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Generalized Sasakian Space-Forms with Beta-Kenmotsu Structure and Ricci Solitons

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Abstract
We explore the properties of almost Ricci solitons and the gradient Ricci solitons on generalized Sasakian-space-forms with beta-Kenmotsu structure. We consider almost Ricci solitons on generalized Sasakian-space-forms with beta-Kenmotsu structure when the soliton vector field coincides with the Reeb vector field, and it is point-wise collinear with the Reeb vector field. We also investigate the properties of generalized Sasakian-space-forms with beta-Kenmotsu structure admitting the gradient Ricci solitons.

Keywords- Almost contact metric manifolds, Generalized Sasakian space-forms, Curvature tensors, Eta-Einstein manifolds, Ricci flow, Almost Ricci soliton.

1. Introduction

Contact geometry in differential geometry is the study of a geometric structure on smooth manifolds that satisfies a criterion known as "complete non-integrability" and is provided by a hyperplane distribution in the tangent bundle. The non-integrability condition translates into a maximal nondegeneracy condition on the differential one-form, which is equivalent to such a distribution being given (at least locally) as the kernel of the form. In contrast, the Frobenius theorem contains two equivalent conditions for a hyperplane distribution to be "completely integrable," that is, to be tangent to a codimension one foliation on the manifold. In many aspects, contact geometry is the odd-dimensional equivalent of symplectic geometry, which is a structure on some even-dimensional manifolds. The mathematical formalism of classical mechanics serves as the inspiration for both contact and symplectic geometry. It allows one to examine either the constant-energy hypersurface, which has odd dimension due to its codimension one, or the even-dimensional phase space of a mechanical system. Geometrical optics, classical mechanics, thermodynamics, geometric quantization, integrable systems, and control theory are just a few of the many physical applications of contact geometry. Applications of contact geometry can also be found in low-dimensional topology; for instance, Kronheimer and Mrowka have used it to demonstrate the property P conjecture. We are motivated to study contact manifolds and their many classes by the uses of contact metric manifolds. The features of a class of almost contact metric manifolds, known as generalized Sasakian space-forms, are investigated in this work.

A heat flow equation is a nonlinear partial differential equation, which was introduced by Grattan-Guinness (1822). A similar nonlinear variant of heat flow equation, named as harmonic map heat flow, was invented by Eells and Sampson (1964). This flow inspired Hamilton to define a geometric flow, named Ricci flow (Hamilton, 1982, 1988). The Ricci flow, often known as Hamilton's Ricci flow, is a partial differential equation for a Riemannian metric that is studied in differential geometry and geometric analysis. Because of formal parallels in the equation's mathematical structure, it is sometimes compared to the diffusion of heat and heat equation. It is nonlinear, though, and displays a number of phenomena that are absent from the analysis of the heat equation. A Ricci flow $\left(\frac{\partial g}{\partial t}=-2Ric,\ g(0)=g_0\right)$ is a nonlinear partial differential equation, where, g, t and Ric are the Riemannian metric, time, and the Ricci tensor of the Riemannian n-manifold M, respectively.

In 2000, the Clay Mathematics Institute observed that there are seven iconic unsolved problems in Mathematics, listed as:

- Hodge conjecture (proposed by William Hodge in 1950).
- Navier-Stokes existence and smoothness conjecture (proposed by Luis Caffarelli in 1960).
- P versus NP problem conjecture (proposed by Stephen Cook and Leonid Levin in 1971).
- Poincaré conjecture (proposed by Henri Poincaré in 1904).
- Riemann hypothesis conjecture (proposed by Bernhard Riemann in 1859).
- Birch and Swinnerton-Dyer conjecture (proposed by Birch and Swinnerton-Dyer in the first half of 1960's).
- Yang-Mills existence and mass gap conjecture (proposed by Yang & Mills about a half century ago).

The above listed millennium problems (except Poincaré conjecture) are the unsolved open challenging problems of science till today with a prize of 7 million US dollars. The Poincaré conjecture (one of the millennium problems) and Geometrization Conjecture have been solved by Perelman (2022, 2003a, 2003b) with the help of Ricci flow. Anderson (2004) has given the Geometrization of 3-manifolds via Ricci flow. Remarks that the Ricci flows have been used as a tool to address many long-standing unsolved problems of mathematics, physics, medical science, engineering, and technology. In this series, Wang et al. (2012) have used the Ricci flow as a tool to parameterize the brain surface conformally. We may use the Ricci flow as a powerful tool to compute the conformal Riemannian metric with prescribed Gaussian curvatures, which has many applications in engineering such as spline construction in geometric modeling, parametrization in graphics, surface parametrization, conformal brain mapping in medical imaging, and so on. A self-similar solution of the Ricci flow is termed as Ricci soliton. Thus, the Ricci soliton has been used as a tool to address several issues of mathematical sciences and allied areas.

