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Maximum Entropy Solution for $M^X/G/1$ Priority Reiterate G-queue Under Working Breakdown and Working Vacation

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Abstract

The maximum entropy principle has grown progressively more pertinent to queueing systems. The principle of maximum entropy (PME) presents an impartial framework as a promising method to examine complex queuing processes. This principle can be employed to assess the most appropriate probability distributions for queueing scenarios in a variety of widespread industrial issues. The aspects of general service bulk arrival retrial G-queue including working vacation, state-dependent arrival, priority users, and working breakdown are all explored in this article. Real-world applications for this kind of waiting line include computer systems, industrial companies, packet-switching networks, and communication facilities, etc. The adverse users (or negative arrivals) can make an appearance when the server (operator) is preoccupied with a positive user. Consumer's arrival patterns follow the Poisson distribution. Priority consumers and regular (ordinary) consumers are the two groups of consumers that are considered in this investigation. Priority consumers do not have to wait in line and are granted a special right of prevention that allows them to receive services before ordinary consumers. Initially, we have estimated performance metrics including orbit size and long-run probabilities in this research work. The maximum entropy approach is then used to give a comparative perusal between the system's exact and estimated waiting times. Apart from that a bi-objective optimization model is developed to diminish both consumers waiting times and estimated costs simultaneously. It is manageable to establish an effective balance between the standard of service and operating expenses using the analytical strategy that has been provided.

Keywords- Retrial G- queue, State-dependent arrival, Working vacation, Working breakdown, Maximum entropy approach, Bi-objective optimization.

1. Introduction

Plenty of researchers have been studying queueing networks using a combination of positive and negative consumers over the past 20 years. Gelenbe (1991a) proposed the G-queues, often known as the queueing system with negative consumers (*deleterious user*). Positive consumers (*favourable user*) join a queue and gets service just like regular consumers on the queueing network. If a negative consumer encounters an empty queue (*waiting line*) then it will dissipate and if it does not, it will decrease the number of favourable consumers in queue by one. Deleterious users have no access to the service. Users who are favourable when they depart one queue to join another can either turn negative or stay positive. Typically, a deleterious user pulls away a favourable user and breaks down the machinery. As an example, deleterious users are expressed as synchronous signals virus on an external framework throughout the communications process. In this, disruptive viruses that impact computers can corrupt the content of the file and harm the process by which the computer functions. Owing to their widespread use in neural networks, communication networks, computers and production systems, G-queues have generated a great deal of interest. A number of systems can be expressed through G-networks. The analogy with neural networks in the article of Gelenbe (1989) served as the inspiration for the initial model in Gelenbe (1991a) where each queue signifies as a neuron and clients indicate excitation (positive) or inhibition (negative) signals. The signals in biophysical neurons can also appear as random trains of impulses with a constant size, resembling consumers moving through a queuing system. In Gelenbe and Tucci (1991b), a G-network application to doubly redundant systems is described. Favourable consumers can be viewed as resource requests whereas deleterious consumers can be thought of as decisions to reject such requests in many resource systems. In this application, work is set up on two different processors and is subsequently cancelled at one of the processors if the work is successfully finished at the other one. Additional applications have also been established such as specific networking issues by Henderson et al. (1994).

Shannon (1948) first established the idea of *entropy* in his pioneering publications on communication theory. *Entropy* is a fundamental quantity related to a random variable in information theory. The *maximum entropy principle* (MEP) permits the selection of the best probability distribution or the one with the greatest entropy amidst a variety of probability distributions that represent the knowledge of state at that time. As demonstrated by Kapur (1989), numerous disciplines including statistical mechanics, business, marketing, economics, statistical thermodynamics, finance and insurance, operations research and reliability theory have benefited from the maximum entropy principle. The maximum entropy concept has been applied extensively in Operations Research, specifically in queueing framework to evaluate probability distribution for various processes in steady-state. The problem domain must be identified before the maximum entropy concept can be applied. This means that in cases including systems (*framework*) that are physical, all the parameters associated with the constraints must be recognised together with the various states in which the framework may exist. For instance, it is presumed that energy, electric charge and other associated parameters related to the framework are known in advance. Mechanics are frequently required for this task. In this phase, it is not presumed which specific state is actually occupied instead it is considered that we do not know and cannot know this with assurance, leading us to focus on the possibility that each of the states is occupied.

The fact is that there is hardly any reliable server (*operator*) in reality. Indeed, the servers may experience prolonged and unpredicted malfunctions while assisting a client. For instance, in production processes a machine may malfunction as a result of issues with the machine or the work. As a consequence of this, there will be a period of delay while the servers are being fixed. Several authors have looked at framework with a repairable server (*authentic operator*) as a reliability model and queueing model. Computer networks and technologies have been utilised since the half of the twentieth century in a variety of fields involving telephone communication, flexible producing goods, supply-chain systems and e-commerce. Typically, these frameworks function under circumstances that are unpredictable and the difference between the variable service capacity and random service demand generates the congestion frequently. Consequently, it is more vital nowadays to conduct studies on the performance indices of these technological devices. There are real-life situations in which the failure of an operator may not prevent the service of a client entirely. For illustration, the performance of the computer system may be slowed down due to the presence of a virus in the framework. The computer system might still be able to complete multiple functions although quite slowly. However, the basic presumption in each of the queueing systems with server failures that have been examined so far is that a server failure entirely reduces the service of the system. The machine replacement problem serves as an additional example. When a machine is in use, it may abruptly malfunction. It is quickly swapped out by another backup device preferably one that operates more slowly. As soon as the malfunctioning device is fixed, it is put back in operation. Server failures have a significant impact on performance metrics of unreliable queuing systems which is well known. For this reason, numerous studies on unreliable (unauthentic) waiting line framework have been conducted over the years. We acknowledge to Thiruvengadam (1963), Mitrany and Avi-Itzhak (1968), and references therein as the initial studies on this topic. However, in real-world circumstances the system requires to have a backup (standby) server just in case the primary server is supposed to fail. While the primary server is being fixed, the backup server provides services to clients. The service rate of the substitute server differs from (probably lower than) that of the primary server. The primary server rejoins the system and is made accessible as shortly the repair is finished. This is the idea of the working breakdown (*working malfunction*) which was initially proposed by Kalidass and Kasturi (2012). The system can handle any emergencies that may arise during the repair period by permitting the substitute server to deliver services, and efficiency of framework is maximised. Additionally, the cost of waiting consumers or jobs is reduced and issues regarding consumers diminish who must wait for the main server to get repaired while the working malfunction service occurs. As a result, a working malfunction service is a more suitable repair policy to waiting framework that are unreliable. For further study go through the references Yang et al. (2002), Ke et al. (2009), Dimitriou and Langaris (2010), Choudhury and Tadj (2011), Choudhury and Ke (2012), Dimitriou (2013) and recent references include Vijayalakshmi et al. (2021), Seenivasan and Abinaya (2022), Liu et al. (2023).

