = \frac{-(1+2\eta)+\sqrt{(1+2\eta)^2+4\varepsilon}}{2\varepsilon}$, $r_2 = \frac{-(1+2\eta)-\sqrt{(1+2\eta)^2+4\varepsilon}}{2\varepsilon}$, $a_1 = \frac{1-exp(r_2)}{exp(r_1)-exp(r_2)}$, and $a_2 = 1 - a_1$.

Table 4 gives MAE and **Table 5** gives ROC of Example 2, for different α, β, N and η values. **Figure 3** provides the behavior of the solution of Example 2 for different values of η. The **Figure 3** shows that the change in parameter values has negligible effect on the solution. This proves the robustness of the method.

Table 4. MAE for Example 2 with $\varepsilon = 0.1$, and different values of η, α, and β.

α and β	η	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
1/4, 1/4	0.01	2.5936E-02	2.2612E-04	2.2581E-06	2.2463E-08
1/6, 1/3		2.3687E-02	2.0783E-04	2.0756E-06	2.0638E-08
1/14, 3/7		2.1146E-02	1.8693E-04	1.8671E-06	1.8550E-08
1/18, 4/9		2.0726E-02	1.8345E-04	1.8323E-06	1.8516E-08
1/24, 11/24		2.0358E-02	1.8040E-04	1.8019E-06	1.8212E-08
1/12, 5/12		2.1462E-02	1.8955E-04	1.8932E-06	1.8812E-08
1/4, 1/4	0.05	2.8553E-02	2.4650E-04	2.4651E-06	2.4533E-08
1/6, 1/3		2.6346E-02	2.2897E-04	2.2895E-06	2.2780E-08
1/14, 3/7		2.3853E-02	2.0894E-04	2.0890E-06	2.0771E-08
1/18, 4/9		2.3441E-02	2.0560E-04	2.0556E-06	2.1056E-08
1/24, 11/24		2.3081E-02	2.0268E-04	2.0263E-06	2.0454E-08
1/12, 5/12		2.4163E-02	2.1144E-04	2.1141E-06	2.1024E-08
1/4, 1/4	0.09	3.1159E-02	2.6805E-04	2.6760E-06	2.6643E-08
1/6, 1/3		2.9000E-02	2.5112E-04	2.5071E-06	2.4954E-08
1/14, 3/7		2.6560E-02	2.3176E-04	2.3141E-06	2.3025E-08
1/18, 4/9		2.6156E-02	2.2854E-04	2.2819E-06	2.3008E-08
1/24, 11/24		2.5804E-02	2.2572E-04	2.2538E-06	2.2726E-08
1/12, 5/12		2.6864E-02	2.3418E-04	2.3382E-06	2.3265E-08

Table 5. ROC for Example 2 with $\varepsilon = 0.1$, and different values of η, α, and β.

α and β	η	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
1/4, 1/4	0.01	2.1601E+00	2.0014E+00	2.0000E+00	2.3791E+00
1/6, 1/3		2.1531E+00	2.0013E+00	2.0000E+00	2.0256E+00
1/14, 3/7		2.1451E+00	2.0010E+00	2.0000E+00	2.0274E+00
1/18, 4/9		2.1438E+00	2.0013E+00	1.9999E+00	2.4984E+00
1/24, 11/24		2.1425E+00	2.0014E+00	2.0000E+00	2.0545E+00
1/12, 5/12		2.1461E+00	2.0011E+00	2.0000E+00	1.6729E+00
1/4, 1/4	0.05	2.1807E+00	1.9993E+00	1.9999E+00	2.3380E+00
1/6, 1/3		2.1737E+00	1.9996E+00	2.0000E+00	2.0236E+00
1/14, 3/7		2.1656E+00	1.9996E+00	2.0000E+00	2.0234E+00
1/18, 4/9		2.1643E+00	1.9999E+00	2.0000E+00	2.4532E+00
1/24, 11/24		2.1631E+00	1.9999E+00	2.0000E+00	2.4383E+00
1/12, 5/12		2.1666E+00	1.9998E+00	2.0000E+00	1.7082E+00
1/4, 1/4	0.09	2.2025E+00	2.0018E+00	2.0000E+00	2.3018E+00
1/6, 1/3		2.1955E+00	2.0018E+00	2.0000E+00	2.0183E+00
1/14, 3/7		2.1874E+00	2.0017E+00	2.0000E+00	2.0197E+00
1/18, 4/9		2.1860E+00	2.0017E+00	2.0000E+00	2.3772E+00
1/24, 11/24		2.1849E+00	2.0017E+00	2.0001E+00	2.0402E+00
1/12, 5/12		2.1885E+00	2.0017E+00	2.0000E+00	1.7380E+00

