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Construction of a Class of Higher-Order Iterative Techniques and its Convergence in Banach Spaces

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Abstract
We propose a novel sixth-order convergence scheme for solving scalar equations, based on the weight function approach. This approach provides us with the flexibility to construct new iterative techniques with the same level of convergence. In addition, we extend the same idea to nonlinear systems of equations with the help of Banach space operators. Further, a challenging semi-local convergence analysis is conducted to establish the theoretical foundation of the scheme. We also demonstrate the applicability and efficiency of the scheme by applying it to three problems in applied science: an integral equation, a boundary value problem (BVP), and the two-dimensional Burger's equation. Our scheme not only achieves smaller absolute residual errors, reduced differences between successive iterations, and requires fewer iterations to reach the desired accuracy compared to existing methods, but it also demonstrates lower CPU time consumption and stable convergence order. Finally, we conclude that our scheme exhibits superior efficiency and compatibility compared to existing methods of the same convergence order.

Keywords- Nonlinear systems, Newton's technique, Banach spaces, Convergence order.

1. Introduction

Mathematical modeling plays not only a crucial role in a variety of mathematical applications but also in physics, economics, chemistry, engineering, statistics, and applied science disciplines (Cercignani, 1988; Grosan and Abraham, 2008). These models help us describe and predict real-world phenomena by transforming them into mathematical equations. Many such models can be simplified into the following system of nonlinear equations (SNES) (after some simplifications and analyses)

$$\mathcal{F}(x) = 0 \tag{1}$$

where, $\mathcal{F}: \Omega \subseteq \mathrm{X} \rightarrow \mathrm{Y}$ is a nonlinear operator and Ω is an open convex subset of a Banach space (BS) X. The images of F in a BS Y. Analytical or closed-form solutions are not always possible for such problems. In this context, iterative methods play a crucial role in estimating solutions through approximation. Newton's method is one of the most popular methods, and its iteration is as follows:

$$x_{k+1} = x_k - \Gamma_k^{-1}\mathcal{F}(x_k), \quad k = 0,1,2, \dots \tag{2}$$

where, $\Gamma_k = \Gamma_k$, x_0 being the starting guess. The inverse of Γ_k ($\Gamma_k^{-1} \in \mathcal{L}(\mathrm{Y}, \mathrm{X})$), is a collection of bounded linear operators from Y into X. Over the past few years, researchers have suggested numerous higher-order iterative methods to address both scalar equations and SNES (Abbasbandy et al., 2016; Deep and Argyros, 2023; Grau-Sánchez et al., 2011a; Grau-Sánchez et al., 2011b; Hueso at el., 2015; Kou et al., 2007; Lotfi et al., 2015; Sharma and Bahl, 2021; Sharma and Gupta, 2014; Wang and Li, 2017; Xiao and Yin, 2016). In the recent years, some scholars also proposed iterative methods for SNES in (Cordero et al., 2024; Cordero et al., 2025; Kumar et al., 2025). The introduction of 6th-order iterative techniques utilizing weight functions has also been a recent trend (Behl et al., 2019; Behl and Argyros, 2020).

Motivated by ongoing advancements in this field, our objective is to develop a general class of 6th-order methods for solving scalar equations and extend this framework to SNES using operators in Banach spaces (BSs). The novelty of the article lies in the fact that this process leads to obtain a priori error estimates, existence and uniqueness of solution and R-order using more interesting semilocal convergence. Our scheme Equation (15) achieves smaller residual errors, reduced differences between iterations, and stable convergence, while requiring the same number of iterations as existing methods. Though Lotfi et al. (2015)'s method consumes less CPU time, our method outperforms in terms of residual errors and iteration differences, demonstrating superior efficiency and compatibility. The recurrence relations (RR) technique and Lipschitz conditions are used in this study to establish the semilocal convergence of Equation (3) for solving a SNES. Numerical experiments are conducted on a variety of nonlinear equations, including Burger's equation, boundary value problems, and integral equations.

The structure of the paper is organized as follows: In Section 2, we introduce a novel 6th-order family for scalar equations and its convergence analysis. In Section 3, the 6th-order family is extended to SNES within Banach spaces (BSs). Additionally, a semilocal convergence analysis is conducted, demonstrating the theoretical foundation and applicability of the proposed method in this generalized setting. Section 4 is dedicated to numerical experiments, where the proposed methods are applied to various real-life science problems, including boundary value problems, integral equations, and Burger's equation. The results are analyzed to validate the convergence behavior of the iterative schemes. Finally, Section 5 provides concluding remarks and summarizes the findings.

1.1 Some Basic Definitions

There are many convergence criteria for iterative methods. Two main convergence approaches are given below:

1) Local convergence: The local convergence analysis (Argyros et al., 2020; Sharma and Deep, 2023; Sharma et al., 2025) provides the bounds on the radius of convergence which is based on the required solution.

2) Semi-local convergence: Semilocal analysis focuses on determining the sufficient conditions, based on the information near the initial point, that guarantee the convergence of the given method (more details can be found in Wang et al. (2011) and Zheng and Gu (2012)). Usually, there are two ways for the semi-local convergence, which are given below:

- **Recurrence relations:** Rall proposed a different approach using RR to ensure method convergence, which has been successfully applied in many contributions (see Wang et al. (2011), Zheng and Gu (2012) and references therein). This technique involves generating a sequence of $x \in \mathbb{R}^+$ ($\mathbb{R}^+$ is set of positive real numbers) that produces an appropriate convergence domain and guarantees the convergence of iterative techniques in Banach spaces (BSs).
- **Majorizing sequences:** Another approach to obtaining the convergence of the iterative technique's sequence is the convergence of majorizing sequences (Argyros et al., 2023).

In our study, we employed semi-local convergence analysis, one of the most rigorous and challenging approaches that guarantees the convergence of iterative methods. We adopted the recurrence relation approach to construct our scheme.

2. A Novel Scheme for Scalar Equation

In this part, the proposed family is formulated as follows:

$$\begin{cases} z_k = x_k - \mathcal{H}(r(x_k))\frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \mathcal{W}(r(x_k))\frac{f(z_k)}{f'(x_k)}, \end{cases} \tag{3}$$

where, $y_k = x_k - \theta\frac{f(x_k)}{f'(x_k)}$, $\theta \in \mathbb{R}$ and $f: \mathcal{D} \subset \mathbb{C} \to \mathbb{C}$ is a sufficiently differentiable function in domain $\mathcal{D}$ enclosing the simple root of a nonlinear equation $f(x) = 0$, $r(x_k) = \frac{f'(y_k)}{f'(x_k)}$. The two maps $\mathcal{H}: \mathbb{C} \to \mathbb{C}$ and $\mathcal{W}: \mathbb{C} \to \mathbb{C}$ are analytic in the surrounding of the point b, where $b = 1$. We can rewrite the $r(x_k)$ as: $r(x_k) = 1 + v$, where, $v = \frac{f'(y_k)-f'(x_k)}{f'(x_k)}$.

With the help of Taylor series expansion, the weight functions $\mathcal{H}(r)$ and $\mathcal{W}(r)$ can be expanded in the neighborhood of the point 1 as follows:

$$\mathcal{H}(r) \simeq \alpha_0 + \alpha_1 v + \frac{\alpha_2}{2!}v^2 + \frac{\alpha_3}{3!}v^3 \tag{4}$$

and

$$\mathcal{W}(r) \simeq \beta_0 + \beta_1 v + \frac{\beta_2}{2!}v^2 \tag{5}$$

where, $\alpha_0 = \mathcal{H}(b)$, $\beta_0 = \mathcal{W}(b)$, $\alpha_1 = \mathcal{H}'(b)$, $\beta_1 = \mathcal{W}'(b)$, $\alpha_2 = \mathcal{H}''(b)$, $\beta_2 = \mathcal{W}''(b)$ *and* $\alpha_3 = \mathcal{H}'''(b)$.

The next Theorem 2.1 illustrates the 6th-OC (OC stands for order of convergence) for the suggested class under specific conditions related to weight functions.

Theorem 2.1: With a simple zero x^* of $f(x) = 0$, let $f: \mathcal{D} \subseteq \mathbb{C} \to \mathbb{C}$ be a suitably differentiable function in $\mathcal{D}$. The iterative algorithm family given by Equation (3) achieves 6th-OC. For this, we have to choose the initial approximation x_0 in the close neighborhood of x^*. Further, the weight functions should satisfy the given conditions:

$$\alpha_0 = 1, \alpha_1 = \frac{-3}{4}, \alpha_2 = \frac{9}{4}, |\alpha_3| < \infty,$$
$$\beta_0 = 1, \beta_1 = \frac{-3}{2}, |\beta_2| < \infty.$$

The following error equation serving as a crucial tool for analyzing the convergence and accuracy of iterative methods, which is given below:

$$e_{k+1} = \frac{1}{729}(21870a_2^5 - 8019a_2^3a_3 + 729a_2a_3^2 + 486a_2^2a_4 - 81a_3a_4 + 1728a_2^5\alpha_3 - 288a_2^3a_3\alpha_3 - 3240a_2^5\beta_2 + 648a_2^3a_3\beta_2 - 72a_2^2a_4\beta_2 - 256a_2^5\alpha_3\beta_2)e_k^6) + O(e_k^7).$$

Proof: Assuming that $f'(x^*) \neq 0$ and developing $f(x_k)$ in the neighborhood of x^* using Taylor's series expansion leads to

$$f(x_k) = f'(x^*)\left[e_k + a_2e_k^2 + a_3e_k^3 + a_4e_k^4 + a_5e_k^5 + a_6e_k^6 + O(e_k^7)\right] \tag{6}$$

where, $a_i = \frac{f^{(i)}(x^*)}{i!f\prime(x^*)}$ for $i = 2,3, \ldots$ Also,

$$f'(x_k) = f'(x^*)\left[1 + (2a_2 + 3a_3e_k + 4a_4e_k^2 + 5a_5e_k^3 + 6a_6e_k^4)e_k + O(e_k^6)\right] \tag{7}$$