Distinct classes of consumers exist in a priority queueing system, and each class is treated differently from the others in terms of priority. The consumer category may influence both the service discipline and the service time distribution. Additionally, the priority discipline that is employed may be non-preemptive or preemptive. The client receiving assistance (service) is permitted to finish generally despite the fact that a higher priority client joins the waiting line in the middle of the service of the clients which is referred to as non-preemptive discipline. When a higher priority customer appears, the service to the existing customer will be interrupted in the preemptive situation. The assistance to the consumers whose assistance was disrupted is resumed from the point when it was discontinued whenever the priority discipline is preemptive. In assessment of computer framework, communication networks and the modelling, priority queues are crucial strategies. Jobs in a computer network can be categorised into various classes. This provides quality of service (QoS) support. As an example, an obvious discrepancy between audio, video and data packets could be apparent and various classes might need different services. In these situations, service discipline is frequently implemented at networks that handle jobs based on their priority. An unreliable server with bulk arrival reiterate waiting framework which have two kinds of subscribers that possess non-preemptive priority had been investigated by Jain and Bhargava (2008). Non-preemptive services controlled by an exponential timer and repeated leisure had been looked by Katayama and Kobayashi (2007). The reader may use these articles for carrying out comprehensive research for priority queue, which are Liu et al. (2009), Liu and Gao (2011), Wu and Lian (2013), Gao (2015) and Rajadurai et al. (2016).

1.1 Literature Review

The major goal of our analysis is to maximize entropy outcomes and provide a comparative analysis amidst the system's exact (*precise*) and approximative (*imminent*) predicted waiting times. The queue size distribution of several waiting line frameworks has been acquired by certain researchers applying maximum entropy principle (MEP). The MEP was employed by Wang et al. (2002) to investigate the M/G/1 waiting line framework in various systems. Additionally, Wang et al. (2007) performed an analysis of $M^X/M/1$ model involving maximum entropy concept of a waiting line framework with vacations. Furthermore, Wang and Huang (2009) conducted a comparison amidst the precise analytical results and approximate results using the maximum entropy approach. Using the entropy concept, Wang et al. (2011) carried out a performance analysis of the N-policy queue. Jain and Upadhyaya (2012) employed MEP for a discrete time unreliable server queue with working vacation (*working leisure*). Maximum entropy theory was used by Jain and Upadhyaya (2012) to give results of estimated waiting time for a bulk reiterate waiting line framework under N-policy. The relationship between maximum entropy method and queueing theory has been very thoroughly and clearly explained by Parkash and Mukesh (2016). Further, Chauhan (2018) has performed a comparative analysis using MEP for the $M^X/(G_1, G_2)/1$ framework with two levels of services. Nithya and Haridass (2020b) conducted a stochastic modelling and the maximum entropy analysis of the $M^X/G/1$ queuing system with vacation interruption, balking and startup. In the assessment of a bulk waiting framework with malfunction, regulated arrival, and numerous vacations, further Nithya and Haridass (2020a) looked at optimisation of costs and the highest level of entropy analysis. Bounkhel et al. (2020) have conducted an entropy analysis of a flexible Markovian waiting line with server malfunction. Upadhyaya et al. (2022) used particle swarm optimization technique to obtain the optimal costs of a reiterate G-queueing framework with working malfunction and working leisure including batch arrival. Moreover, Malik et al. (2021b) have looked into maximum entropy findings and optimization results for $M^X/G/1$ reiterate G-queue with delayed repair.

We confront various queueing situations every day where clients must wait for service and it is not prompt. In the past two decades, there has been a lot of relevant research on the subject of reiterate queues in queueing theory. Several investigators have made significant contributions by focusing on the reiterate queues concept. Arrivals who observe the operator busy enter the reiterate group (orbit) to attempt again for their inquires in arbitrary fashion and at various intervals. This is a characteristic of *reiterate queueing systems*. As an illustration, in the Internet Protocol, Transmission Control Protocol (TCP) is used to retransmit missing messages or packets. Several studies have been conducted on reiterate queues during the last several decades. These mathematical models are used in application areas in the performance evaluation of a wide range of systems in data distributed networks, industrial engineering, communication fields and traffic management on high-speed networks. For deep knowledge, see the book written by Falin and Templeton (1997) and the references by Yang and Templeton (1987). Many authors have made significant contributions in this domain. Both Upadhyaya (2014) and Rajadurai et al. (2018b) used the supplementary variable technique to generate all of the favourable performance indices for only one operator reiterate framework. In the aforementioned paper, a bulk arriving structure with a modified vacation policy was put into consideration, whereas in the subsequent paper, the idea of server malfunction, repair and numerous vacation policies was considered. The outcomes for a non-continuous reiterate model with geometric advent including J-vacation policy were later calculated by Upadhyaya (2018). Malik et al. (2021a) have thrown light on the cost analysis of a Geo/G/1 reiterate model with the help of particle swarm optimization and genetic algorithm techniques.

In *G-queues*, a deleterious user can only join when the operator is already full of a favourable user, which causes them to exit the system and disrupts service. A virus invading a computer system is a pretty common example. Such crowds can be seen at places like banks, hospitals, and shopping centres. Gelenbe (1991a) first proposed this concept. Discrete time reiterate G-queue with stochastic decomposition law was determined by Wang and Zhang (2009) after they added deleterious users to the reiterate queue. Wu and Yin (2011) investigated only one server reiterate queue with unfavourable users and non-exhaustive random vacations susceptible to operator failures and maintenance. Reliability estimates and queueing measures for a bulk advent G-queue under general reiterate times, malfunction, and single vacation policy have been provided by Yang et al. (2013). Li and Zhang (2017) explored an M/G/1 reiterate G-queue with general reiterate times and working malfunction. Only one operator reiterate G-queue involving second phase assistance was recently explored by Upadhyaya (2020) who also discussed its use in local network environments. For further references reader may go through Demircioglu et al. (2021), Muthusamy et al. (2022) and Manoharan and Subathra (2023).