The Ricci soliton equation on an n-dimensional Riemannian manifold M is given by,

$$S+\frac{1}{2}\mathcal{L}_V g+\mathfrak{N}g=0 \tag{1}$$

where, $\mathcal{L}$ is the Lie derivative operator of the Riemannian metric g, V is a soliton vector field, S is Ricci tensor and $\mathfrak{N}$ is a soliton constant. We symbolize the Ricci soliton with $(g, V, \mathfrak{N})$. If we choose $\mathfrak{N}$ as a smooth function on M, then the Ricci soliton Equation (1) becomes almost Ricci soliton. Almost Ricci soliton $(g, V, \mathfrak{N})$ is expanding, shrinking or steady provided that $\mathfrak{N}$ is, respectively, positive, negative or zero. The geometrical and physical properties of spacetimes equipped with Ricci solitons and the gradient Ricci solitons have been explored by several researchers. Chaubey and Suh (2023) studied the properties of Fischer-Marsden Conjecture, Ricci Bourguignon solitons and gradient Bourguignon solitons on generalized Sasakian space-forms. They have proved many interesting results, which play an important role in the study and development of this topic. Some properties of Ricci solitons and curvature properties of cosymplectic manifolds have been studied by Ayar and Chaubey (2019). In this series, the properties of Ricci solitons have been explored by several authors including (Chaubey and Vîlcu, 2022; De et al., 2020, 2021a, 2021b; Haseeb et al., 2023; Hui et al., 2018; Pankaj et al., 2021; Pokhariyal et al., 2018; Siddiqi et al., 2022; Suh and Chaubey, 2023; Turan et al., 2019; Yadav et al., 2018, 2023; Yadav and Chaubey, 2020). It is remarked that the Ricci solitons and gradient Ricci solitons have been extensively used in Mathematical Physics, especially in the theory of relativity and cosmology. Haseeb et al. (2022) and Siddiqi et al. (2022) have studied the properties of solitons on space-times, and they proved some interesting results. There could be possible applications of the techniques and results proved in this paper are likely to be applicable in studies like (Hong and Van De Walle, 2012).

Let us choose $V = Df$, where D is the gradient operator of g and f is some smooth function on M, in Equation (1). Then we have the following equation of gradient almost Ricci soliton.

$$Hessf + S + \mathfrak{N}g = 0 \tag{2}$$

where, $Hess\ f$ is the Hessian of smooth function f on M.

The above studies motivated us to explore the properties of $(2n+1)$-dimensional generalized Sasakian-space-forms with β-Kenmotsu structure $M(f_1, f_2, f_3)$ if the Riemannian metric is almost Ricci soliton and gradient Ricci soliton, respectively.

To achieve our main goals, we organize the manuscript as follows: After the introduction part, we gather the basic results of almost contact metric manifolds and their different classes in Section 2. We also gave some known definitions and curvature identities. In the next section, we study the geometrical properties of almost Ricci solitons on $M(f_1, f_2, f_3)$. Special attention is given for the cases when the soliton vector field V of almost Ricci soliton $(g, V, \mathfrak{N})$ coincides with Reeb vector field ξ of M, and it is point-wise collinear with the Reeb vector field ξ, respectively. Section *4* deals with the study of gradient Ricci solitons on $M(f_1, f_2, f_3)$.

2. Almost Contact Metric Manifolds

A differentiable manifold M of dimension $(2n+1)$ equipped with an almost contact structure and contact structure has been studied by Boothby and Wang (1958). Authors have studied these structures by considering topological notions. Sasaki (1960) has investigated the contact and almost contact structures by adopting the notion of tensor calculus. Remark that the contact geometry can be used as a tool to address the issues of physics, like control theory, quantization, thermodynamics, geometric optics, integrable systems, classical mechanics and many other branches. Many authors have established remarkable results on almost contact metric structure and its classes. In this manuscript, we establish some results of a class of almost contact metric manifolds under certain geometric flows.

Consider a differentiable manifold M with a dimension of $(2n+1)$ that is of class C^∞, along with a global differentiable 1-form η for which $\eta \wedge (d\eta)^n \neq 0$ holds true at all points on M. Then M satisfies

$$\xi \otimes \eta - I = \phi^2, \eta(\xi) = 1 \tag{3}$$

and

$$g(\mathfrak{T}_1, \mathfrak{T}_2) = g(\phi\mathfrak{T}_1, \phi\mathfrak{T}_2) + \eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2), \;\; g(\mathfrak{T}_1, \xi) = \eta(\mathfrak{T}_1) \tag{4}$$

for $\mathfrak{T}_1$ and $\mathfrak{T}_2$ belonging to the set $\mathfrak{X}(M)$, is termed as almost contact metric manifold.

where, I represents the identity transformation, ϕ refers to the structure tensor of type (1,1), ξ is the unit vector field of type (1,0), and g is a compatible Riemannian metric on M. For instance, see Blair (1976). The configuration (ϕ, ξ, η, g) on the manifold M is referred to as an almost contact metric structure. From Equation (3), it can be readily observed that

$$rank\,(\phi) = 2n, \;\; \phi\xi = 0, \;\text{and}\; \eta \circ \phi = 0 \tag{5}$$

Also, Equation (3) to Equation (5) infer that $g(\phi\mathfrak{T}_1, \mathfrak{T}_2) + g(\mathfrak{T}_1, \phi\mathfrak{T}_2) = 0$ for all $\mathfrak{T}_1$, $\mathfrak{T}_2 \in \mathfrak{X}(M)$.