From first step of (5), we have

$$y_k = x^* + ((1-\theta) + \theta a_2e_k - 2\theta(a_2^2 - a_3)e_k)e_k + \theta(4a_2^3 - 7a_2a_3 + 3a_4)e_k^4 - 2\theta(4a_2^4 - 10a_2^2a_3 + 3a_3^2 + 5a_2a_4 - 2a_5)e_k^5 \quad +\theta(16a_2^5 - 52a_2^3a_3 + 33a_2a_3^2 + 28a_2^2a_4 - 17a_3a_4 - 13a_2a_5 + 5a_6)e_k^6 + O(e_k^7).$$

Taking $\theta = \frac{2}{3}$, we obtain

$$y_k = x^* + \frac{1}{3}e_k + \frac{2}{3}a_2e_k^2 - \frac{4}{3}(a_2^2 - a_3)e_k^3 + \frac{2}{3}(3a_4 + 4a_2^3 - 7a_2a_3)e_k^4 - \frac{4}{3}(4a_2^4 - 10a_2^2a_3 + 3a_3^2 + 5a_2a_4 - 2a_5)e_k^5 + \frac{2}{3}(16a_2^5 - 52a_2^3a_3 + 33a_2a_3^2 + 28a_2^2a_4 - 17a_3a_4 - 13a_2a_5 + 5a_6)e_k^6 + O(e_k^7) \tag{8}$$

Expanding $f'(y_k)$ about x^* and adopting Equation (8), we yield

$$f'(y_k) = \frac{2}{3}a_2e_k + \frac{1}{3}(4a_2^2 + a_3)e_k^2 + \frac{4}{27}(-18a_2^3 + 27a_2a_3 + a_4)e_k^3 + \frac{1}{81}(432a_2^4 - 864a_2^2a_3 + 216a_3^2 + 396a_2a_4 + 5a_5)e_k^4 + \frac{2}{81}(-432a_2^5 + 1080a_2^3a_3 - 486a_2a_3^2 - 540a_2^2a_4 + 234a_3a_4 + 236a_2a_5 + a_6)e_k^5 + O(e_k^6) \tag{9}$$

Then, we yield

$$r_k = \frac{f\prime(y_k)}{f\prime(x_k)} = \frac{-4}{3}a_2e_k + \frac{4}{3}(3a_2^2 - 2a_3)e_k^2 + \frac{8}{27}(-36a_2^3 + 45a_2a_3 - 13a_4)e_k^3 + \frac{4}{81}(540a_2^4 - 999a_2^2a_3 + 216a_3^2 + 363a_2a_4 - 100a_5)e_k^4 + \frac{4}{81}(-1296a_2^5 + 3186a_2^3a_3 - 1485a_2a_3^2 - 1320a_2^2a_4 + 567a_3a_4 + 453a_2a_5 - 121a_6)e_k^5 + O(e_k^6).$$

By Taylor's series for weight function $\mathcal{H}$ about $r_k = b$, where $b = 1$, we have

$$\mathcal{H}(r_k) = \alpha_0 + \alpha_1(r_k - b) + \alpha_2(r_k - b)^2 + \alpha_3(r_k - b)^3 + O(e_k^5)$$
$$= \alpha_0 - \frac{4}{3}a_2\alpha_1e_k + \frac{4}{9}(9a_2^2\alpha_1 - 6a_3\alpha_1 + 2a_2^2\alpha_2)e_k^2 + \frac{8}{81}(108a_2^3\alpha_1 - 135a_2a_3\alpha_1 + 39a_4\alpha_1 + 54a_2^3\alpha_2 - 36a_2a_3\alpha_2 + 4a_2^3\alpha_3)e_k^3 + \frac{4}{81}(540a_2^4\alpha_1 - 999a_2^2a_3\alpha_1 + 216a_3^2\alpha_1 + 363a_2a_4\alpha_1 - 100a_5\alpha_1 + 450a_2^4\alpha_2 - 576a_2^2a_3\alpha_2 + 72a_3^2\alpha_2 + 104a_2a_4\alpha_2 + 72a_2^4\alpha_3 - 48a_2^2a_3\alpha_3)e_k^4 + O(e_k^5) \tag{10}$$

Consequently, the second substep of Equation (3) yields

$$z_k - x^* = \left(\frac{1}{3} - \alpha_0\right)e_k + \left(\frac{2}{3}a_2 + a_2\alpha_0 + \frac{4}{3}a_2\alpha_1\right)e_k^2 + \frac{2}{9}(-6a_2^2 + 6a_3 - 9a_2^2\alpha_0 + 9a_3\alpha_0 - 24a_2^2\alpha_1 + 12a_3\alpha_1 - 4a_2^2\alpha_2)e_k^3 + \frac{1}{81}(216a_2^3 - 378a_2a_3 + 162a_4 + 324a_2^3\alpha_0 - 567a_2a_3\alpha_0 + 243a_4\alpha_0 + 1404a_2^3\alpha_1 - 1512a_2a_3\alpha_1 + 312a_4\alpha_1 + 504a_2^3\alpha_2 - 288a_2a_3\alpha_2 + 32a_2^3\alpha_3)e_k^4 + O(e_k^5).$$

Taking $\alpha_0 = 1, \alpha_1 = \frac{-3}{4}$ and $\alpha_2 = \frac{9}{4}$, above equation becomes

$$z_k - x^* = \frac{1}{81}(405a_2^3 - 81a_2a_3 + 9a_4 + 32a_2^3\alpha_3)e_k^4 + \frac{2}{81}(-1458a_2^4 + 1296a_2^2a_3 - 81a_3^2 - 90a_2a_4 + 12a_5 - 160a_2^4\alpha_3 + 96a_2^2a_3\alpha_3)e_k^5 + O(e_k^6) \quad (11)$$

Again, by Taylor's series for weight function $\mathcal{W}$ about $r_k = b$,

$$\mathcal{W}(r_k) = \mathcal{W}(b) + \mathcal{W}'(b)(r_k - b) + \mathcal{W}''(b)(r_k - b)^2 + O(e_k^5)$$
$$= \beta_0 - \frac{4}{3}a_2\beta_1e_k + \frac{4}{9}(9a_2^2\beta_1 - 6a_3\beta_1 + 2a_2^2\beta_2)e_k^2 + O(e_k^3) \quad (12)$$

Adopting Taylor's series of $f(z_k)$ about x^*, we yield

$$f(z_k) = \frac{1}{81}(405a_2^3 - 81a_2a_3 + 9a_4 + 32a_2^3\alpha_3)e_k^4 - \frac{2}{81}(1458a_2^4 - 1296a_2^2a_3 + 81a_3^2 + 90a_2a_4 - 12a_5 + 160a_2^4\alpha_3 - 96a_2^2a_3\alpha_3)e_k^5 + O(e_k^6) \quad (13)$$

Further, employing Equations (12) and (13) in the last step of Equation (3), we obtain

$$x_{k+1} = \frac{1}{81}(405a_2^3 - 81a_2a_3 + 9a_4 + 32a_2^3\alpha_3 - 405a_2^3\beta_0 + 81a_2a_3\beta_0 - 9a_4\beta_0 - 32a_2^3\alpha_3\beta_0)e_k^4 + \frac{2}{243}(-4374a_2^4 + 3888a_2^2a_3 - 243a_3^2 - 270a_2a_4 + 36a_5 - 80a_2^4\alpha_3 + 288a_2^2a_3\alpha_3 + 5589a_2^4\beta_0 - 4131a_2^2a_3\beta_0 + 243a_3^2\beta_0 + 297a_2a_4\beta_0 - 36a_5\beta_0 + 576a_2^4\alpha_3\beta_0 - 288a_2^2a_3\alpha_3\beta_0 + 810a_2^4\beta_1 - 162a_2^2a_3\beta_1 + 18a_2a_4\beta_1 + 64a_2^4\alpha_3\beta_1)e_k^5 + \frac{1}{729}(123930a_2^5 - 190998a_2^3a_3 + 48114a_2a_3^2 + 34182a_2^2a_4 - 5346a_3a_4 - 2430a_2a_5 + 378a_6 + 17856a_2^5\alpha_3 - 21312a_2^3a_3\alpha_3 + 3456a_2a_3^2\alpha_3 + 2496a_2^2\alpha_4 - 190998a_2^5\beta_0 + 251505a_2^3a_3\beta_0 - 53217a_2a_3^2\beta_0 - 37746a_2^2a_4\beta_0 + 5589a_3a_4\beta_0 + 2862a_2a_5\beta_0 - 378a_6\beta_0 - 24768a_2^5\alpha_3\beta_0 + 25632a_2^3a_3\alpha_3\beta_0 - 3456a_2a_3^2\alpha_3\beta_0 - 2496a_2^2a_4\alpha_3\beta_0 - 59292a_2^5\beta_1 + 45684a_2^3a_3\beta_1 - 3888a_2a_3^2\beta_1 - 2700a_2^2a_4\beta_1 + 216a_3a_4\beta_1 + 288a_2a_5\beta_1 - 5760a_2^5\alpha_3\beta_1 + 3072a_2^3a_3\alpha_3\beta_1 - 3240a_2^5\beta_2 + 648a_2^3a_3\beta_2 - 72a_2^2a_4\beta_2 - 256a_2^5\alpha_3\beta_2)e_k^6 + O(e_k^7).$$

For the family to be 6th-OC, we take $\beta_0 = 1$ and $\beta_1 = \frac{-3}{2}$. With these values, the above equation yields

$$x_{k+1} = \frac{1}{729}(21870a_2^5 - 8019a_2^3a_3 + 729a_2a_3^2 + 486a_2^2a_4 - 81a_3a_4 + 1728a_2^5\alpha_3 - 288a_2^3a_3\alpha_3 - 3240a_2^5\beta_2 + 648a_2^3a_3\beta_2 - 72a_2^2a_4\beta_2 - 256a_2^5\alpha_3\beta_2)e_k^6 + O(e_k^7) \quad (14)$$

This demonstrates the convergence of our scheme to the sixth order. As a result, the required conditions have been met, and the proof is complete.