Working vacation (WV) is a type of vacation (leisure) policy in which operator sustains to run at a minimal level of service rather than entirely ceasing operations during the vacation period. Several useful systems including network services, web services, file transfer services, and mail services, among others can be used with this waiting line model. It was first applied on an M/M/1 waiting line framework by Servi and Finn (2002). The server cannot resume normal operation during the working leisure until the vacation time is over. Also, even during working leisure period, the server has the option to stop the leisure and switch to the typical busy state if there are consumers at the instant when the service is completed. The term *vacation interruption* refers to this type of policy. Servi and Finn (2002) gave M/M/1/WV waiting line which was transformed into an M/G/1/WV waiting line by Wu and Takagi (2006). An M/M/1 waiting line with working leisure was used by Liu et al. (2007) to derive the results of stochastic decomposition. Zhang and Xu (2008) investigated an M/M/1 waiting line with numerous working leisure and the N-policy using a quasi-birth- and-death (QBD) process and a matrix-geometric solution approach. Rajadurai et al. (2015) and Rajadurai et al. (2016) created reiterate queueing models with the idea of working leisure and vacation interruption in cases of malfunction. We can consult a thorough analysis of Zhang and Liu (2015) and Gao et al. (2014) for more details. Recently, Rajadurai et al. (2018a) conducted research on a reiterate waiting line with only one server, working leisure and interruption of vacation. It is a frequent misconception that the attendant (or server) will stop assisting anytime the queue with server failure. Kim and Lee (2014) explored the M/G/1 waiting line framework in the context of catastrophes (disasters) and failures. Readers may refer to Upadhyaya and Kushwaha (2020) and Agarwal et al. (2021) for a thorough evaluation of the literature review.

The majority of research on optimal design of queueing model emphasizing on a single objective problem with an expenditure or profit function as the goal of optimisation is given by Tadj and Choudhury (2005). Nevertheless, the simultaneous optimisation of many objective functions is a requirement for many real-world optimisation issues (Song and Roy-Chowdhury, 2008). The process of making judgements requires careful evaluation of many factors, especially when objective must be met with a finite amount of resources. There is currently few research on multiple objective models used in queueing framework (e.g., Berman et al., 2006; Liou, 2015; Vahdani and Mohammadi, 2015; Tavakkoli-Moghaddam et al., 2017). A bi-objective hierarchical hub location issue with the goal of simultaneously minimising total cost and total travel time was recently proposed by Khodemani-Yazdi et al. (2019). Chaleshtori et al. (2020) employed a bi-objective optimisation method to a multi-layer location-allocation problem with jockeying. In recent years, a bi-objective analysis and performance evaluation for F-Policy queue with alternating service rates has been done by Wu et al. (2023).

1.2 Research Gap and Contributions

The review of the literature mentioned above indicates that there are considerable research gaps with regard to the G-queue and maximum entropy approach. The goal of this endeavour seeks to bridge that gap and the present effort is focused on delivering performance evaluation by minimising the cost for state-dependent, batch advent reiterates G- queue with working malfunction and working leisure with priority users. By including the idea of priority users with working malfunction in single working leisure with arrivals in batches, we have done generalization of the work of both Rajadurai et al. (2016) and Rajadurai (2018). As far as the author is aware, there is no work that combines reiterate G-queue with working malfunction, working leisure with priority users using the method of supplementary variable technique.

In this study, we use the Maximum Entropy Principle to analyse the queueing characteristics of the batch advent (*arrival*) reiterate G-queueing model by including numerous eminent aspects such as priority consumers, state-dependent, working malfunction and working leisure. The probability of various states is determined using SVT (Supplementary Variable Technique), and approximate results of the expected system size and waiting time are also derived. By employing the MEP, a comparative investigation is executed between the precise (exact) results and imminent (approximate) results. Some numeral results also have been given to validate our model. A bi-objective optimization is also carried out to estimate the expected cost and queue length. The mathematical findings and queuing analysis of the model offer a particular and intriguing applications in the computer processing framework and in working of *BusStop Service* model, as described in section 3 later. Managers who are capable of designing a system with economic management can benefit from our concept.

The whole article is segmented into the subsequent sections. Section 2 covers the explanation of model and practical implications are provided in sections 3. Section 4 further provides steady state response and boundary conditions. In section 5, we procure the probability generating functions for system state and solutions of the steady state equations. After this, the derivation of system performance metrics and average system size is done in section 6. Analysis of the maximum entropy strategy is addressed to stimulate our model with the real world in section 7. Section 8 sets out appropriate numerical results. Section 9 includes the bi-objective optimization which is used for calculating the expected cost and queue length. The conclusion and potential outcomes have been outlined in section 10.

2. Justification of the Model

This article entails working malfunction and working leisure with priority clients on only one server M^X/G/1 framework (*system*) where clients arrive in batches. Let $\mathcal{N}(t)$ be the total number of clients present overall at time t. The following is the description of the model in more detail:

Arriving Schedule- Relying on the operator (or *server*), In accordance with the compound Poisson process arrival happens in batches as shown below.

- $ð_0$= for server (operator) functioning in regular service status or when they are not being used.
- $ð_1$= for server (operator) being loaded with priority clients and also for ordinary clients.
- $ð_2$= for server (operator) being engaged with working malfunction and on working leisure.