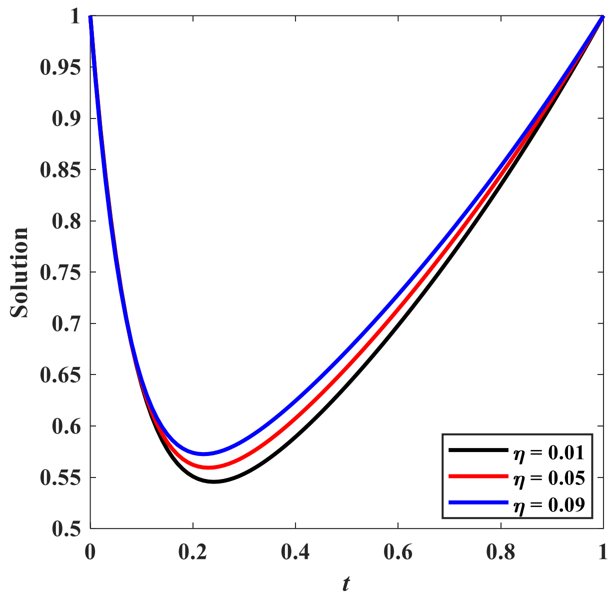

Figure 3. Solution of Example 2 for $\varepsilon = 10^{-1}, \delta = 2\varepsilon$, and different values of η.

Example 3: Consider the following singularly perturbed differential equation with mixed shifts

$$\varepsilon w_\varepsilon''(t) + w_\varepsilon'(t) - 2w_\varepsilon(t-\delta) - 5w_\varepsilon(t) + w_\varepsilon(t+\eta) = 1,$$

with the boundary conditions

$$w_\varepsilon(t) = 1, -\delta \le t \le 0 \text{ and } w_\varepsilon(t) = 1, 1 \le t \le \eta.$$

The exact solution of the problem is given by $w_\varepsilon(t) = a_1\, exp(r_1 t) + a_2\, exp(r_2 t)$,

where, $r_1 = \frac{-(1+2\delta+\eta)+\sqrt{(1+2\delta+\eta)^2+24\varepsilon}}{2\varepsilon}, r_2 = \frac{-(1+2\delta+\eta)-\sqrt{(1+2\delta+\eta)^2+24\varepsilon}}{2\varepsilon}, a_1 = \frac{1-exp(r_2)}{exp(r_1)-exp(r_2)}$, and $a_2 = 1 - a_1$.

Table 6. MAE for Example 3 with $\varepsilon = 0.1$ and $\eta = 0.8\varepsilon$, and different values of δ, α, and β.

δ	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
0.01	3.3654E-02	3.1254E-04	3.1271E-06	3.1263E-08
0.05	3.8935E-02	3.6406E-04	3.6375E-06	3.6365E-08
0.09	4.4494E-02	4.1912E-04	4.1821E-06	4.1842E-08

Table 7. ROC for Example 3 with $\varepsilon = 0.1,\ \eta = 0.8\varepsilon$, and different values of δ, α, and β.

δ	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
0.01	2.0837E+00	1.9986E+00	2.0000E+00	2.0118E+00
0.05	2.0557E+00	2.0013E+00	2.0000E+00	2.0100E+00
0.09	2.0283E+00	2.0024E+00	2.0000E+00	1.9937E+00

Tables 6 and **7** presents the MAE and ROC of Example 3 for different values of δ, and **Tables 8** and **9** presents the MAE and ROC of Example 3 for different values of η. The behavior of the solution for different values of δ, and η are given in **Figures 4** and **5**.

Table 8. MAE for Example 3 with $\varepsilon = 0.1,\ \delta = 0.5\varepsilon$, and different values of η, α, and β.