3. Extension of Family of 6th-Order Method to BSs and its Convergence Analysis

Now, the extension of family given by Equation (3) presented in Section 2 to BSs is presented to solve Equation (1) preserving the OC as seen in scalar equations. The scheme in BSs is written as

$$\begin{cases} z_k = x_k - \mathcal{H}(r(x_k))\Gamma_k\mathcal{F}(x_k), \\ x_{k+1} = z_k - \mathcal{W}(r(x_k))\Gamma_k\mathcal{F}(z_k), \end{cases} \quad (15)$$

where, $y_k = x_k - \theta\Gamma_k\mathcal{F}(x_k), \quad \Gamma_k = \Gamma_k^{-1}, for\ k \in \mathbb{N},$

$$\mathcal{H}(r(x_k)) = \alpha_0 I + \alpha_1 v_k + \frac{1}{2!}\alpha_2 v_k^2 + \frac{1}{3!}\alpha_3 v_k^3,$$

and

$$\mathcal{W}(r(x_k)) = \beta_0 I + \beta_1 v_k + \frac{1}{2!}\beta_2 v_k^2.$$

where, $v_k = \Gamma_k[\mathcal{F}'(y_k) - \mathcal{F}'(x_k)]$ and I is identity operator on X.

3.1 Preliminary Results

At x_0, we have $\mathcal{F}'(x_0)^{-1} = \Gamma_0 \in \mathcal{L}(\mathrm{Y},\mathrm{X})$. In addition, we suppose that the inverse of $\mathcal{F}'$ occurs at some $x_0 \in \Omega$. The collection of linear operators from $\mathcal{Y}$ into $\mathcal{X}$ is known as $\mathcal{L}(\mathrm{Y},\mathrm{X})$. The following hypotheses are made:

$(C1)$ $\| \Gamma_0 \| \leq \beta$,
$(C2)$ $\| \Gamma_0 \mathcal{F}(x_0) \| \leq \eta$,
$(C3)$ $\| A_2(x) \| \leq M, x \in \Omega, where \quad A_2(x) = \mathcal{F}''(x)$,
$(C4)$ $\| A_3(x) \| \leq N, x \in \Omega, where \quad A_3(x) = \mathcal{F}'''(x)$,
$(C5)$ Then, a positive real number L exists such that

$$\| A_3(x) - A_3(y) \| \leq L \| x - y \|, \forall x, y \in \Omega \tag{16}$$

In the lemmas that follow, we begin by examining an approximation of the operator $\mathcal{F}$ which will be utilized in subsequent conclusions.

Lemma 3.1: Considering that $\mathcal{F}: \Omega \subset \mathrm{X} \to \mathrm{Y}$ is a continuously 3rd-order Fréchet differentiable nonlinear operator. In addition, we assume that Ω is an open convex set. Further, $\mathcal{X}$ and $\mathcal{Y}$ are two BSs. Then, the following item holds

$$\begin{aligned}\mathcal{F}(z_k) &= \tfrac{1}{2}A_3(x_k)(s_{x_k})^2(z_{s_k}) + \tfrac{1}{2}A_2(s_k)(z_{s_k})^2 + A_2(x_k)(s_{x_k})v_k^2\left[\tfrac{9}{8} + \tfrac{\alpha_3}{2}\left(\tfrac{v_k}{2} + \tfrac{1}{3}\right)\right](s_{x_k}) + \\ &\tfrac{1}{2}\int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^3(1-t)^2 dt - \tfrac{1}{3}\int_0^1 \left[A_3\left(x_k + \tfrac{2}{3}t(s_{x_k})\right) - A_3(x_k)\right](s_{x_k})^3(1 - \\ &t)dt + \int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^2(1-t)dt(z_{s_k}) + \int_0^1 [A_2(s_k + t(z_{s_k})) - \\ &A_2(s_k)](z_{s_k})^2(1-t)dt + \left[\tfrac{2}{3}\int_0^1 \left(A_2\left(x_k + \tfrac{2}{3}t(s_{x_k})\right) - A_2(x_k)\right)(s_{x_k})dt\right]\left(\tfrac{9}{8}v_k + \tfrac{1}{6}\alpha_3 v_k^2\right)(s_{x_k})\end{aligned} \tag{17}$$

where, $s_{x_k} = s_k - x_k$, $z_{s_k} = z_k - s_k$ and $s_k = x_k - \Gamma_k \mathcal{F}(x_k)$.

Proof: Utilizing the Taylor expansion, we get

$$\mathcal{F}(z_k) = \mathcal{F}(s_k) + \mathcal{F}'(s_k)(z_{s_k}) + \tfrac{1}{2}A_2(s_k)(z_{s_k})^2 + \int_0^1 [A_2(s_k + t(z_{s_k})) - A_2(s_k)](z_{s_k})^2(1-t)dt \tag{18}$$

Now, $z_{s_k} = (I - \mathcal{H}(r(x_k)))\Gamma_k \mathcal{F}(x_k) = \left((1-\alpha_0)I - (\alpha_1 v_k + \frac{1}{2!}\alpha_2 v_k^2 + \frac{1}{3!}\alpha_3 v_k^3)\right)\Gamma_k \mathcal{F}(x_k)$.

Taking $\alpha_0 = 1$, we obtain

$$z_{s_k} = -(\alpha_1 v_k + \tfrac{1}{2!}\alpha_2 v_k^2 + \tfrac{1}{3!}\alpha_3 v_k^3)\Gamma_k \mathcal{F}(x_k) = f_1(v_k)(s_{x_k}) \tag{19}$$

where,

$$f_1(v_k) = \alpha_1 v_k + \tfrac{1}{2!}\alpha_2 v_k^2 + \tfrac{1}{3!}\alpha_3 v_k^3 \tag{20}$$

Again by Taylor expansion and using the fact that $\mathcal{F}(x_k) = -\Gamma_k(s_{x_k})$, we yield

$$\mathcal{F}(s_k) = \frac{1}{2}A_2(x_k)(s_{x_k})^2 + \frac{1}{6}A_3(x_k)(s_{x_k})^3 + \frac{1}{2}\int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^3(1-t)^2 dt \quad (21)$$

$$\mathcal{F}'(s_k) = \Gamma_k + A_2(x_k)(s_{x_k}) + \frac{1}{2}A_3(x_k)(s_{x_k})^2 + \int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^2(1-t)dt \quad (22)$$

and

$$\mathcal{F}'(y_k) = \Gamma_k + \theta A_2(x_k)(s_{x_k}) + \frac{\theta^2}{2}A_3(x_k)(s_{x_k})^2 + \theta^2\int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](1-t)dt(s_{x_k})^2 \quad (23)$$

Substituting Equations (19), (22) and (23) in the second term on the R.H.S. of Equation (18), we attain

$$\begin{aligned}\mathcal{F}'(s_k)(z_{s_k}) &= (\mathcal{F}'(s_k) - \Gamma_k)(z_{s_k}) + \Gamma_k(z_{s_k}) \\ &= A_2(x_k)(s_{x_k})(z_{s_k}) + \frac{1}{2}A_3(x_k)(s_{x_k})^2(z_{s_k}) + \int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^2(1-t)dt(z_{s_k}) + (\mathcal{F}'(y_k) - \Gamma_k)\left(\alpha_1 + \frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k}) \\ &= A_2(x_k)(s_{x_k})(z_{s_k}) + \frac{1}{2}A_3(x_k)(s_{x_k})^2(z_{s_k}) + \int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^2(1-t)dt(z_{s_k}) + \alpha_1\theta A_2(x_k)(s_{x_k})^2 + \frac{\alpha_1\theta^2}{2}A_3(x_k)(s_{x_k})^3 + \alpha_1\theta^2\int_0^1 \big[A_3(x_k + t(s_{x_k})) - A_3(x_k)\big](s_{x_k})^3(1-t)dt + (\mathcal{F}'(y_k) - \Gamma_k)\left(\frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k})\end{aligned} \quad (24)$$

Substituting Equations (21) and (24) in Equation (18), we get

$$\mathcal{F}(z_k) = \left(\frac{1}{2} + \alpha_1\theta\right)A_2(x_k)(s_{x_k})^2 + \left(\frac{1}{6} + \frac{\alpha_1\theta^2}{2}\right)A_3(x_k)(s_{x_k})^3 + \frac{1}{2}\int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^3(1-t)^2dt + \int_0^1 [A_3(x_k + t(s_{x_k})) - A_3(x_k)](s_{x_k})^2(1-t)dt(z_{s_k}) + \alpha_1\theta^2\int_0^1 \big[A_3(x_k + t(s_{x_k})) - A_3(x_k)\big](s_{x_k})^3(1-t)dt + \int_0^1 [A_2(s_k + t(z_{s_k})) - A_2(s_k)](z_{s_k})^2\mathcal{F}(1-t)dt\frac{1}{2}A_3(x_k)(s_{x_k})^2(z_{s_k}) + \frac{1}{2}A_2(s_k)(z_{s_k})^2 + A_2(x_k)(s_{x_k})(z_{s_k}) + (\mathcal{F}'(y_k) - \Gamma_k)\left(\frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k}) \quad (25)$$

Taking $\alpha_1\theta + \frac{1}{2} = 0$ and $\frac{\alpha_1\theta^2}{2} + \frac{1}{6} = 0$, we obtain

$$\theta = \frac{2}{3} \quad \text{and} \quad \alpha_1 = \frac{-3}{4} \quad (26)$$

Now consider last two terms of Equation (25)

$$\begin{aligned}&A_2(x_k)(s_{x_k})(z_{s_k}) + (\mathcal{F}'(y_k) - \Gamma_k)\left(\frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k}) = A_2(x_k)(s_{x_k})\left[\frac{-3}{4}v_k + \frac{1}{2}\alpha_2 v_k^2 + \frac{1}{6}\alpha_3 v_k^3\right](s_{x_k}) + [A_2(x_k)(y_{x_k}) + \int_0^1 (A_2(x_k + t(y_{x_k})) - A_2(x_k))(y_{x_k})dt]\left(\frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k}) \\ &= \left(\frac{-3}{4} + \frac{\alpha_2}{3}\right)A_2(x_k)(s_{x_k})v_k(s_{x_k}) + A_2(x_k)(s_{x_k})v_k^2\left(\frac{\alpha_2}{2} + \frac{\alpha_3}{3}\left(\frac{v_k}{2} + \frac{1}{3}\right)\right)(s_{x_k}) + \left[\frac{2}{3}\int_0^1 (A_2(x_k + t(y_{x_k})) - A_2(x_k))(s_{x_k})dt\right]\left(\frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k})\end{aligned}$$

where, $y_{x_k} = y_{x_k}$.