Further, P (X = k) = $\mathcal{D}_k$ the probability distribution of bulk size with $\sum_{k=1}^{\infty}\mathcal{D}_k$= 1, and X is defined as a random variable to reflect the bulk size. As well, we take into account the random variable $\mathfrak{Y}(t)$ defined as:

$$\mathfrak{Y}(t)=\begin{cases} 0, \text{ while the operator is not active and under both working leisure and working disruption state.} \\ 1, \text{ each time the operator is dormant and perform normal service.} \\ 2, \text{ if an operator is requested by a priority consumer without restricting a regular consumer at time } 't' \text{ within a normal service period.} \\ 3, \text{ if an operator is dealing with an ordinary consumer and in normal service period at a time } 't'. \\ 4, \text{ while the operator is dispersed and at a time } 't' \text{ in lower service period.} \end{cases}$$

Retrial (*reiterate*) schedule- The client visits a retry group called orbit while the server is overloaded, inoperable, or on working leisure. Clients from the orbit continue to want service until they receive it. We presume that the only client who is permitted to access to the server (*operator*) is the one at the front of the line. Moreover, let the inter reiterate time be determined by a random variable $\mathfrak{R}$ with distribution $\mathfrak{R}(t)$ and its Laplace Stieltjes Transform (LST) $\mathfrak{R}^*(\vartheta)$.

Regular (*normal*) service schedule- During typical peak seasons, the service times follow a general distribution. It is represented by the random variable $\mathcal{U}_b$ using the distribution function (d.f) $\mathcal{U}_b(t)$ and L.S.T $\mathcal{U}_b^*(\vartheta)$.

Working breakdown (*working malfunction*) and Working vacation (*working leisure*) schedule- In the circumstances that the framework has no consumers, the operator departs for a working leisure. The operator does not fully stop providing service during this form of vacation; instead, it provides service at a rate that is significantly reduced. The exponential distribution with parameter 'ς' is supposed to follow by this period. If a server notices that the framework is vacant after a working leisure period is over, it will either wait for a new consumer with possibility 'p' or prolong its leisure with possibility q; q = 1 − p. Even though, if there are consumers at the completion of a working leisure, a vacation interruption may occur, in which case the server quickly switches back to its active status. The server adheres to a general distribution with random variable $\mathcal{U}_w$ and distribution function $\mathcal{U}_w(t)$ while it is on working leisure. The first and second moments are b_w and $(b_w)^2$ correspondingly, and the corresponding LST is $\mathcal{U}^*_w(\vartheta)$.

The hazard rates for the server's various states, including reiterate, busy (with priority and ordinary clients), working leisure are as specified.

$$\mathfrak{r}(s)=\frac{d\,\mathfrak{R}(s)}{1-\mathfrak{R}(s)}; \qquad \omega_b(s)=\frac{d\mathcal{U}_b(s)}{1-\mathcal{U}_b(s)}; \qquad \omega_{\wp}(s)=\frac{d\mathcal{U}_{\wp}(s)}{1-\mathcal{U}_{\wp}(s)}; \qquad \omega_w(s)=\frac{d\mathcal{U}_w(s)}{1-\mathcal{U}_w(s)}$$

Let us use the abbreviations $\mathfrak{R}_0(t)$, $\mathcal{U}_{\wp}(t)$, $\mathcal{U}_b(t)$ and $\mathcal{U}_w(t)$ to indicate the lapsed reiterate time, lapsed service time, lapsed time of priority clients, lapsed time of working leisure respectively. To acquire the bivariate Markov Process $\{\mathfrak{Y}(t),\mathcal{N}(t); t\geq 0\}$, where state of operator is represented by $\{\mathfrak{Y}(t)\}$ as mentioned before.

Other underlying presumptions of the model include:
$\mathcal{D}'(1)=E(X)$, $\mathcal{D}''(1)=E(X^2)$

The limiting possibilities of the server's various states are provided by $Q_0(t) = \hat{P}\,\{\mathfrak{Y}(t)=0,\mathcal{N}(t)=0\}$ and $\hat{P}_0(t) = \hat{P}\,\{\mathfrak{Y}(t)=1,\mathcal{N}(t)=0\}$.
Let $Q_0 = lim_{t\to\infty} Q_0(t)$ and $\hat{P}_0= lim_{t\to\infty}\hat{P}_0(t)$.

Notations Description

Symbols	Description
$\tilde{\beta}$	Arrival rate of priority consumer
$\tilde{\vartheta}$	Vacation rate
$\tilde{\alpha}$	Negative consumer advent rate
μ_1	Regular assistance rate
μ_2	Lower assistance rate during working malfunction and working leisure
ς	Working malfunction rate

$\tilde{\gamma}$	Parameter used in Laplace transform for reiterate using exponential distribution.
$\hat{P}_0(t)$	Possibility that the framework is dormant (idle) at time '*t*'.
$Q_0(t)$	Possibility that the framework is dormant (idle) at time '*t*' and the operator is working malfunction and working leisure. (Lower speed service).
$\Re(g, t)$	Possibility that there are altogether n consumers in the orbit and that the observed consumer's passing reiterate time occurs between g and $g+dg$ at time t.
$\Phi_{1,n}(g, t)$	Possibility that there are altogether n consumers in the orbit and that the observed priority consumer's passing normal service time occurs between g and $g+dg$ at time t.
$\Phi_{b,n}(g, t)$	Possibility that there are altogether n consumers in the orbit and that the observed consumer's passing normal service time occurs between g and $g+dg$ at time t.
$\mathcal{B}_n(g, t)$	Possibility that there are altogether n consumers in the orbit and that the observed consumer's passing lower speed service time occurs between g and $g+dg$ at time t.

3. Practical Application

The telecommunication industry can be benefited greatly from this research. For an illustration, the wireless body sensor network (WBSN), which enables service delivery to consumers utilising an internet platform has recently piqued the interest of the *BusStop Service* system. A system that seeks to promote public transportation in urban area is proposed by the BusStop service. The task of gathering and storing information regarding public transportation and disseminating it to users falls under the purview of the BusStop service. The BusStop service is built on the features that cellular phone provide network. USSD (*Unstructured Supplementary Data)* messages are used by the BusStop to activate the service. When necessary, the service finds the subscriber and generates a response that includes the subscriber's position. The user receives a response through SMS (*Short Message Service*) in the last step. The system's primary purpose is to notify users about the timetable of selected lines, the routes of certain lines, and the number of lines that operate close to their location. Based on the executed tasks displayed in Figure 1, the BusStop service is divided into three sections.

(i) End Users: Cellular phones are employed by end users to communicate with the service over the UTRAN/GERAN (*UMTS Terrestrial Radio Access Network /GSM EDGE Radio Access Network*) access network.