A contact metric manifold M is defined as an almost contact metric manifold where $d\eta(\mathfrak{T}_1, \mathfrak{T}_2) = g(\mathfrak{T}_1, \phi\mathfrak{T}_2)$ holds for all $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{X}(M)$, with d representing the exterior derivative operator. M satisfying the tensorial expression $[\phi, \phi] = -2d\eta \otimes \eta$ where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ, is called as a normal contact metric manifold. A contact metric manifold M is Sasakian if and only if $R(\mathfrak{T}_1, \mathfrak{T}_2)\xi = \eta(\mathfrak{T}_2)\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\mathfrak{T}_2$ holds for all $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{X}(M)$ (Blair, 1976). Here R denotes the Riemann curvature tensor corresponding to the Levi-Civita connection ∇.

The various space-forms, such as real space-forms, Sasakian space-forms, Kenmotsu space-forms, and cosymplectic space-forms, served as inspiration for Alegre et al. (2004), who defined a new space-form known as a generalized Sasakian-space-form. Thus, an almost contact metric manifold of $(2n+1)$ dimensions with the global contact form η $(\eta \wedge (d\eta)^n \neq 0)$, the structure tensor ϕ, and the unit vector field ξ satisfying the following curvature identity:

$$R(\mathfrak{T}_1, \mathfrak{T}_2)\mathfrak{T}_3 = f_1\{g(\mathfrak{T}_2, \mathfrak{T}_3)\mathfrak{T}_1 - g(\mathfrak{T}_1, \mathfrak{T}_3)\mathfrak{T}_2\} + f_2\{g(\mathfrak{T}_1, \phi\mathfrak{T}_3)\phi\mathfrak{T}_2 - g(\mathfrak{T}_2, \phi\mathfrak{T}_3)\phi\mathfrak{T}_1 + 2g(\mathfrak{T}_1, \phi\mathfrak{T}_2)\phi\mathfrak{T}_3\} + f_3\{\eta(\mathfrak{T}_1)\eta(\mathfrak{T}_3)\mathfrak{T}_2 - \eta(\mathfrak{T}_2)\eta(\mathfrak{T}_3)\mathfrak{T}_1 + \eta(\mathfrak{T}_2)g(\mathfrak{T}_1, \mathfrak{T}_3) - \eta(\mathfrak{T}_1)g(\mathfrak{T}_2, \mathfrak{T}_3)\xi\} \tag{6}$$

for all $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3 \in \mathfrak{X}(M)$, is referred to as a generalized Sasakian-space-form, in which the smooth functions on M are f_1, f_2, and f_3, and g is the Riemannian metric. The curvature tensor with regard to the Levi-Civita connection ∇ and the collection of all smooth vector fields of M are indicated here by the symbols R and $\mathfrak{X}(M)$, respectively. We use $M(f_1, f_2, f_3)$ as a generalized Sasakian-space-form throughout the paper. In particular, the generalized Sasakian-space-form becomes Sasakian-space-form and Kenmotsu-space-form, respectively, if we select $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, and $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$. On $M(f_1, f_2, f_3)$, it is found that the smooth function f_2 vanishes if and only if the manifold is conformally flat (Kim, 2006). Numerous geometers have examined the characteristics of $M(f_1, f_2, f_3)$ in various settings. For instance, we refer to Alegre and Carriazo (2008), Alegre and Carriazo (2011) Carriazo et al. (2013), Chaubey and Yadav (2018), Chaubey and Yildiz (2019), Falcitelli (2010), Li et al. (2022), Sular and Özgür (2012). Pandey and Gupta (2009) and Pandey and Mohammad (2020) studied the properties of Kenmotsu manifolds.

From Equation (6), it can be seen that $M(f_1, f_2, f_3)$ satisfies,

$$S(\mathfrak{T}_1, \mathfrak{T}_2) = (3f_2 + 2nf_1 - f_3)g(\mathfrak{T}_1, \mathfrak{T}_2) - \big((2n-1)f_3 + 3f_2\big)\eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2) \tag{7}$$

which becomes

$$Q\mathfrak{T}_1 = -(3f_2 + (2n-1)f_3)\eta(\mathfrak{T}_1)\xi + (2nf_1 + 3f_2 - f_3)\mathfrak{T}_1 \quad (8)$$

where, $S(\mathfrak{T}_1, \mathfrak{T}_2) = \sum_{i=1}^{2n+1} g(R(e_i, \mathfrak{T}_1)\mathfrak{T}_2, e_i)$ is the Ricci tensor of $M(f_1, f_2, f_3)$ for a set of orthonormal vectors $\{e_i,\ i = 1, 2, \cdots, (2n+1)\}$, and Q is the Ricci operator such that $S(\mathfrak{T}_1, \mathfrak{T}_2) = g(Q\mathfrak{T}_1, \mathfrak{T}_2)$. Put $\mathfrak{T}_3 = \xi$ in Equation (6) and following Equation (3) to Equation (5) we have

$$R(\mathfrak{T}_1, \mathfrak{T}_2)\xi = (f_1 - f_3)\{\eta(\mathfrak{T}_2)\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\mathfrak{T}_2\} \quad (9)$$

which can be written as

$$R(\xi, \mathfrak{T}_1)\mathfrak{T}_2 = (f_1 - f_3)\{g(\mathfrak{T}_1, \mathfrak{T}_2)\xi - \eta(\mathfrak{T}_2)\mathfrak{T}_1\}.$$