Taking $\frac{-3}{4} + \frac{\alpha_2}{3} = 0$, we obtain $\alpha_2 = \frac{9}{4}$. Then

$$A_2(x_k)(s_{x_k})(z_{s_k}) + (\mathcal{F}'(y_k) - \Gamma_k)\left(\frac{1}{2}\alpha_2 v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k}) = A_2(x_k)(s_{x_k})v_k^2\left(\frac{9}{8} + \frac{\alpha_3}{3}\left(\frac{v_k}{2} + \frac{1}{3}\right)\right)(s_{x_k}) + \left[\frac{2}{3}\int_0^1 (A_2(x_k + t(y_{x_k})) - A_2(x_k))(s_{x_k})dt\right]\left(\frac{9}{8}v_k + \frac{1}{6}\alpha_3 v_k^2\right)(s_{x_k}) \quad (27)$$

Equation (17) renders easily by substituting Equation (27) in Equation (25).

Lemma 3.2: If the postulates of Lemma 3.1 are satisfied, then we yield

$$\mathcal{F}(x_{k+1}) = A_2(x_k)[f_1(v_k)\Gamma_k\mathcal{F}(x_k)\mathcal{W}(r(x_k)) + \Gamma_k\mathcal{F}(x_k)f_2(v_k)]\Gamma_k\mathcal{F}(z_k) - \frac{\beta_2}{2}\Gamma_k v_k^2\Gamma_k\mathcal{F}(z_k + \frac{1}{2}A_2(z_k)(x_{k+1} - z_k)^2 + \left[\int_0^1 \left(A_2(x_k + \frac{2}{3}t(s_{x_k})) - A_2(x_k)\right)(s_{x_k})dt\right]\Gamma_k\mathcal{F}(z_k) + \int_0^1 (A_2(x_k + t(z_k - x_k)) - A_2(x_k))(z_k - x_k)dt(x_{k+1} - z_k) + \int_0^1 (A_2(z_k + t(x_{k+1} - z_k)) - A_2(z_k))(1-t)dt(x_{k+1} - z_k)^2 \quad (28)$$

Proof: We obtain the following expression by adopting Taylor expansion

$$\mathcal{F}(x_{k+1}) = \mathcal{F}(z_k) + \mathcal{F}'(z_k)(x_{k+1} - z_k) + \frac{1}{2}A_2(z_k)(x_{k+1} - z_k)^2 + \int_0^1 (A_2(z_k + t(x_{k+1} - z_k)) - A_2(z_k))(1-t)dt(x_{k+1} - z_k)^2 \quad (29)$$

Taking $\beta_0 = 1$, we also notice that

$$x_{k+1} - z_k + \Gamma_k\mathcal{F}(z_k) = -\left(\beta_1 v_k + \frac{1}{2}\beta_2 v_k^2\right)\Gamma_k\mathcal{F}(z_k) \quad (30)$$

Now, using Taylor expansion, Equation (30) and taking $\beta_1 = \frac{-3}{2}$, we obtain

$$\begin{aligned}\mathcal{F}(z_k) + \mathcal{F}'(z_k)(x_{k+1} - z_k) &= \left(\mathcal{F}'(z_k) - \mathcal{F}'(x_k)\right)(x_{k+1} - z_k) + \Gamma_k(x_{k+1} - z_k + \Gamma_k\mathcal{F}(z_k)) \\ &= A_2(x_k)[f_1(v_k)\Gamma_k\mathcal{F}(x_k)\mathcal{W}(r(x_k)) + \Gamma_k\mathcal{F}(x_k)f_2(v_k)]\Gamma_k\mathcal{F}(z_k) \\ &\quad - \frac{\beta_2}{2}\Gamma_k v_k^2\Gamma_k\mathcal{F}(z_k) + \frac{1}{2}A_2(z_k)(x_{k+1} - z_k)^2 + \left[\int_0^1 \left(A_2(x_k + \frac{2}{3}t(s_{x_k})) - A_2(x_k)\right)dt(s_{x_k})\right]\Gamma_k\mathcal{F}(z_k) + \int_0^1 \left(A_2\left(x_k + t(z_k - x_k)\right) - A_2(x_k)\right)dt(z_k - x_k)(x_{k+1} - z_k)\end{aligned} \quad (31)$$

Substituting Equation (31) in (30), we get Equation (28).

Let us denote $|\alpha_3| = a$, and $|\beta_2| = b$. Next, some scalar functions are introduced which will be used in subsequent results.

$$g_{a,b}(t) = \frac{1}{162}[162 + 81t + 81t^2 + 8at^3 + t\left(1 + t + \frac{2bt^2}{9}\right) \times \left(81 + 81t + 8at^2 + \frac{1}{324}(162 + 81t + 81t^2 + 8at^3)\right)] \quad (32)$$

$$h_{a,b}(t) = \frac{1}{1 - tg_{a,b}(t)} \quad (33)$$

$$p_1(t) = 1 + \frac{t}{2} + \frac{t^2}{2} + \frac{4at^3}{81} \quad (34)$$

$$p_2(t) = 1 + t + \frac{2bt^2}{9} \quad (35)$$

$$p_3(t) = \frac{t}{2} + \frac{t^2}{2} + \frac{4at^3}{81} \quad (36)$$

$$p_4(t) = t + \frac{2bt^2}{9} \quad (37)$$

$$f_{a,b}(t_1, t_2, t_3) = \frac{t_1^3}{2} + \frac{4at_1^3(t_1+1)}{81} + \frac{t_1 t_2}{3} + \frac{8at_1^2 t_2}{243} + \frac{5t_3}{18} + \frac{(t_2+t_3)p_3(t_1)}{2} + \frac{t_1 p_3(t_1)^2}{2} + \frac{t_2 p_3(t_1)^3}{2} \quad (38)$$

$$\phi_{a,b}(t_1, t_2, t_3) = f_{a,b}(t_1, t_2, t_3)\left(\frac{2bt_1^2}{9} + t_1(p_3(t_1)p_2(t_1) + p_4(t_1)) + \frac{t_1 p_2(t_1)^2}{2} + \frac{2t_2}{3} + t_2 p_1(t_1)^2 p_2(t_1) + \frac{t_2 p_2(t_1)^3 f_{a,b}(t_1, t_2, t_3)}{2}\right) \quad (39)$$

Using the notations $\eta_0 = \eta, \delta_0 = \delta, a_0 = M\delta\eta, e_0 = N\delta\eta^2, c_0 = L\delta\eta^3$, the subsequent sequences are defined for $k \geq 0$

$$\delta_{k+1} = \delta_k h_{a,b}(a_k) \tag{40}$$

$$\eta_{k+1} = \eta_k h_{a,b}(a_k)\phi_{a,b}(a_k, e_k, c_k) \tag{41}$$

$$a_{k+1} = a_k h_{a,b}(a_k)^2 \phi_{a,b}(a_k, e_k, c_k) \tag{42}$$

$$e_{k+1} = e_k h_{a,b}(a_k)^3 \phi_{a,b}(a_k, e_k, c_k)^2 \tag{43}$$

$$c_{k+1} = c_k h_{a,b}(a_k)^4 \phi_{a,b}(a_k, e_k, c_k)^3 \tag{44}$$

for $k = 0,1, \ldots.$

Let $z_{a,b}(t) = tg_{a,b}(t) - 1$. Since $z(0) = -1$, that means we have atleast one positive root. Let ρ represent the lowest positive zero of $tg_{a,b}(t) - 1$.

Lemma 3.3: Suppose that the real functions $g_{a,b}$, $h_{a,b}$ and $\phi_{a,b}$ are defined as in Equations (32), (33) and (39), respectively. Then,

(i) The functions $g_{a,b}(t)$ and $h_{a,b}(t)$ are both increasing, and greater than 1 $\forall t \in (0, \rho)$,
(ii) The function $\phi_{a,b}(t_1, t_2, t_3)$ is increasing for $t_1 \in (0, \rho), t_2 > 0, t_3 > 0$.

Proof: The proof is obvious.

Lemma 3.4: Suppose that the real functions $g_{a,b}$, $h_{a,b}$ and $\phi_{a,b}$ are defined as in Equations (32), (33) and (39), respectively. If $0 < a_0 < \rho$ and

$$h_{a,b}(a_0)^2 \phi_{a,b}(a_0, e_0, c_0) < 1 \tag{45}$$

then for all $k \geq 0$, we have
(i) $h_{a,b}(a_k) > 1$ and $h_{a,b}(a_k)\phi_{a,b}(a_k, e_k, c_k) < 1$,
(ii) the sequences $\{\eta_k\}, \{a_k\}, \{e_k\}, \{c_k\}$ and $\{h_{a,b}(a_k)\phi_{a,b}(a_k, e_k, c_k)\}$ are decreasing,
(iii) $a_k g_{a,b}(a_k) < 1$ and $h_{a,b}(a_k)^2 \phi_{a,b}(a_k, e_k, c_k) < 1 \quad for \quad k \geq 0$.