(ii) BusStop Service: The logic of the service is carried out via a Web application that is run on an application server. A JDBC (*Java Database Connectivity*) library serves to facilitate the communication between application and database. Information on public transportation is saved to a database. In order to ensure interaction between the operator and the user and to identify users, the web application makes use of Telco 2.0 APIs (*Application Programming Interface*).

(iii) GMS/UMTS Operator (*Global System for Mobile Communication/ The Universal Mobile Telecommunications System*): Using interfaces exposed in Internet Telco 2.0, this operator connects the end user to the BusStop service. Let us just presume that a BusStop framework provides its service and a variety of other services over the cellular phone, allowing a consumer to call and acquire the routes, schedules, and the position of the stoppage for themselves (ordinary consumers using text message). A controller (one server) is on duty to answer all incoming calls on this BusStop framework.

Whenever consumers use voice message (as voice messages have priority over the other via cellular phone) then this type of consumers is called 'priority consumers'. When a consumer calls, there is a chance that the line will be busy. If that happens, the consumer may call again after some time has passed (reiterate). While the controller is inspecting other sectors of the BusStop framework, he lacks the ability to answer calls (in leisure mode). In these circumstances, the operator controller (substitute server) typically serves but relatively more slowly than usual (working leisure). It is possible for a customer to lose service during a phone call owing to an improper signal, poor network coverage, or malware attack (negative consumers). During rush hours like weekends, we can see a large influx of calls, in which case the controller will need to work more quickly or the waiting line may become congested (state dependent situation). The aforementioned condition is a great fit for our model under examination. As a result, the system manager will be benefited immensely from all the results and findings offered in our work. In section 8, a numerical instance is offered to support the scenario that was just described.

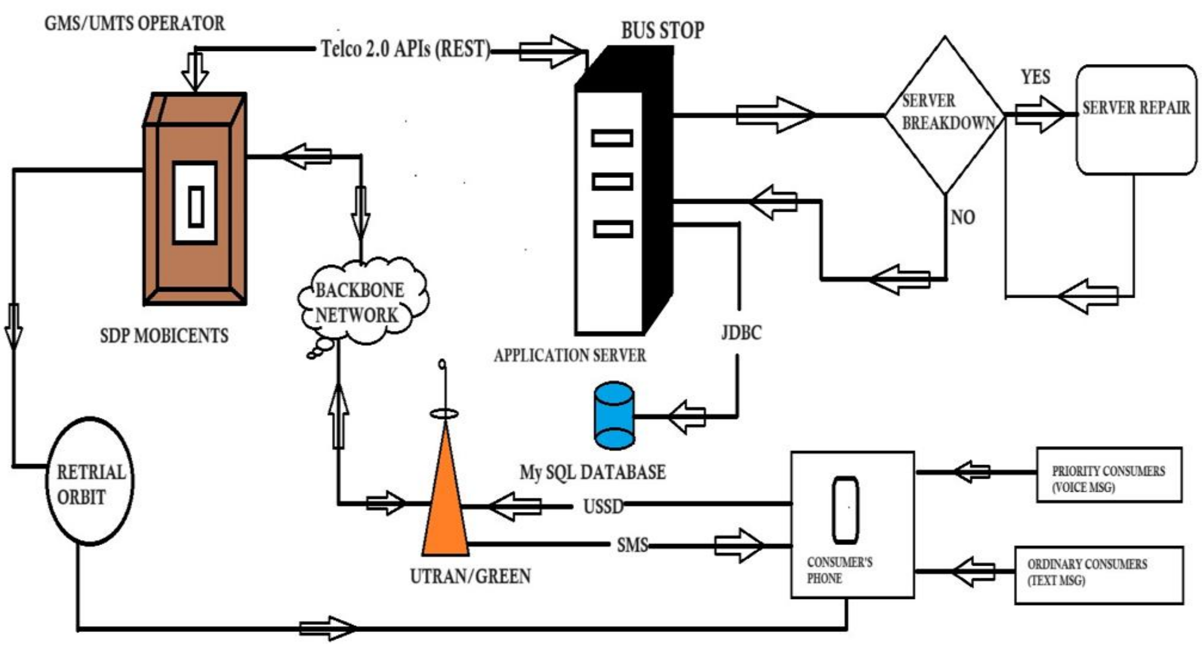

Figure 1. Working of bus stop service system.

This paradigm is also relevant to health industries, car parking structures, simple mail transfer protocol (SMTP), and other areas.

4. Steady State Equations

The equations that regulate the dynamics of the framework are attained by implementing the supplementary variable technique and they are as follows:

$$(\eth_0+\tilde{\beta})\hat{P}_0 = \varsigma\, \mathcal{B}_0 \tag{1}$$

$$(\eth_0 + \tilde{\beta} + \vartheta^{\sim} + \varsigma)\, \mathcal{B}_0 = \vartheta\, \mathcal{B}_0 + \int_0^\infty \Phi_{b,0}(g)\, \omega_b(g)\, dg + \int_0^\infty \mathcal{B}_0(g)\, \omega(g)\, dg + \int_0^\infty \Phi_{1,0}(g)\, \omega_p(g)\, dg + (\alpha^{\sim}+\tilde{\beta}) \int_0^\infty \Phi_{b,n}(g)\, dg + \alpha^{\sim}\int_0^\infty \Phi_{1,n}(g)\, dg,\ n\geq 0 \tag{2}$$

$$\frac{d}{dg}\mathfrak{R}_n(g) + (\eth_0 + \tilde{\beta} + \mathfrak{r}(g))\, \mathfrak{R}_n(g) = 0,\ n\geq 1 \tag{3}$$

$$\frac{d}{dg}\Phi_{1,0}(g) + (\eth_1 + \tilde{\alpha} + \omega_p(g))\, \Phi_{1,0}(g) = 0 \tag{4}$$

$$\frac{d}{dg}\Phi_{1,n}(g) + (\eth_1 + \tilde{\alpha} + \omega_p(g))\Phi_{1,n}(g) = \eth_1 \sum_{k=1}^{n} \mathcal{D}_k \Phi_{1,n-k}(g),\ n\geq 1 \tag{5}$$

$$\frac{\partial}{\partial s}\Phi_{b,n}(s) + (\eth_1+\tilde{\beta} + \alpha^{\sim}+\omega_b(s))\Phi_{b,0}(s) = 0 \quad (6)$$

$$\frac{\partial}{\partial s}\Phi_{b,n}(s) +(\eth_1 + \tilde{\beta} + \alpha^{\sim}+ \omega_b(s))\Phi_{b,n}(s) = \eth_1 \sum_{k=1}^{n} \mathcal{D}_k \Phi_{b,n-k}(s),\ n\geq 1 \quad (7)$$

$$\frac{d}{dg}\mathcal{B}_0(g) + (\eth_2 + \vartheta^{\sim} + \varsigma + \omega_w(g))\mathcal{B}_0(g)=0,\ n=0 \quad (8)$$

$$\frac{d}{dg}\mathcal{B}_n(g) +(\eth_2 + \vartheta^{\sim} + \varsigma + \omega_w(g))\mathcal{B}_n(g)= \eth_2 \sum_{k=1}^{n} \mathcal{D}_k \mathcal{B}_{n-k}(g),\ n\geq 1 \quad (9)$$