Additionally, it is noted that the following identity are satisfied by $M(f_1, f_2, f_3)$.

$$S(\mathfrak{T}_1, \xi) = 2n(f_1 - f_3)\eta(\mathfrak{T}_1), \text{ for all } \mathfrak{T}_1 \in \mathfrak{X}(M) \quad (10)$$

The contraction of Equation (7) over the vector fields $\mathfrak{T}_1$ and $\mathfrak{T}_2$ takes the form

$$r = 2n\{3f_2 + (2n+1)f_1 - 2f_3\}.$$

From Equation (10), it is obvious that $2n(f_1 - f_3)$ is an eigenvalue of the Ricci operator Q corresponding to the eigenvector ξ. Suppose a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure is denoted by $M(f_1, f_2, f_3)$. Then we have

$$\nabla_{\mathfrak{T}_1}\xi = \beta(\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\xi) \iff (\nabla_{\mathfrak{T}_1}\eta)(\mathfrak{T}_2) = \beta\big(g(\mathfrak{T}_1, \mathfrak{T}_2) - \eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2)\big) \quad (11)$$

which gives $(\nabla_\xi \eta)(\mathfrak{T}_1) = (\nabla_{\mathfrak{T}_1}\eta)(\xi) = 0$. From Equation (11), it is clear that $(\nabla_{\mathfrak{T}_1}\eta)(\mathfrak{T}_2) - (\nabla_{\mathfrak{T}_2}\eta)(\mathfrak{T}_1) = 0$, that is, the 1-form η is closed. Also, we notice that

$$(\mathcal{L}_\xi g)(\mathfrak{T}_1, \mathfrak{T}_2) = g(\nabla_{\mathfrak{T}_1}\xi, \mathfrak{T}_2) + g(\mathfrak{T}_1, \nabla_{\mathfrak{T}_2}\xi) = 2\beta[g(\mathfrak{T}_1, \mathfrak{T}_2) - \eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2)] \quad (12)$$

In particular, $M(f_1, f_2, f_3)$ reduces to the generalized Sasakian-space-form with cosymplectic and Kenmotsu structures, respectively, if we select $\beta = 0$ and $\beta = 1$ on $M(f_1, f_2, f_3)$. Alegre and Carriazo (2008) demonstrated that the relation $f_1 - f_3 + \xi(\beta) + \beta^2 = 0$ is satisfied by a generalized Sasakian-space-form with β-Kenmotsu structure. We assume that β is constant on $M(f_1, f_2, f_3)$ throughout the manuscript.

Chaubey and Suh (2023) proved that the following:

Lemma 2.1 An $M(f_1, f_2, f_3)$ satisfies

(i) $(\nabla_{\mathfrak{T}_1}Q)(\xi) = 2n\,\beta\,(f_1 - f_3)\eta(\mathfrak{T}_1)\xi - \beta\,Q\mathfrak{T}_1 + 2n\mathfrak{T}_1(f_1 - f_3)\xi + 2n(f_1 - f_3)\beta\mathfrak{T}_1 - 2n(f_1 - f_3)\beta\eta\,(\mathfrak{T}_1)\xi$,

(ii) $(\nabla_\xi Q)(\mathfrak{T}_1) = \xi(2n\,f_1 + 3f_2 - f_3)\mathfrak{T}_1 - \xi\,(3f_2 + (2n-1)f_3)\eta(\mathfrak{T}_1)\xi$,

(iii) $(\nabla_\xi Q)(\mathfrak{T}_1) - (\nabla_{\mathfrak{T}_1}Q)(\xi) = [\xi((2n-1)f_3 + 3f_2) + \beta(3f_2 + (2n-1)f_3)](\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\xi)$,

(iv) $dr(\xi) = -n\beta[(2n-1)f_3 + 3f_2] = (n-1)\xi(3f_2 + (2n-1)f_3)$,

(v) $\mathfrak{T}_1(2nf_1 + 3f_2 - f_3) = \mathfrak{T}_1(3f_2 + (2n-1)f_3)$

for all $\mathfrak{T}_1 \in \mathfrak{X}(M)$.

From Lemma 2.1 (iv), it is obvious that the scalar curvature tensor of $M(f_1, f_2, f_3)$ is constant if and only if $3f_2 + (2n-1)f_3$ is constant. Thus, we can state:

Lemma 2.2 An $M(f_1, f_2, f_3)$ possesses a constant scalar curvature if and only if $3f_2 + (2n-1)f_3$ is constant.