Proof: Lemma 3.3 and Equation (45) imply that $h_{a,b}(a_0) > 1$ and $h_{a,b}(a_0)\phi_{a,b}(a_0, e_0, c_0) < 1$. It is also clear from relations given by Equations (41-45) that $\eta_1 < \eta_0$, $a_1 < a_0$, $e_1 < e_0$ and $c_1 < c_0$. Also, $h_{a,b}(a_1)\phi_{a,b}(a_1, e_1, c_1) < h_{a,b}(a_0)\phi_{a,b}(a_0, e_0, c_0)$ and thus (ii) holds true for $k = 0$. As a result it implies that $a_1 g(a_1) < a_0 g(a_0) < 1$ and $h_{a,b}(a_1)^2 \phi_{a,b}(a_1, e_1, c_1) < h_{a,b}(a_0)^2 \phi_{a,b}(a_0, e_0, c_0) < 1$, which proves the third part for $k = 0$. Using first part of Lemma 3.3 and induction, the Lemma 3.4 is verified for all $k \geq 0$.

Lemma 3.5: Assume the maps $g_{a,b}, h_{a,b}, \phi_{a,b}$ are defined as in Equations (32), (33) and (39), respectively.In addition, we consider $\theta \in (0,1)$, then for $t \in (0, \rho)$, $g_{a,b}(\theta t) < g_{a,b}(t)$, $h_{a,b}(\theta t) < h_{a,b}(t)$, $p_1(\theta t) < p_1(t)$, $p_2(\theta t) < p_2(t)$, $p_3(\theta t) < \theta p_3(t)$, $p_4(\theta t) < \theta p_4(t)$, $f_{a,b}(\theta t_1, \theta^2 t_2, \theta^3 t_3) < \theta^3 f_{a,b}(t_1, t_2, t_3)$ *and* $\phi_{a,b}(\theta t_1, \theta^2 t_2, \theta^3 t_3) < \theta^5 \phi_{a,b}(t_1, t_2, t_3)$.

Proof: For the values $\theta \in (0,1)$ and $t \in (0, \rho)$, by using Equations (32) - (39), the Lemma 3.5 follows easily.

Lemma 3.6: By adopting the Lemma 3.4, suppose $\gamma = h_{a,b}(a_0)\phi_{a,b}(a_0 e_0, c_0)$, $\Delta = 1/h_{a,b}(a_0)$. Then, we have

$$h_{a,b}(a_k)\phi_{a,b}(a_k, e_k, c_k) \le \Delta\gamma^{6^k}, k \ge 0 \tag{46}$$

and

$$\prod_{i=0}^{k} h_{a,b}(a_i)\phi_{a,b}(a_i, e_i, c_i) \le \Delta^{k+1}\gamma^{\frac{6^{k+1}-1}{5}} \tag{47}$$

Proof: Since

$$a_1 = a_0 h_{a,b}(a_0)^2 \phi_{a,b}(a_0, e_0, c_0) = \gamma a_0,$$

$$e_1 = e_0 h_{a,b}(a_0)^3 \phi_{a,b}^2(a_0, e_0, c_0) < \gamma^2 e_0$$

and

$$c_1 = c_0 h_{a,b}(a_0)^4 \phi_{a,b}^3(a_0, e_0, c_0) < \gamma^3 c_0.$$

Now

$$\begin{aligned} h_{a,b}(a_1)\phi_{a,b}(a_1, e_1, c_1) &< h_{a,b}(\gamma a_0)\phi_{a,b}(\gamma a_0, \gamma^2 e_0, \gamma^3 c_0) \\ &< \gamma^5 h_{a,b}(a_0)\phi_{a,b}(a_0, e_0, c_0) = \gamma^{6^1-1} h_{a,b}(a_0)\phi_{a,b}(a_0, e_0, c_0) = \Delta\gamma^{6^1}. \end{aligned}$$

Suppose for $i \ge 1$, $h_{a,b}(a_i)\phi_{a,b}(a_i, e_i, c_i) \le \Delta\gamma^{6^i}$. Then, above lemma renders the following

$$\begin{aligned} &h_{a,b}(a_{i+1})\phi_{a,b}(a_{i+1}, e_{i+1}, c_{i+1}) \\ &\quad < h_{a,b}(a_i)\phi_{a,b}(a_i h_{a,b}(a_i)^2 \phi_{a,b}(a_i, e_i, c_i), e_i h_{a,b}(a_i)^3 \phi_{a,b}(a_i, e_i, c_i)^2, c_i h_{a,b}(a_i)^4 \phi_{a,b}(a_i, e_i, c_i)^3) \\ &\quad = \Delta\gamma^{6^{i+1}}. \end{aligned}$$

Thus $h_{a,b}(a_k)\phi_{a,b}(a_k, e_k, c_k) \le \Delta\gamma^{6^k}$ holds for all $k \ge 0$. Thus

$$\prod_{i=0}^{k} h_{a,b}(a_i)\phi_{a,b}(a_i, e_i, c_i) \le \prod_{i=0}^{k} \Delta\gamma^{6^i} \le \Delta^{k+1}\gamma^{\sum_{i=0}^{k} 6^i} \le \Delta^{k+1}\gamma^{\frac{6^{k+1}-1}{5}}.$$

Thus, Equation (47) is proved.

Lemma 3.7: Using the postulates of Lemma 3.4 and assuming $\gamma = h_{a,b}(a_0)^2\phi_{a,b}(a_0, e_0, c_0), \Delta = 1/h_{a,b}(a_0)$, the sequence $\{\eta_k\}$ satisfies,

$$\eta_k \le \eta\delta^k\gamma^{\frac{6^k-1}{5}}, k \ge 0 \tag{48}$$

and thus, the sequence $\{\eta_k\} \to 0$ and

$$\sum_{i=k}^{k+m-1} \eta_i \le \eta\Delta^k\gamma^{\frac{6^k-1}{5}} M \quad \forall \quad k \ge 0 \quad \text{and} \quad m \ge 1 \tag{49}$$

where, $M = \left(\frac{1-(\Delta\gamma^{6^k})^m}{1-\Delta\gamma^{6^k}}\right)$.

Proof: From Equations (41) and (46), we have

$$\begin{aligned}\eta_k &= \eta_{k-1} h_{a,b}(a_{k-1})\phi_{a,b}(a_{k-1}, e_{k-1}, c_{k-1})\\ &= \eta_{k-2} h_{a,b}(a_{k-2})\phi_{a,b}(a_{k-2}, e_{k-2}, c_{k-2}) h_{a,b}(a_{k-1})\phi_{a,b}(a_{k-1}, e_{k-1}, c_{k-1})\\ &= \cdots\\ &= \eta \textstyle\prod_{i=0}^{k-1} h_{a,b}(a_i)\phi_{a,b}(a_i, e_i, c_i)\\ &\leq \eta \Delta^k \gamma^{\frac{6^k-1}{5}}.\end{aligned}$$

Since $\delta < 1$ and $\gamma < 1$, it shows that $\eta_k \to 0$ as $k \to \infty$. Now

$$\begin{aligned}&\textstyle\sum_{i=k}^{k+m-1} \eta_i \leq \sum_{i=k}^{k+m-1} \eta \Delta^i \gamma^{\frac{6^i-1}{5}}\\ &\leq \eta \gamma^{\frac{6^k-1}{5}} \textstyle\sum_{i=0}^{m-1} \Delta^{k+i} (\gamma^{6^k})^{\frac{6^i-1}{5}} \leq \eta \Delta^k \gamma^{\frac{6^k-1}{5}} \sum_{i=0}^{m-1} \Delta^i (\gamma^{6^k})^i\\ &\leq \eta \Delta^k \gamma^{\frac{6^k-1}{5}} M.\end{aligned}$$

Using the fact that $\Delta < 1$ and $\gamma < 1$, a convergent series verifying:

$$\sum_{k=0}^{\infty} \eta_k \leq \frac{1}{1-\Delta\gamma}\eta$$

is obtained.

3.2 Recurrence Relations for Our Technique

It comprises of the derivation of RR for the scheme given by Equation (15) under the postulates taken in consideration in the previous section. Let $B(x,r)$ and $\overline{B(x,r)}$ defined as $\{y \in \mathcal{X}: \| y - x \| < r\}$ and $\{y \in \mathcal{X}: \| y - x \| \leq r\}$ respectively.

For $n = 0$, the existence of y_0, z_0 and s_0 is implied if Γ_0 exists and we have

$$\| s_0 - x_0 \| = \| \Gamma_0 \mathcal{F}(x_0) \| \leq \eta.$$

$$\| y_0 - x_0 \| = \frac{2}{3} \| \Gamma_0 \mathcal{F}(x_0) \| \leq \frac{2}{3}\eta.$$

Also, we get

$$\| v_0 \| \leq \| \Gamma_0 \| \| \mathcal{F}'(y_0) - \mathcal{F}'(x_0) \| \leq \frac{2}{3},$$

so that

$$\| z_0 - x_0 \| = \| \mathcal{H}(u(x_0)) \| \| \Gamma_0 \mathcal{F}(x_0) \| \leq \left(1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{4}{81}|\alpha_3| a_0^3\right)\eta = p_1(a_0)\eta \tag{50}$$

and

$$\begin{aligned}\| x_1 - z_0 \| &= \| \mathcal{W}(u(x_0)) \| \| \Gamma_0 \mathcal{F}(z_0) \| \leq \left(1 + a_0 + \frac{2}{9}|\beta_2| a_0^2\right) \| \Gamma_0 \mathcal{F}(z_0) \|\\ &= p_2(a_0) \| \Gamma_0 \mathcal{F}(z_0) \|\end{aligned} \tag{51}$$

Also, we have

$$\| z_0 - s_0 \| \leq \left(\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{4}{81}|\alpha_3| a_0^3\right)\eta = p_3(a_0)\eta \tag{52}$$

$$\| f_1(v_0) \| \leq p_3(a_0) \tag{53}$$

$$\| f_2(v_0) \| \leq p_4(a_0) \tag{54}$$

Lemma 3.1 and Lemma 3.2 render

$$\| \Gamma_0 \mathcal{F}(z_0) \| \leq f_{a,b}(a_0, e_0, c_0)\eta \tag{55}$$

and
$$\| \Gamma_0 \mathcal{F}(x_1) \| \leq \phi_{a,b}(a_0, e_0, c_0)\eta \tag{56}$$