At g=0 and s=0 the following boundary conditions apply:

$$\Re_n(0) = \int_0^\infty \Phi_{1,n}(g)\,\omega_p(g)\,dg + \int_0^\infty \Phi_{b,n}(g)\,\omega_b(g)\,dg + \int_0^\infty \mathcal{B}_n(g)\,\omega_w(g)\,dg,\ n\geq 1 \quad (10)$$

$$\Phi_{1,n}(g) = \tilde{\beta}\int_0^\infty \Re_n(g)\,dg,\ n\geq 1 \quad (11)$$

$$\Phi_{b,0}(0) = \int_0^\infty \Re_1(g)\,\mathfrak{r}(g)\,dg+ (\vartheta^{\sim}+\varsigma)\int_0^\infty \mathcal{B}_0(g)\,dg +(\eth_0 + \tilde{\beta})\,\hat{P}_0 \quad (12)$$

$$\Phi_{b,n}(0) = \int_0^\infty \Re_{n+1}(g)\,\mathfrak{r}(g)\,dg + \eth_0\int_0^\infty \Re_n(g)\,dg + (\vartheta^{\sim}+\varsigma)\int_0^\infty \mathcal{B}_n(g)dg,\ n\geq 1 \quad (13)$$

$$\mathcal{B}_n(0) =\begin{cases} (\eth_0 + \tilde{\beta}) = \mathcal{B}_0, n = 0 \\ 0, n \geq 1 \end{cases} \quad (14)$$

Here is the normalizing condition:

$$\hat{P}_0 + \mathcal{B}_0+\int_0^\infty \Re_n(g)\,dg +\sum_{n=0}^{\infty} \left[\int_0^\infty \Phi_{1,n}(g)\,dg + \int_0^\infty \Phi_{b,n}(g)\,dg + \int_0^\infty \mathcal{B}(g)\,dg\right] = 1 \quad (15)$$

5. Probability Generating Functions for System State and Solution of the Steady State Equations

The steady state solution of the equations given above is determined using the generating function technique. In a probability theory, the partial generating function of a discrete random variable is defined as the power series representation of the probability mass function of the random variable. We define generating functions of system state that are listed below for $|q| \leq 1$:

$\Re(g, q) =\sum_{n=1}^{\infty} \Re_n(g)\, q^n$, $\Re(0, q) =\sum_{n=1}^{\infty} \Re_n(0)\, q^n$, $\Phi_1(g, q)= \sum_{n=1}^{\infty} \Phi_{1,n}(g)\, q^n$

$\Phi_1(0, q)= \sum_{n=1}^{\infty} \Phi_{1,n}(0)\, q^n$, $\Phi_b(g, q)= \sum_{n=1}^{\infty} \Phi_{b,n}(g)\, q^n$, $\Phi_b(0, q)= \sum_{n=1}^{\infty} \Phi_{b,n}(0)\, q^n$

$\mathcal{B}(g, q) =\sum_{n=1}^{\infty} \mathcal{B}_n(g)\, q^n$, $\mathcal{B}(0, q) =\sum_{n=1}^{\infty} \mathcal{B}_n(0)\, q^n$.

In this section, we establish a PGF expression for the number of consumers in the priority reiterate disruptive advent waiting list framework with working leisure, working disruption and in orbit at arbitrary interval.

***Theorem* 5.1** It refers to the partial generating functions at random defined intervals whenever the operator is dormant during the reiterate time, on working leisure and working malfunction with normal service, congested due to priority consumers and normal service, loaded due to ordinary consumers and normal service, and absorbed due to lower-speed service periods, which are given as,

$$\Re(g, q) = \Re(0, q)\,[1- \Re(g)]\, e^{-(\eth 0 + \tilde{\beta})} \quad (16)$$

$$\Phi_1(g, q)= \Phi_1(0, q)\,[1-\mathcal{V}_p(g)]\, e^{-\mathcal{U}_p(q)g} \quad (17)$$

$$\Phi_b(s, q)= \Phi_b(0, q)\,[1-\mathcal{V}_b(s)]\, e^{-\mathcal{U}_b(q)s} \quad (18)$$

$$\mathcal{B}(g, q)= \mathcal{B}(0, q)\,[1-\mathcal{V}_w(g)]\, e^{-\mathcal{U}_w(q)g} \quad (19)$$

$\mathcal{U}_p(q)$= ð$_1$ (1-$\mathcal{D}(q)$) + $\tilde{\alpha}$; $\mathcal{U}_b(q)$= ð$_1$ (1-$\mathcal{D}(q)$) + $\tilde{\alpha}$+ $\tilde{\beta}$; $\mathcal{U}_w(q)$= ð$_2$ (1-$\mathcal{D}(q)$) + $\tilde{\vartheta}$+ ς,

$$\Phi_b(0,q) = \frac{\Re(0,q)}{q}[\Re^*(ð_0+\tilde{\beta}) + ð_0\, q\, \overline{\Re}^*(ð_0+\tilde{\beta})] + (ð_0+\tilde{\beta})\,\mathcal{B}_0\,\mathcal{H}(q) + (ð_0+\tilde{\beta})\,\hat{P}_0 \tag{20}$$

$$\mathcal{H}(q) = \frac{(\vartheta+\varsigma)\left(1-\mathcal{V}_w^*(\mathcal{U}_w(q))\right)}{\mathcal{U}_w(q)};\ \mathcal{E}(q) = \mathcal{V}_b^*\mathcal{U}_b(q) + \mathcal{S}(q) \tag{21}$$