η	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
0.01	3.4298E-02	3.1862E-04	3.1890E-06	3.1882E-08
0.05	3.6920E-02	3.4420E-04	3.4420E-06	3.4412E-08
0.09	3.9615E-02	3.7078E-04	3.7036E-06	3.7028E-08

Table 9. ROC for Example 3 with $\varepsilon = 0.1,\ \eta = 0.8\varepsilon$, and different values of δ, α, and β.

η	$N = 10$	$N = 100$	$N = 1000$	$N = 10000$
0.01	2.0802E+00	1.9981E+00	2.0000E+00	1.9906E+00
0.05	2.0662E+00	1.9995E+00	2.0000E+00	1.9915E+00
0.09	2.0523E+00	2.0019E+00	2.0000E+00	1.9919E+00

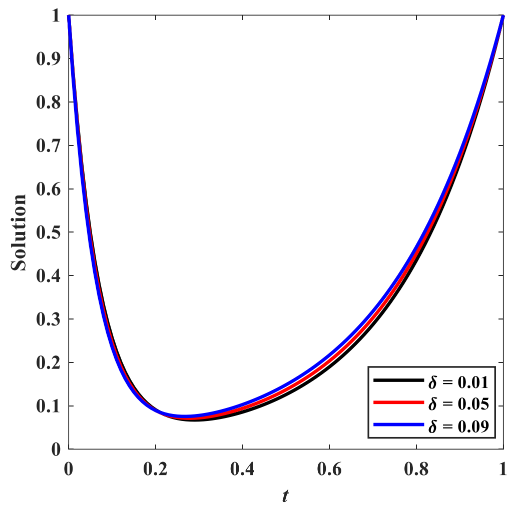

Figure 4. Solution of Example 3 for $\varepsilon = 10^{-1}, \eta = 0.8\varepsilon, N = 100$, and different values of δ.

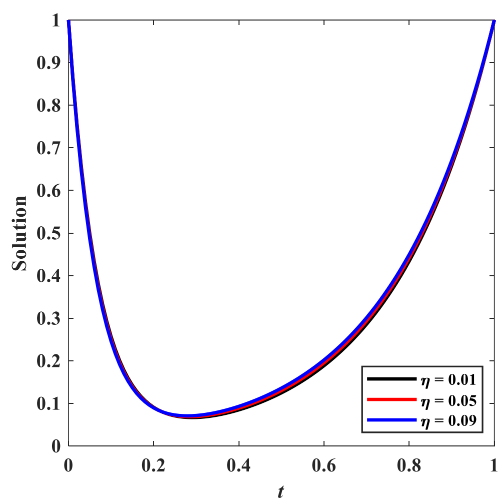

Figure 5. Solution of Example 3 for $\varepsilon = 10^{-1},\ \delta = 0.5\varepsilon,\ N = 100$, and different values of η.

We have solved three numerical examples with left boundary layer. From the tabulated values, it can be easily seen that the MAE decreases as N increases. In Example 3, from **Figures 4** and **5**, it is seen that the delay and advance parameters have similar effect on the solution. This implies that the variation caused by delay and advance parameters in the solution are very well handled by the proposed method for different values of α and β. From **Tables 1, 4, 6** and **Table 8**, presenting MAE, it is seen that there is no abrupt change in the error values proving the stability of the method. **Tables 2, 5, 7** and **9** show that the numerical order of convergence is 2.

5. Conclusion

The non-polynomial spline method is developed and analyzed for SPDDE with mixed shifts. The method presents a promising pathway to tackle the complex structure of SPDDEs. The proposed approach effectively captures the boundary layers and ensures high accuracy. One of the key features of the proposed method is second-order convergence, which is verified by thorough error analysis and numerical experiments. The scheme's stability is also confirmed by stability analysis under standard perturbation and shift parameters, making it highly reliable. Numerical experiments confirm the robustness and efficiency of the method, showing superior accuracy compared to traditional numerical schemes. The MAE values presented in the tables show the consistency of the method throughout the domain. The non-polynomial spline is very effective and computationally simple to apply. The findings indicate that the non-polynomial spline can be utilized to solve various types of SPDEs, SPDDEs with different delay structures, and other complexities. In the future, non-polynomial splines can be explored for SPDDEs with fixed delay, non-linear SPDDEs, and higher-order SPDDEs.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Acknowledgments

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors. The authors would like to thank the editor and anonymous reviewers for their comments that help improve the quality of this work.

AI Disclosure

The authors declare that no assistance is taken from generative AI to write this article.

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