Again, by Taylor expansion, we have
$$\mathcal{F}(z_0) = \mathcal{F}(x_0) + \mathcal{F}'(x_0)(z_0 - x_0) + \int_0^1 \left[\mathcal{F}'\left(x_0 + t(z_0 - x_0)\right) - \mathcal{F}'(x_0)\right]$$
$$= \mathcal{F}(x_0) + \mathcal{F}'(x_0)(s_0 - x_0) + \mathcal{F}'(x_0)(z_0 - s_0) + \int_0^1 \left[\mathcal{F}'\left(x_0 + t(z_0 - x_0)\right) - \mathcal{F}'(x_0)\right]dt(z_0 - x_0) \tag{57}$$

then from Equations (50) and (52), we obtain
$$\| \Gamma_0 \mathcal{F}(z_0) \| \leq \left(p_3(a_0) + \frac{1}{2}a_0 p_1^2(a_0)\right)\eta \tag{58}$$

From Equation (51), we get
$$\| x_1 - z_0 \| \leq p_2(a_0)\left(p_3(a_0) + \frac{1}{2}a_0 p_1^2(a_0)\right)\eta \tag{59}$$

From Equations (50) and (59), we have
$$\| x_1 - x_0 \| \leq \| x_1 - z_0 \| + \| z_0 - x_0 \| \leq g_{a,b}(a_0)\eta \tag{60}$$

From the previous section supposition $h_{a,b}(a_0)^2\phi_{a,b}(a_0, e_0, c_0) < 1$, hence it implies $x_1 \in B(x_0, R\eta)$. Now

$$\| I - \Gamma_0 \mathcal{F}'(x_1) \| \leq \| \Gamma_0 \| \| \mathcal{F}'(x_0) - \mathcal{F}'(x_1) \| \leq a_0 g_{a,b}(a_0) < 1.$$

According to the Banach lemma, the inverse of $\mathcal{F}'(x_1)$ (Γ_1) possible and
$$\| \Gamma_1 \| \leq \frac{\delta_0}{1 - a_0 g_{a,b}(a_0)} = h_{a,b}(a_0)\delta_0 = \delta_1 \tag{61}$$

Hence, we obtain
$$\| s_1 - x_1 \| = \| \Gamma_1 \mathcal{F}(x_1) \| \leq h_{a,b}(a_0)\phi_{a,b}(a_0, e_0, c_0)\eta_0 = \eta_1 \tag{62}$$

Since $g_{a,b}(a_0) > 1$, we find that
$$\| s_1 - x_0 \| \leq \| x_1 - x_0 \| + \| s_1 - x_1 \| \leq R\eta \tag{63}$$

which shows that s_1, y_1 lies in $B(x_0, R\eta)$. Thus, the following lemma can be obtained by induction.

Lemma 3.8: With considerations of Lemma 3.2 and hypotheses C1–C5 satify, then the following relations are valid for all $k \geq 0$:

(I) There exists γ_k and $\| \gamma_k \| \leq h_{a,b}(a_{k-1}) \| \gamma_{k-1} \|$
(II) $\| \gamma_k \mathcal{F}(x_k) \| \leq \eta_k$
(III) $M \| \Gamma_k \| \| \gamma_k \mathcal{F}(x_k) \| \leq a_k$
(IV) $N \| \Gamma_k \| \| \gamma_k \mathcal{F}(x_k) \|^2 \leq e_k$
(V) $L \| \Gamma_k \| \| \gamma_k \mathcal{F}(x_k) \|^3 \leq c_k$
(VI) $\| x_{k+1} - x_k \| \leq g_{a,b}(a_k)\eta_k$
(VII) $\| x_{k+1} - x_0 \| \leq R\eta$, where $R = \frac{g_{a,b}(a_0)}{1 - h_{a,b}(a_0)\phi_{a,b}(a_0, e_0, c_0)}$.

Proof: The proof of (I) − (VI) follows easily using an inductive procedure and formerly described development. In order to prove (VII), using (VI) and Lemma 3.7, we have

$$\| x_{k+1} - x_0 \| \leq \sum_{i=0}^{k} \| x_{i+1} - x_i \| \leq \sum_{i=0}^{k} g_{a,b}(a_i)\eta_i$$
$$\leq g_{a,b}(a_0) \textstyle\sum_{i=0}^{k} \eta_i \leq R\eta.$$

3.3 Semi-Local Convergence (SC)

This section comprises of SC establishment of the method given by Equation (3) along with domain of existence-uniqueness. Additionally, the priori error bounds of the solution are also depicted.

Theorem 3.1: Let $\mathcal{F}$ denote a nonlinear operator as defined in Equation (1). This operator is continuously differentiable up to the 3rd-order within an open subset Ω_0 that includes x_0 and postulates $(C_1) - (C_5)$ hold. Considering $a_0 = M\delta\eta, e_0 = N\delta\eta^2$ and $c_0 = L\delta\eta^3$ such that a_0 lies in $(0, \rho)$, ρ being the smallest positive zero of $tg_{a,b}(t) - 1$ and $g_{a,b}, h_{a,b}$ and $\phi_{a,b}$ are defined by Equations (32), (33) and (39), respectively.

Let $\overline{B(x_0, R\eta)} \in \Omega$ where $R = \frac{g_{a,b}(a_0)}{1-h_{a,b}(a_0)\phi_{a,b}(a_0,e_0,c_0)}$ with $\Delta = \frac{1}{h_{a,b}(a_0)}$ and $\gamma = h_{a,b}^2(a_0)\phi_{a,b}(a_0, e_0, c_0)$.

Then the sequence $\{x_k\}$, initializing from x_0, obtained by scheme (3) converges to a required solution x^*. For any $a, b \in \mathbb{R}^+$, the R-OC of (3) is at least six. The iterates x_k, y_k, z_k, x^* lie in $\overline{B(x_0, R\eta)}$. In addition, the solution x^* is unique and lies in $B(x_0, \frac{2}{M\delta} - R\eta) \cap \Omega$.

A priori error estimate is given as

$$\| x_k - x_0 \| \leq g_{a,b}(a_0)\Delta^k \gamma^{\frac{6^k-1}{5}} \frac{1}{1-\Delta\gamma^{6^k}}\eta \tag{64}$$

Proof: We shall demonstrate that $\{x_k\}$ is a Cauchy sequence. By adopting Lemma 3.3 and Lemma 3.4, it follows that $g_{a,b}(a_k) \leq g_{a,b}(a_n)$ for all $k \leq n$. Thus, we have

$$\| x_{n+k} - x_k \| \leq \textstyle\sum_{i=k}^{k+n-1} \| x_{i+1} - x_i \| \leq \sum_{i=k}^{k+n-1} g_{a,b}(a_i)\eta_i$$
$$\leq g_{a,b}(a_0) \textstyle\sum_{i=k}^{k+n-1} \eta_i \leq \Delta^k \gamma^{\frac{6^k-1}{5}} \frac{1-(\Delta\gamma^{6^k})^n}{1-\Delta\gamma^{6^k}} \eta_0 \tag{65}$$

so that, $\{x_k\}$ is a cauchy sequence. Therefore, it reaches to the required solution x^*. By taking $n \to \infty$ we obtain the estimation:

$$\| x_k - x_0 \| \leq g_{a,b}(a_0)\Delta^k \gamma^{\frac{6^k-1}{5}} \frac{1}{1 - \Delta\gamma^{6^k}}\eta.$$

Taking $k = 0$ in Equation (65) and $n \to \infty$, we have

$$\| x^* - x_k \| \leq R\eta_0.$$

Then, $x^* \in \overline{B(x_0, R\eta_0)}$.

Firstly, we illustrate that x^* is a solution of (1). Since
$\| \Gamma_n \| \leq \| \Gamma_n - \Gamma_0 \| + \| \Gamma_0 \| \leq M \| x_n - x_0 \| + \| \Gamma_0 \| \leq MR\eta_0 + \| \Gamma_0 \|,$

and boundedness of Γ_n and convergence of $\| \Gamma_n \mathcal{F}(x_n) \|$ to 0 follow immediately. This implies that

$$\| \mathcal{F}(x_n) \| \leq \| \mathcal{F}'(x_n) \| \| \Gamma_n \mathcal{F}(x_n) \| \to 0,$$

and, by the continuity of $\mathcal{F}$, we get that x^* is a solution of (1).

For uniqueness of x^*, firstly we see that $y^* \in B\left(x_0, \frac{2}{M\delta} - R\eta\right) \cap \Omega$, by Lemma

$$R\eta < \frac{1}{a_0}\eta < \left(\frac{2}{a_0} - R\right)\eta = \frac{2}{M\delta} - R\eta,$$

and $B\left(x_0, \frac{2}{M\delta} - R\eta\right) \cap \Omega \supseteq \overline{B(x_0, R\eta)}$.

Let $y^* \in B\left(x_0, \frac{2}{M\delta} - R\eta\right) \cap \Omega$ is another zero of $\mathcal{F}(x)$. By Taylor theorem, we have

$$\int_0^1 \mathcal{F}'(x^* + t(y^* - x^*))dt(y^* - x^*) = \mathcal{F}(y^*) - \mathcal{F}(x^*) = 0 \tag{66}$$

As

$$\begin{aligned} \| \Gamma_0 \| \| \int_0^1 [\mathcal{F}'(x^* + t(y^* - x^*)) - \mathcal{F}'(x_0)]dt \| &\leq M\delta \int_0^1 \| x^* - x_0 + t(y^* - x^*) \| dt \\ &\leq M\delta \int_0^1 (t \| y^* - x_0 \| + (1-t) \| x^* - x_0 \|)dt \\ &< \frac{M\delta}{2}\left(R\eta + \frac{2}{M\delta} - R\eta\right) = 1 \end{aligned} \tag{67}$$

As a result of Banach's lemma, it can be inferred that $\left(\int_0^1 (\mathcal{F}'(x^* + t(y^* - x^*))dt\right)^{-1}$ exists and which confirm the $x^* = y^*$.