$\mathcal{S}(q) = (\tilde{\alpha}+\tilde{\beta})\,\overline{\mathcal{V}_b^*}(\mathcal{U}_b(q))$ and also $\mathcal{G}(q) = \mathcal{V}_p^*(\mathcal{U}_p(q)) + \tilde{\alpha}\,\overline{\mathcal{V}_p^*}(\mathcal{U}_p(q))$,

$$\Re(0,q) = \Phi_1(0,q)\,\mathcal{G}(q) + \Phi_b(0,q)\,\mathcal{E}(q) + (ð_0+\tilde{\beta})\,\mathcal{B}_0\,\mathcal{V}_w^*(\mathcal{U}_w(q)) - (ð_0+\tilde{\beta})\,\mathcal{B}_0 - \varsigma\,\mathcal{B}_0 \tag{22}$$

$$\Phi_1(0,q) = \tilde{\beta}\,\Re(0,q)\,\overline{\Re}^*(ð_0+\tilde{\beta})\,\overline{\mathcal{V}^*}(\tilde{\alpha}) \tag{23}$$

$$\mathcal{B}(0,q) = (ð_0+\tilde{\beta})\,\mathcal{B}_0 \tag{24}$$

Proof: We obtain the above equations from Equations (16) to (24) by multiplying equations (2) to (14) by the appropriate power of q and adding over 'n' as well as solving equations (1) and (10).

***Theorem* 5.2** For each of the following periods: inactive in the reiterate period with working leisure and working malfunction, operator is dormant in normal service period, disperse with priority users with normal service period, engaged with ordinary users with normal service period, operator is bustling with lower speed service, the marginal probability generating functions, respectively are found as,

$$\Re(q) = \frac{Num(q)}{Deno(q)} \tag{25}$$

$$\Phi_1(q) = \tilde{\beta}\,\overline{\mathcal{V}_p^*}(\tilde{\alpha})\,\frac{Num(q)}{Deno(q)} \tag{26}$$

$$\Phi_b(q) = \Phi_b(0,q)\,\overline{\mathcal{V}_b^*}(\mathcal{U}_b(q)) \tag{27}$$

$$\mathcal{B}(q) = \mathcal{B}(0,q)\,\overline{\mathcal{V}_w^*}(\mathcal{U}_w(q)) \tag{28}$$

$$Num(q) = q\,\mathcal{B}_0\,\overline{\Re}^*(ð_0+\tilde{\beta})\{((ð_0+\tilde{\beta})\,\mathcal{H}(q)+\varsigma)\,\mathcal{E}(q) + (ð_0+\tilde{\beta})(\overline{\mathcal{V}_w^*}(\mathcal{U}_w(q))-1) - \varsigma\} \tag{29}$$

$$Deno(q) = q - q\,\tilde{\beta}\,\overline{\Re}^*(ð_0+\tilde{\beta})\,\mathcal{G}(q) - \{\Re^*(ð_0+\tilde{\beta}) + ð_0\,q\,\overline{\Re}^*(ð_0+\tilde{\beta})\}\,\mathcal{E}(q).$$

$$Num'(1) = \mathcal{B}_0\,(1-\Re^*(ð_0+\tilde{\beta})\,\mathcal{D}'(1)\,\{(1-\mathcal{V}_w^*(\tilde{\vartheta}+\varsigma))[-ð_{1+}\frac{\partial_2}{\tilde{\vartheta}+\varsigma}] - \frac{\varsigma}{ð0+\tilde{\beta}}\,ð_1\}.$$

$$Deno'(1) = (1+\Re^*(ð_0+\tilde{\beta})\,ð_1\,\mathcal{D}'(1) + \overline{\Re}^*(ð_0+\tilde{\beta})(ð_0\,ð_1\,\mathcal{D}'(1) - ð_0+\tilde{\beta}\,ð_1\,\mathcal{D}'(1) - \tilde{\beta}))).$$

$$Num_b'(1) = First' \times \overline{\mathcal{V}_b^*}(ð_0+\tilde{\beta}).$$

$$First' = \mathcal{B}_0\,\{[((ð_0+\tilde{\beta})\,ð_2\,\mathcal{D}'(1)\,\mathcal{V}_w^*(\tilde{\vartheta}+\varsigma) + \overline{\mathcal{V}_w^*}((\tilde{\vartheta}+\varsigma) + (ð_0+\varsigma)(1-\mathcal{V}_w^*(\tilde{\vartheta}+\varsigma)+\varsigma) - ð_2\,\mathcal{D}'(1)\,(ð_0+\tilde{\beta})\,((\mathcal{V}_w^*)'(\tilde{\vartheta}+\tilde{\beta})\,\Re^*(ð_0+\tilde{\beta}) + ð_0\,\overline{\Re}^*(ð_0+\tilde{\beta}) + ð_0\,((ð_0+\tilde{\beta})\,\overline{\Re}^*(ð_0+\tilde{\beta})(\mathcal{V}_w^*((\tilde{\vartheta}+\varsigma)-1)-\varsigma)] - \tilde{\beta}\,\overline{\Re}^*(ð_0+\tilde{\beta})\{(ð_0+\tilde{\beta})\,(1-\mathcal{V}_w^*(\tilde{\vartheta}+\varsigma)+\varsigma) - ð_1\,\mathcal{D}'(1)\,((ð_0+\tilde{\beta})(1-\mathcal{V}_w^*(\tilde{\vartheta}+\varsigma)+\varsigma) + ð_2\,\mathcal{D}'(1)\,(ð_0+\tilde{\beta})\,((\mathcal{V}_w^*)'(\tilde{\vartheta}+\varsigma) + \overline{\mathcal{V}_w^*}((\tilde{\vartheta}+\varsigma))\} \tag{30}$$

Proof: On integrating the equations with respect to $'g'$ from (15) to (22) and considering that the partial generating function can be expressed as follows: $\Re(g) = \int_0^\infty \Re(g,q)\,dg$, $\Phi_1(q)=\int_0^\infty \Phi_1(g,q)\,dg$, $\Phi_b(q)=\int_0^\infty \Phi_b(g,q)\,dg$, $\mathcal{B}(q)=\int_0^\infty \mathcal{B}(g,q)\,dg$, we set q =1 in Equations (23) to (29) and we use the L' Hospital rule where necessary, Utilising the normalizing condition, we acquire $[\hat{P}_0 + \mathcal{B}_0 + \Re(1) + \Phi_1(1) + \Phi_b(1) + \mathcal{B}(1)] = 1$, which allows us to evaluate the possibility that the operator is stagnant (idle) while the consumer is not exist in the orbit.