A $(2n+1)$-dimensional almost contact metric manifold M is said to be an η-Einstein manifold if its non-vanishing Ricci tensor S satisfies,

$$S = b\,\eta \otimes \eta + a\,g \tag{13}$$

where, a and b are smooth functions on M. If we take $b = 0$ in Equation (13) then the η-Einstein manifold reduces to the Einstein manifold.

Definition 2.1 A smooth function Ω on an m-dimensional Riemannian manifold is said to be

- harmonic if and only if it satisfies the Laplace equation $\Delta\,\Omega = 0$.
- super-harmonic if and only if $\Delta\,\Omega \leq 0$.
- sub-harmonic if and only if $\Delta\,\Omega \geq 0$.

3. Almost Ricci Solitons on Generalized Sasakian-Space-Forms with β-Kenmotsu Structure

Let a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure admit almost Ricci soliton $(g, \xi, \mathfrak{N})$. Then an almost Ricci soliton Equation (1) reduces to

$$(\mathcal{L}_\xi g)(\mathfrak{T}_1, \mathfrak{T}_2) + 2S(\mathfrak{T}_1, \mathfrak{T}_2) + 2\,\mathfrak{N}\,g(\mathfrak{T}_1, \mathfrak{T}_2) = 0,$$

which in view of Equation (7) and Equation (12) becomes

$$2\beta\left(g(\mathfrak{T}_1, \mathfrak{T}_2) - \eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2)\right) + 2(2nf_1 + 3f_2 - f_3)g(\mathfrak{T}_1, \mathfrak{T}_2) - 2(3f_2 + (2n-1)f_3)\eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2) + 2\mathfrak{N}g(\mathfrak{T}_1, \mathfrak{T}_2) = 0 \tag{14}$$

$\forall\, \mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{X}(M)$. Setting $\mathfrak{T}_2 = \xi$ in Equation (14) and then following Equation (3) and Equation (4) we lead to

$$(2nf_1 - 2nf_3)\eta(\mathfrak{T}_1) + \mathfrak{N}\,\eta(\mathfrak{T}_1) = 0,$$

which becomes

$$\mathfrak{N} = -2n(f_1 - f_3) = 2n\,\beta^2 \geq 0 \tag{15}$$

This equation together with our hypothesis (β is constant on $M(f_1, f_2, f_3)$) infers that the soliton function $\mathfrak{N}$ of $(g, \xi, \mathfrak{N})$ is constant. Remark that the almost Ricci soliton equation with $\mathfrak{N}$ = constant reduces to a Ricci soliton equation. Thus, we conclude the following:

Theorem 3.1 An almost Ricci soliton $(g, \xi, \mathfrak{N})$ on $M(f_1, f_2, f_3)$ is a Ricci soliton.

The contraction of Equation (14) over the vector fields $\mathfrak{T}_1$ and $\mathfrak{T}_2$ gives

$$2n\,\beta + 2n(2n+1)f_1 + 6nf_2 - 4nf_3 + \mathfrak{N}(2n+1) = 0 \tag{16}$$

From Equation (15) and Equation (16), we notice that

$$\beta = 3f_2 - (2n-1)f_3.$$

Thus, we state our results as follows.

Theorem 3.2 Let the soliton vector field V of an almost Ricci soliton $(g, V, \mathfrak{N})$ coincide with the Reeb vector field ξ of generalized Sasakian-space-forms with β-Kenmotsu structure. Then $\beta = 3f_2 - (2n-1)f_3$.

In consequence of Theorem 3.1, Theorem 3.2 and Lemma 2.2, we list the following corollaries.

Corollary 3.1 Let the Riemannian metric g of a $(2n+1)$-dimensional generalized Sasakian-space-form with β-Kenmotsu structure $M(f_1, f_2, f_3)$ be almost Ricci soliton $(g, \xi, \mathfrak{N})$. Then the scalar curvature of $M(f_1, f_2, f_3)$ is constant.

Corollary 3.2 Let a generalized Sasakian space-form with cosymplectic structure admit an almost Ricci soliton $(g, \xi, \mathfrak{N})$. Then $(g, \xi, \mathfrak{N})$ is steady and $3f_2 - (2n-1)f_3 = 0$.

Corollary 3.3 Let a generalized Sasakian space-form with Kenmotsu structure admit an almost Ricci soliton $(g, \xi, \mathfrak{N})$. Then $(g, \xi, \mathfrak{N})$ is expanding and $3f_2 - (2n-1)f_3 = 1$.

Next, we suppose that $M(f_1, f_2, f_3)$ admits an almost Ricci soliton $(g, \xi, \mathfrak{N})$, where the soliton vector field V of $(g, \xi, \mathfrak{N})$ is pointwise collinear with the Reeb vector field ξ of M, that is, $V = a\xi$ for some smooth function a on M. The covariant derivative of this expression along $\mathfrak{T}_1$ gives,

$$\nabla_{\mathfrak{T}_1} V = \mathfrak{T}_1(a)\xi + \beta a\,(\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\xi) \tag{17}$$