4. Numerical Results

Here, we conducted a computational analysis to illustrate their practical relevance. We compared our scheme with existing 6th-order methods, selecting the following approaches for evaluation: method given by Equation (8) from Abbasbandy et al. (2016), method given by Equation (14) from Hueso et al. (2015), and method given by Equation (6) from Wang and Li (2017). Additionally, we included method given by Equation (5) proposed by Lotfi et al. (2015) in our comparison.

To conduct our computational examination, we selected three problems. In Example 4.1, we addressed a nonlinear integral equation (NIE), with the numerical outcomes depicted in **Table 1**. The numerical findings for the SC of the boundary value problem (BVP) discussed in Example 4.2 and computational outcomes are mentioned in **Table 2**. Finally, we analyzed the well-known Burgers' equation in Example 4.3, with the corresponding numerical findings mentioned in **Table 3**.

In addition, we mentioned the computational OC (COC), which has been computed using the following formulas:

$$\xi = \frac{\ln \frac{\|e_{\iota+1}\|}{\|e_\iota\|}}{\ln \frac{\|e_\iota\|}{\|e_{\iota-1}\|}}, \quad e_\iota = x_\iota - x_*, \quad for \iota = 1,2, \ldots$$

or approximated COC ($ACOC$) Grau-Sánchez et al. 2011(a and b) by:

$$\xi^* = \frac{\ln\frac{\|x_{\iota+1}-x_\iota\|}{\|x_\iota - x_{\iota-1}\|}}{\ln\frac{\|x_\iota - x_{\iota-1}\|}{\|x_{\iota-1}-x_{\iota-2}\|}}, \quad for \iota = 2,3,\ldots$$

The termination criteria for the programming process are as follows: (i) $\| x_{\iota+1} - x_\iota \| < \epsilon$, and (ii) $\| F(x_\iota) \| < \epsilon$, where $\epsilon = 10^{-150}$. All calculations were carried out using Mathematica 11 with multi-precision arithmetic. We used the following weight functions:

$$\begin{aligned} \mathcal{H}(r(x_k)) &= I - \frac{3}{4}v_k + \frac{9}{4}v_k^2, \\ \mathcal{W}(r(x_k)) &= I - \frac{3}{2}v_k. \end{aligned}$$

We can also select alternative weight functions, provided they satisfy the conditions outlined in the Lemma and Theorem. This approach allows us to generate numerous new 6th-order iterative schemes.

The following describes the features of the computer machine that is used for programming:

- Brand name: HP.
- Operating System Build: 19045.2006.
- Computer RAM: 8.00 GB.
- Processor: i7(Intel)
- Window Edition: Window 10, 64-bit.

Example 4.1: We adopt the succeeding NIE:

$$\mathcal{F}(x)(s) = x(s) - 1 + \frac{1}{2}\int_0^1 s\, cos(x(t))dt \tag{68}$$

where, $s \in [0,1], \mathcal{X} = \mathrm{C}[0,1]$, the space of continuous functions on $[0,1]$ and $x \in \Omega = B(0,2) \subset \mathcal{X}$. Here, we use max-norm,

$$\| x \| = \max_{s\in[0,1]} |x(s)|.$$

We can easily see that the 1st, 2nd and 3rd order Fréchet derivatives of $\mathcal{F} \in \Omega$. Therefore, we have

$$\| A_2(x) \| \leq \frac{1}{2} \equiv M, \quad x \in \Omega, \quad \| A_3(x) \| \leq \frac{1}{2} \equiv N, \quad x \in \Omega,$$
$$\| A_3(x) - A_3(y) \| \leq \frac{1}{2} \| x - y \|, \quad x, y \in \Omega.$$

Table 1. Comparison of error bounds for Example 4.1.

k	Method (15)	Wang et al. (2011)	Zheng and Gu (2012)
1	2.9×10^{-4}	8.0×10^{3}	4.0×10^{-4}
2	1.4×10^{-22}	4.5×10^{-10}	7.0×10^{-21}
3	1.2×10^{-128}	2.0×10^{-48}	1.6×10^{-120}

Our method (15) exhibits lower error bounds as compared to the existing method.

Choosing $x(t) = 4/3$ as initial approximation, then we obtain

$$\| \mathcal{F}(x_0) \| \leq \frac{1}{2}\cos\frac{4}{3}.$$

Also, $\| I - \mathcal{F}'(x_0) \| \leq \frac{1}{2}\sin\frac{4}{3}$, then by Banach lemma, Γ_0 exists and

$$\| \Gamma_0 \| \leq \frac{2}{2-\sin\frac{4}{3}} \equiv \delta.$$

and it follows that

$$\| \Gamma_0 \mathcal{F}(x_0) \| \leq \frac{\cos\frac{4}{3}}{2-\sin\frac{4}{3}} \equiv \eta.$$

Therefore, we have ,

$$a_0 = M\delta\eta \leq \frac{cos\frac{4}{3}}{(2-sin\frac{4}{3})^2},$$

$$e_0 = N\delta\eta^2 \leq \frac{cos^2\frac{4}{3}}{(2-sin\frac{4}{3})^3},$$

$$c_0 = L\delta\eta^3 \leq \frac{cos^3\frac{4}{3}}{(2-sin\frac{4}{3})^4}.$$

Then, we can compute

$$a_0 = 0.222571 \ldots < \rho = 0.508641 \ldots,$$

and

$$h^2(a_0)p_2(a_0, e_0, c_0) \simeq 0.00292569 < 1.$$

Thus, the postulates of Theorem 3.1 are satisfied and hence, using the theoretical results proved in Section 3, the solution x^* lies in $\overline{B}(x_0, R\eta)$, where $\overline{B}(x_0, R\eta) = \overline{B}(4/3, 0.301185 \ldots)$. It is also confirmed that solution is unique and lies in $B(4/3, 1.75494) \cap \Omega$. The error bounds comparison between the method in Equation (15) and the 6th-order methods by Wang et al. (2011) and Zheng and Gu (2012) is shown in **Table 1** which depicts the superiority of the proposed method.

Example 4.2: Boundary value problems (BVPs) are among the fundamental challenges in Mathematics, Physics, and Engineering (Sharma and Gupta, 2014). These problems involve solving differential equations that are subject to conditions set at specific points, typically at the boundaries of a given domain. For this reason, we selected the succeeding BVP (refer to (Kou et al., 2007) for further details):

$$u'' + a^2 u'^2 = -1 \tag{69}$$

with $u(0) = 0,\ u(1) = 1$. By partitioning the closed interval $[0,1]$ into ℓ segments, we obtain

$$u_0 = u(\gamma_0) = 0, u_1 = u(\gamma_1), u_2 = u(\gamma_2), \ldots, u_\ell = u(\gamma_\ell) = 1,$$

where, $\gamma_{\ell+1} = \gamma_\ell + h, h = \frac{1}{\ell}$.

By applying discretization approach, we get

$$u'_\tau = \frac{u_{\tau+1} - u_{\tau-1}}{2h}, \quad u''_\tau = \frac{u_{\tau-1} - 2u_\tau + u_{\tau+1}}{h^2}, \quad \tau = 1,2,3, \ldots, p-1,$$

which proceed to the following SNES

$$u_{\tau-1} - 2u_\tau + u_{\tau+1} + \frac{1}{8}(u_{\tau+1}^2 + u_{\tau-1}^2 - 2u_{\tau+1}u_{\tau-1}) + h^2 = 0 \tag{70}$$

with $\mu = \frac{1}{2}$. For $p = 111$, the SNES x^* given in Equation (70) mentioned above, the required solution is shown below:

$$\begin{aligned} x^* = (\ & 0.01520\cdots, 0.03027\cdots, 0.04520\cdots, 0.05999\cdots, 0.07465\cdots, 0.08917\cdots, \\ & 0.1035\cdots, 0.1178\cdots, 0.1319\cdots, 0.1459\cdots, 0.1598\cdots, 0.1735\cdots, 0.1871\cdots, \\ & 0.2006\cdots, 0.2139\cdots, 0.2272\cdots, 0.2403\cdots, 0.2533\cdots, 0.2661\cdots, 0.2789\cdots, \\ & 0.2915\cdots, 0.3040\cdots, 0.3164\cdots, 0.3286\cdots, 0.3407\cdots, 0.3528\cdots, 0.3647\cdots, \\ & 0.3765\cdots, 0.3881\cdots, 0.3997\cdots, 0.4111\cdots, 0.4225\cdots, 0.4337\cdots, 0.4448\cdots, \\ & 0.4558\cdots, 0.4667\cdots, 0.4774\cdots, 0.4881\cdots, 0.4986\cdots, 0.5091\cdots, 0.5194\cdots, \\ & 0.5296\cdots, 0.5398\cdots, 0.5498\cdots, 0.5597\cdots, 0.5695\cdots, 0.5792\cdots, 0.5888\cdots, \\ & 0.5983\cdots, 0.6076\cdots, 0.6169\cdots, 0.6261\cdots, 0.6352\cdots, 0.6442\cdots, 0.6530\cdots, \\ & 0.6618\cdots, 0.6705\cdots, 0.6791\cdots, 0.6875\cdots, 0.6959\cdots, 0.7042\cdots, 0.7124\cdots, \\ & 0.7205\cdots, 0.7284\cdots, 0.7363\cdots, 0.7441\cdots, 0.7518\cdots, 0.7594\cdots, 0.7669\cdots, \\ & 0.7743\cdots, 0.7816\cdots, 0.7889\cdots, 0.7960\cdots, 0.8030\cdots, 0.8100\cdots, 0.8168\cdots, \\ & 0.8236\cdots, 0.8302\cdots, 0.8368\cdots, 0.8433\cdots, 0.8497\cdots, 0.8560\cdots, 0.8622\cdots, \\ & 0.8683\cdots, 0.8743\cdots, 0.8802\cdots, 0.8861\cdots, 0.8918\cdots, 0.8975\cdots, 0.9031\cdots, \\ & 0.9085\cdots, 0.9139\cdots, 0.9192\cdots, 0.9245\cdots, 0.9296\cdots, 0.9346\cdots, 0.9396\cdots, \\ & 0.9445\cdots, 0.9493\cdots, 0.9540\cdots, 0.9586\cdots, 0.9631\cdots, 0.9675\cdots, 0.9719\cdots, \\ & 0.9761\cdots, 0.9803\cdots, 0.9844\cdots, 0.9884\cdots, 0.9924\cdots, 0.9962\cdots)^{tr} \end{aligned}$$

The (COC), iteration count, error differences across successive iterations, CPU time and absolute residual errors for Example 4.2 are illustrated in **Table 2**, with initial choice $x_0 = (0.9, 0.9 \overset{110}{\cdots}, 0.9)^{tr}$.