Using Equation (7) and Equation (17) in the almost Ricci soliton equation,

$$g(\nabla_{\mathfrak{T}_1} V, \mathfrak{T}_2) + g(\mathfrak{T}_1, \nabla_{\mathfrak{T}_2} V) + 2S(\mathfrak{T}_1, \mathfrak{T}_2) + 2\mathfrak{N}\, g(\mathfrak{T}_1, \mathfrak{T}_2) = 0.$$

we achieve

$$\mathfrak{T}_1(a)\eta(\mathfrak{T}_2) + 2\beta a\,\{g(\mathfrak{T}_1, \mathfrak{T}_2) - \eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2)\} + \{\mathfrak{T}_2(a)\eta(\mathfrak{T}_1) + 2[2nf_1 + 3f_2 - f_3]g(\mathfrak{T}_1, \mathfrak{T}_2) - 2\big(3f_2 + (2n-1)f_3\big)\eta(\mathfrak{T}_1)\eta(\mathfrak{T}_2) + 2\mathfrak{N}g(\mathfrak{T}_1, \mathfrak{T}_2) = 0 \tag{18}$$

Substitute $\mathfrak{T}_2 = \xi$ in the above equation, we find

$$\mathfrak{T}_1(a) + \xi(a)\eta(\mathfrak{T}_1) - 2\big[2nf_1 - 2nf_2 - \mathfrak{N}\big]\eta(\mathfrak{T}_1) = 0 \tag{19}$$

Again putting $\mathfrak{T}_1 = \xi$ in the above equation, we get

$$\xi(a) = 2n(f_1 - f_3) - \mathfrak{N} \tag{20}$$

In view of Equation (19) and Equation (20), we conclude

$$\mathfrak{T}_1(a) - \big[2n\big(f_1 - f_3\big) - \mathfrak{N}\big]\eta(\mathfrak{T}_1) = 0 \tag{21}$$

Equation (20) and Equation (21) give

$$\mathfrak{T}_1(a) = \xi(a)\xi \iff Da = \xi(a)\xi \tag{22}$$

where, D denotes the gradient operator of g. Equation (22) shows that the gradient of the smooth function a is point-wise collinear with the Reeb vector field ξ. The covariant derivative of Equation (22) gives,

$$\nabla_{\mathfrak{T}_1} Da = \mathfrak{T}_1\big(\xi(a)\big)\xi + \xi(a)\beta\,(\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\xi),$$

which reduces to (taking contraction over $\mathfrak{T}_1$)

$$\Delta a = \xi(\xi(a)) + 2n\,\beta\,\xi(a) = \xi(\xi(a) + 2n\,\beta\,a).$$

Thus, we can state the following:

Theorem 3.3 Let $M(f_1, f_2, f_3)$ admit an almost Ricci soliton $(g, V, \mathfrak{N})$, where $V = a\,\xi$. Then the gradient of the smooth function a is point-wise collinear with the Reeb vector field ξ. Also, $(g, V, \mathfrak{N})$ is expanding, shrinking or steady if $2n(f_1 - f_3) > \xi(a)$, $2n(f_1 - f_3) < \xi(a)$ or $2n(f_1 - f_3) = \xi(a)$, respectively, and $\Delta a = \xi(\xi(a) + 2n\,\beta\,a)$.

Definition 2.1 together with Theorem 3.6 state the following:

Corollary 3.4 Let $M(f_1, f_2, f_3)$ admit an almost Ricci soliton $(g, V, \mathfrak{N})$, where the soliton vector field V is point-wise collinear with the Reeb vector field ξ, that is $V = a\,\xi$. Then the smooth function a is

- harmonic if and only if $\xi(a) + 2n\,\beta\,a$ = constant,
- sub-harmonic if and only if $\xi(\xi(a) + 2n\,\beta\,a) \geq 0$,
- super-harmonic if and only if $\xi(\xi(a) + 2n\,\beta\,a) \leq 0$.

Now, contracting Equation (18) over $\mathfrak{X}_1$ and $\mathfrak{X}_2$ we lead

$$2(n+1)f_1 + 3f_2 - 3f_3 + \beta\,a\ - 2n\,\beta^2\xi(a) = 0.$$

From Equation (21) we have

$$d\,a\ = b\ \wedge \eta.$$

where, $b = 2n(f_1 - f_3) - \mathfrak{N}$ and d represents the exterior derivative of g. Taking exterior derivative of last equation and then following $d^2 = 0$ and $d\eta = 0$, we find

$$db(\mathfrak{X}_1)\eta(\mathfrak{X}_2) - db(\mathfrak{X}_2)\eta(\mathfrak{X}_1) = 0, \qquad db = -d\,\mathfrak{N},$$

which gives $D\,\mathfrak{N}\ = \xi(\mathfrak{N})\xi$. This shows that the gradient of soliton function $\mathfrak{N}$ is point-wise collinear with the Reeb vector field ξ. It can be easily shown that the soliton function $\mathfrak{N}$ of almost Ricci soliton $(g, V, \mathfrak{N})$ satisfies the Poisson equation $\Delta\mathfrak{N} = \Psi$, where $\Psi = \xi(\xi(\mathfrak{N})) + 2n\beta\xi(\mathfrak{N})$. Now, we state:

Corollary 3.5 Let a generalized Sasakian-space-form with β-Kenmotsu structure admit an almost Ricci soliton $(g, V, \mathfrak{N})$. If the soliton vector field V of $(g, V, \mathfrak{N})$is point-wise collinear with the Reeb vector field, ξ, then the soliton function $\mathfrak{N}$ satisfies the Poisson equation $\Delta\mathfrak{N} = \Psi$.

4. Gradient Ricci Solitons on $M(f_1, f_2, f_3)$

This section deals with the study of generalized Sasakian-space-forms with β -Kenmotsu structure admitting a gradient Ricci soliton. We have from Equation (2),

$$\nabla_{\mathfrak{X}_1} Df = -Q\mathfrak{X}_1 - \mathfrak{N}\mathfrak{X}_1 \tag{23}$$

Differentiating Equation (23) covariantly along the vector field $\mathfrak{X}_1$, we have

$$\nabla_{\mathfrak{X}_2}\nabla_{\mathfrak{X}_1} Df = -\left(\nabla_{\mathfrak{X}_2} Q\right)(\mathfrak{X}_1) - Q\left(\nabla_{\mathfrak{X}_2}\mathfrak{X}_1\right) - \mathfrak{N}\nabla_{\mathfrak{X}_2}\mathfrak{X}_1 \tag{24}$$

Swapping out $\mathfrak{X}_1$ and $\mathfrak{X}_2$ in Equation (24), we get

$$\nabla_{\mathfrak{X}_1}\nabla_{\mathfrak{X}_2} Df = -\left(\nabla_{\mathfrak{X}_1} Q\right)(\mathfrak{X}_2) - Q\left(\nabla_{\mathfrak{X}_1}\mathfrak{X}_2\right) - \mathfrak{N}\nabla_{\mathfrak{X}_1}\mathfrak{X}_2 \tag{25}$$

Using Equation (23) to Equation (25) in the curvature identify

$$R(\mathfrak{X}_1, \mathfrak{X}_2)Df = \nabla_{\mathfrak{X}_1}\nabla_{\mathfrak{X}_2} Df - \nabla_{\mathfrak{X}_2}\nabla_{\mathfrak{X}_1} Df\ - \nabla_{[\mathfrak{X}_1,\ \mathfrak{X}_2]} Df,$$

we find

$$R(\mathfrak{T}_1,\mathfrak{T}_2)Df = \left(\nabla_{\mathfrak{T}_2}Q\right)(\mathfrak{T}_1) - \left(\nabla_{\mathfrak{T}_1}Q\right)(\mathfrak{T}_2) \tag{26}$$

The covariant derivative of Equation (8) gives
$\left(\nabla_{\mathfrak{T}_1}Q\right)(\mathfrak{T}_2) = \mathfrak{T}_1(3f_2 + (2n-1)f_3)(\mathfrak{T}_2 - \eta(\mathfrak{T}_2)\xi) - (3f_2 + (2n-1)f_3)\beta\,[g(\mathfrak{T}_1,\mathfrak{T}_2)\xi + \eta(\mathfrak{T}_2)\mathfrak{T}_1 - 2\eta(\mathfrak{T}_2)\eta(\mathfrak{T}_1)\xi]$

which gives
$\left(\nabla_{\mathfrak{T}_1}Q\right)(\mathfrak{T}_2) - \left(\nabla_{\mathfrak{T}_2}Q\right)(\mathfrak{T}_1) = \mathfrak{T}_1(3f_2 + (2n-1)f_3)(\mathfrak{T}_2 - \eta(\mathfrak{T}_2)\xi) - (3f_2 + (2n-1)f_3)\beta\,[\eta(\mathfrak{T}_2)\mathfrak{T}_1 - \eta(\mathfrak{T}_1)\mathfrak{T}_2]$,

reduces to
$$g\left(\left(\nabla_{\mathfrak{T}_1}Q\right)(\mathfrak{T}_2) - \left(\nabla_{\mathfrak{T}_2}Q\right)(\mathfrak{T}_1), \xi\right) = 0 \tag{27}$$

In consequence of Equation (27), Equation (26) assumes the form
$$g(R(\mathfrak{T}_1,\mathfrak{T}_2)Df, \xi) = 0 \tag{28}$$

Again Equation (6) together with Equation (3) to Equation (5) takes the form
$$g(R(\mathfrak{T}_1,\mathfrak{T}_2)Df, \xi) = (f_1 - f_3)\{\mathfrak{T}_2(f)\eta(\mathfrak{T}_1) - \mathfrak{T}_1(f)\eta(\mathfrak{T}_2)\} \tag{29}$$

Equation (28) and Equation (29) infer that either $f_1 = f_3$ or $\mathfrak{T}_2(f)\eta(\mathfrak{T}_1) = \mathfrak{T}_1(f)\eta(\mathfrak{T}_2)$. In first case, the manifold under consideration becomes a generalized Sasakian-space-form with cosymplectic structure. Let us consider the second case $\mathfrak{T}_2(f)\eta(\mathfrak{T}_1) = \mathfrak{T}_1(f)\eta(\mathfrak{T}_2)$, where, $f_1 \neq f_3$. Taking $\mathfrak{T}_2 = \xi$ in last equation, we obtain
$$\mathfrak{T}_1(f) = \xi(f)\eta(\mathfrak{T}_1) \iff Df = \xi(f)\xi \tag{30}$$

This shows that the gradient of the gradient function f is point-wise collinear with the Reeb vector field ξ. Thus, we can state the following:

Theorem 4.1 Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form with β-Kenmotsu structure. If $M(f_1, f_2, f_3)$ admits a gradient Ricci soliton, then either $M(f_1, f_2, f_3)$ is either a generalized Sasakian-space-form with cosymplectic structure or the gradient of gradient function of the gradient Ricci soliton is point-wise collinear with the Reeb vector field of the manifold.