Table 2. Computational results of Example 4.2.

Methods	$\parallel \mathcal{F}(x_3) \parallel$	$\parallel x_4 - x_3 \parallel$	n	CPU Timing	ξ^*
Abbasbandy et al. (2016)]	1.0×10^{-187}	7.6×10^{-187}	3	274.018	6.2268
Lotfi et al. (2015)	8.5×10^{-146}	1.7×10^{-143}	3	110.052	5.4277
Hueso et al. (2015)	4.0×10^{-136}	3.2×10^{-135}	3	175.773	5.2326
Method (15)	4.2×10^{-197}	3.2×10^{-196}	3	155.117	6.2146

(Our scheme given by Equation (15) not only achieves smaller absolute residual errors, reduced differences between successive iterations, and requires same number of iterations to reach the desired accuracy compared to existing methods, but it also demonstrates stable convergence order. In addition, there is no doubt that method proposed by Lotfi et al. (2015) consuming lower CPU time consumption as compared to our and existing ones but absolute residual errors and reduced differences between successive iterations belong to our method. Finally, we conclude that our scheme exhibits superior efficiency and compatibility compared to existing methods of the same convergence order.)

Example 4.3: The two-dimensional Burger's equation is among the most known equations in Physics, Mathematics, Engineering and applied sciences. It describes the evolution of velocity fields over the time, incorporating both convection and diffusion processes. As a result, we examine the 2-D Burger's equation (Xiao and Yin, 2016) given by

$$\frac{\partial^2 p}{\partial^2 u} - \frac{\partial p}{\partial t} + p\frac{\partial p}{\partial u} + g(u,t) = 0,$$

where, $(u,t) \in [0,1] \times [0,1]$ and boundary hypotheses

$$p(0,t) = p(1,t) = 0, p(u,0) = 10(u^2 - u), p(u,1) = \frac{10}{e}(u^2 - u),$$

where, $g(u,t) = -10e^{-2t}[e^t(u^2 - u + 2) + 10u(2u^2 - 3u + 1)]$. The meaning of $p_{i,j} = p(u_i, t_j)$ is the estimated answer at the grid points of the mesh. Further, $\mathcal{M}$ and $\mathcal{N}$ represent the number of steps in the u-direction and t-direction, respectively, with corresponding step sizes h and k. By applying a finite difference discretization, we obtain a SNES derived from the given partial differential equation. Further, we utilize the central difference and backward difference schemes as follows:

$$\frac{\partial^2 p}{\partial^2 u} = \frac{\partial^2 p}{\partial^2 u}(u_i, t_j) = \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{h^2},$$

and

$$\frac{\partial p}{\partial u} = \frac{p_{i+1,j} - p_{i-1,j}}{2h}, \quad \frac{\partial p}{\partial t} = \frac{p_{i,j+1} - p_{i,j-1}}{2k}.$$

To derive a complicated SNES of size 144×144, we choose $\mathcal{M} = \mathcal{N} = 12$. The desired solution, denoted as x^* given below:

$$\begin{aligned} x^* = (\; & -0.6575\cdots, -0.6088\cdots, -0.5637\cdots, -0.5220\cdots, -0.4833\cdots, -0.4475\cdots, \\ & -0.4144\cdots, -0.3837\cdots, -0.3553\cdots, -0.3290\cdots, -0.3046\cdots, -0.2821\cdots, -1.205\cdots, \\ & -1.116\cdots, -1.033\cdots, -0.9570\cdots, -0.8862\cdots, -0.8205\cdots, -0.7598\cdots, -0.7035\cdots, \\ & -0.6514\cdots, -0.6032\cdots, -0.5585\cdots, -0.5172\cdots, -1.643\cdots, -1.522\cdots, -1.409\cdots, \\ & -1.305\cdots, -1.208\cdots, -1.118\cdots, -1.036\cdots, -0.9594\cdots, -0.8883\cdots, -0.8226\cdots, \\ & -0.7617\cdots, -0.7053\cdots, -1.972\cdots, -1.826\cdots, -1.691\cdots, -1.566\cdots, -1.450\cdots, \\ & -1.342\cdots, -1.243\cdots, -1.151\cdots, -1.066\cdots, -0.9871\cdots, -0.9140\cdots, -0.8464\cdots, \\ & -2.191\cdots, -2.029\cdots, -1.879\cdots, -1.740\cdots, -1.611\cdots, -1.492\cdots, -1.381\cdots, \\ & -1.279\cdots, -1.184\cdots, -1.096\cdots, -1.015\cdots, -0.9404\cdots, -2.301\cdots, -2.130\cdots, \\ & -1.973\cdots, -1.827\cdots, -1.691\cdots, -1.566\cdots, -1.450\cdots, -1.343\cdots, -1.243\cdots, \\ & -1.151\cdots, -1.066\cdots, -0.9875\cdots, -2.301\cdots, -2.130\cdots, -1.973\cdots, -1.827\cdots, \\ & -1.691\cdots, -1.566\cdots, -1.450\cdots, -1.343\cdots, -1.243\cdots, -1.151\cdots, -1.066\cdots, \\ & -0.9875\cdots, -2.191\cdots, -2.029\cdots, -1.879\cdots, -1.740\cdots, -1.611\cdots, -1.492\cdots, \\ & -1.381\cdots, -1.279\cdots, -1.184\cdots, -1.096\cdots, -1.015\cdots, -0.9405\cdots, -1.972\cdots, \\ & -1.826\cdots, -1.691\cdots, -1.566\cdots, -1.450\cdots, -1.342\cdots, -1.243\cdots, -1.151\cdots, \\ & -1.066\cdots, -0.9871\cdots, -0.9140\cdots, -0.8464\cdots, -1.643\cdots, -1.522\cdots, -1.409\cdots, \\ & -1.305\cdots, -1.208\cdots, -1.119\cdots, -1.036\cdots, -0.9594\cdots, -0.8884\cdots, -0.8226\cdots, \\ & -0.7617\cdots, -0.7053\cdots, -1.205\cdots, -1.116\cdots, -1.033\cdots, -0.9571\cdots, -0.8862\cdots, \\ & -0.8206\cdots, -0.7598\cdots, -0.7036\cdots, -0.6515\cdots, -0.6032\cdots, -0.5585\cdots, -0.5172\cdots, \\ & -0.6575\cdots, -0.6088\cdots, -0.5637\cdots, -0.5220\cdots, -0.4834\cdots, -0.4476\cdots, -0.4144\cdots, \\ & -0.3837\cdots, -0.3553\cdots, -0.3290\cdots, -0.3046\cdots, -0.2821\cdots)^{tr} \end{aligned}$$

The (COC), iteration count, error differences across successive iterations, CPU time and absolute residual errors for Example 4.3 are illustrated in **Table 3**, by choosing $x_0 = (-1, -1, \overset{141}{\cdots}, -1)^{tr}$.

(Our scheme given by Equation (15) achieves smaller residual errors, reduced differences between iterations, and stable convergence, while requiring the same number of iterations as existing methods. Though Lotfi et al.'s method consumes less CPU time, our method outperforms in terms of residual errors and iteration differences, demonstrating superior efficiency and compatibility.)

Table 3. Computational results of Example (4.3).

Methods	$\parallel \mathcal{F}(x_3) \parallel$	$\parallel x_4 - x_3 \parallel$	n	CPU Timing	ξ^*
Abbasbandy et al. (2016)]	2.4×10^{-208}	9.1×10^{-211}	3	1595.51	6.2955
Lotfi et al. (2015)	1.0×10^{-147}	2.0×10^{-149}	4	665.138	5.1818
Hueso et al. (2015)	1.7×10^{-145}	1.3×10^{-147}	4	1122.6	5.4736
Method (15)	3.9×10^{-217}	1.4×10^{-219}	3	822.798	6.2224

5. Conclusion

We introduced a novel 6th-order convergence (6th-OC) scheme for solving scalar equations by adopting the weight function approach. This approach provides flexibility in constructing new schemes of the same order. The proposed method is extended to SNES using the BS operator. Additionally, we suggested its semi-local convergence, which is rigorously analyzed based on the recurrence relations approach. To validate the efficiency and effectiveness of our scheme, we have applied it to three problems in applied science: an integral equation, a boundary value problem transformed into a 121×121 and the two-dimensional Burger's equation has been converted into a 144×144 SNES. The results obtained demonstrates that our technique outperforms existing schemes of similar convergence order.

Our scheme achieved smaller absolute residual errors, reduced differences between successive iterations, and required fewer iterations to reach the desired accuracy. Further, it also demonstrated lower CPU time consumption and stable convergence order. All the above advantages make it a robust and efficient method for solving complex nonlinear problems. Therefore, we conclude that our scheme has broader applications in applied mathematics and computational science.

Conflict of Interest
The authors declare that they do not have conflict of interests.

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The authors declare that no assistance is taken from generative AI to write this article.

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