Taking covariant derivative of Equation (30) along the vector field $\mathfrak{T}_1$ and then following Equation (3) to Equation (5) and Equation (8), we lead to
$$\begin{aligned}\mathfrak{R} &= -\left[\frac{r}{2n} + \beta(\xi(f) + \beta)\right]\\ &= -[2nf_1 + 3f_2 - f_3 + \beta\,\xi(f)]\end{aligned} \tag{31}$$

Well-known that a gradient Ricci soliton is expanding, steady or shrinking provided that $\mathfrak{R}$ is positive, zero and negative. These facts together with Equation (31) conclude the following results.

Corollary 4.1 Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form with β-Kenmotsu structure. If $M(f_1, f_2, f_3)$ admits a gradient Ricci soliton, then the soliton is expanding, shrinking, and steady if $2nf_1 + 3f_2 - f_3 + \beta\,\xi(f) < 0, 2nf_1 + 3f_2 - f_3 + \beta\,\xi(f) > 0$, and $2nf_1 + 3f_2 - f_3 + \beta\,\xi(f) = 0$, respectively.

5. Results Discussion

To generalize the notion of different space forms, Alegre et al. (2004) have established the notion of generalized Sasakian space-forms and proved its existence by proving some non-trivial examples. Since then, the properties of generalized Sasakian-space-forms have been studied by many geometers, including Alegre et al. (2004), Alegre and Carriazo (2008, 2011), Chaubey and Yadav (2018), Chaubey and Yildiz (2019). Recently, Chaubey and Suh (2023) explored the properties of Ricci-Bourguignon solitons and the Fischer-Marsden Conjecture on generalized Sasakian-space-forms with β-Kenmotsu structure. To the best of our knowledge, the study of almost Ricci solitons and gradient almost Ricci solitons are not studied on generalized Sasakian-space-forms with β-Kenmotsu structure. This manuscript is dedicated to filling this gap, and we explored the geometrical properties of a class of almost contact metric manifolds endowed with almost Ricci solitons and gradient almost Ricci solitons. We have derived the sufficient condition for which almost Ricci solitons on $M(f_1, f_2, f_3)$ to be Ricci solitons, which has been used a tool to solve one of the millennium problems, named Poincaré Conjecture proposed in 1904. We established the relations between the smooth functions f_1, f_2 and f_3 of $M(f_1, f_2, f_3)$, which may be used in the classification of generalized Sasakian space forms. We noticed that $M(f_1, f_2, f_3)$ holds a set of partial differential equations, including the Poisson equation, which has many applications in engineering, and allied areas. The sufficient conditions for which the solitons on $M(f_1, f_2, f_3)$ to be expanding, shrinking, or steady are established. In this sequel, it is proved that $M(f_1, f_2, f_3)$ endowed with gradient almost Ricci solitons is either cosymplectic manifold (play a central role in mathematical physics) or the gradient of the soliton function is pointwise colinear with the Reeb vector field of $M(f_1, f_2, f_3)$. This work will be helpful to researchers working in the area of geometric flows and their solutions. The results of this manuscript may be considered as a basic result for further study of generalized Sasakian-space-forms with β-Kenmotsu structure.

6. Conclusion

The Ricci flow, often known as Hamilton's Ricci flow, is a specific partial differential equation for a Riemannian metric in the mathematical domains of differential geometry, geometric analysis, and mathematical physics. Because of formal parallels in the equation's mathematical structure, it is sometimes compared to the diffusion of heat and heat equation. It is nonlinear, though, and displays a number of phenomena that are absent from the analysis of the heat equation. A self-similar solution of Ricci flow equation is termed as Ricci soliton Equation (1). Ricci flows and solitons have been used to address many long-standing problems of science, technology and applied areas. For example, Poincaré conjecture (one of the millennium problems announced by the Clay Mathematical Institute), differential sphere conjecture, Willium Thurston’s conjecture, etc. Surface parameterization, surface matching, manifold splines, and the creation of geometric structures on generic surfaces are just a few of the many uses for Ricci flow in graphics, geometric modeling, and medical imaging. Due to its various applications in different eras, it attracts researchers to do research in this area. If we consider the soliton constant as a smooth function in the Ricci soliton Equation (1), then we recover the expression of almost Ricci soliton equation. Noted that the Ricci flows and their solutions are capable to solve many long standings as well as new challenges of science, technology and medical science.

Conflict of Interest

The authors declare no conflict of interest.

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