6. System Performance Metrics

The following performance metrics will be established using the probability generating function of the queue length. We initially calculated the long run probabilities under various service provider consideration conditions with objective to estimate the operating of a queueing framework in steady state, from which we then extracted other significant performance indicators.

6.1 Steady State Possibilities

(i) Let $\mathfrak{R}$ be the steady state possibility of the operator being stagnant (dormant) during reiterate time, then

$$\mathfrak{R} = \mathfrak{R}(1) = \frac{\mathcal{B}0\ (1-\mathfrak{R}^*(ð0+\tilde{\beta})\ \mathcal{D}\prime(1)\ \{(1-\mathcal{V}_w^*\ (\vartheta^{\sim}+\varsigma))[-ð1+\frac{\partial_2}{\vartheta^{\sim}+\varsigma}]-\frac{\varsigma}{ð0+\tilde{\beta}}ð1\}}{(1+\mathfrak{R}^*(ð0+\tilde{\beta})\ ð1\ \mathcal{D}\prime(1)+\bar{\mathfrak{R}}*(ð0+\tilde{\beta})(ð0\ ð1\ \mathcal{D}\prime(1)-ð0+\tilde{\beta}\ ð1\ \mathcal{D}\prime(1)-\tilde{\beta})))} \tag{31}$$

(ii) Let Φ_1 be the steady state possibility of the operator being engaged with a priority consumer and in a normal service period at a time, then

$$\Phi_1 = \Phi_1(1) = \tilde{\beta}\ \overline{\mathcal{V}_{\wp}^*}(\tilde{\alpha}) \times \frac{\mathcal{B}0\ (1-\mathfrak{R}^*(ð0+\tilde{\beta})\ \mathcal{D}\prime(1)\ \{(1-\mathcal{V}_w^*\ (\vartheta^{\sim}+\varsigma))[-ð1+\frac{\partial_2}{\vartheta^{\sim}+\varsigma}]-\frac{\varsigma}{ð0+\tilde{\beta}}ð1\}}{(1+\mathfrak{R}^*(ð0+\tilde{\beta})\ ð1\ \mathcal{D}\prime(1)+\bar{\mathfrak{R}}*(ð0+\tilde{\beta})(ð0\ ð1\ \mathcal{D}\prime(1)-ð0+\tilde{\beta}\ ð1\ \mathcal{D}\prime(1)-\tilde{\beta})))} \tag{32}$$

(iii) Let Φ_b be the steady state possibility of the operator being engaged with an ordinary consumer and in normal service period at a time, then

$$\Phi_b = \Phi_b(1) = \frac{Num_b'(1)}{Deno'(1)} \times \overline{\mathcal{V}_b^*}(\mathcal{U}_b\ (q)) \tag{33}$$

where, $Num_b'(1) = First' \times \overline{\mathcal{V}_b^*}\ (ð_0+\tilde{\beta})$

$$First' = \mathcal{B}_0\ \{\ [\ (\ (ð_0+\tilde{\beta})\ ð_2\mathcal{D}'(1)\ \mathcal{V}_w^*\ (\vartheta^{\sim}+\varsigma)+\overline{\mathcal{V}_w^*}((\vartheta^{\sim}+\varsigma)+(ð_0+\varsigma)(1-\mathcal{V}_w^*\ (\vartheta^{\sim}+\varsigma)+\varsigma)-ð_2\ \mathcal{D}'(1)\ (ð_0+\tilde{\beta})\ ((\mathcal{V}_w^*)'\ (\vartheta^{\sim}+\tilde{\beta})\ \mathfrak{R}^*(ð_0+\tilde{\beta})+ð_0\ \bar{\mathfrak{R}}^*(ð_0+\tilde{\beta})+ð_0\ ((ð_0+\tilde{\beta})\ \bar{\mathfrak{R}}^*(ð_0+\tilde{\beta})(\mathcal{V}_w^*((\vartheta^{\sim}+\varsigma)-1)-\varsigma)]-\tilde{\beta}\ \bar{\mathfrak{R}}^*(ð_0+\tilde{\beta})\ \{(ð_0+\tilde{\beta})\ (1-\mathcal{V}_w^*(\vartheta^{\sim}+\varsigma)+\varsigma)-ð_1\ \mathcal{D}'(1)\ ((ð_0+\tilde{\beta})(1-\mathcal{V}_w^*(\vartheta^{\sim}+\varsigma)+\varsigma)+ð_2\ \mathcal{D}'(1)\ (ð_0+\tilde{\beta})\ ((\mathcal{V}_w^*)'(\vartheta^{\sim}+\varsigma)+\overline{\mathcal{V}_w^*}((\vartheta^{\sim}+\varsigma))\} \tag{34}$$

where,

$$Deno'(1) = (1+\mathfrak{R}^*(ð_0+\tilde{\beta})\ ð_1\ \mathcal{D}'(1)+\bar{\mathfrak{R}}^*(ð_0+\tilde{\beta})(ð_0\ ð_1\ \mathcal{D}'(1)-ð_0+\tilde{\beta}\ ð_1\ \mathcal{D}'(1)-\tilde{\beta})))$$

(iv) Let $\mathcal{B}$ be the steady state possibility of the operator being engaged during lower speed service period at time, then

$$\mathcal{B} = \mathcal{B}(1) = (ð_0+\tilde{\beta})\ \mathcal{B}_0\left\{\frac{1-\mathcal{B}_w^*(\vartheta+\varsigma)}{(\vartheta+\varsigma)}\right\} \tag{35}$$

(v) Let $\mathcal{B}_{w\,v}$ be the steady state possibility of the operator being on working malfunction and working leisure, then

$$\mathcal{B}_{w\,v} = \mathcal{B} + \mathcal{B}_0 = \frac{\mathcal{B}_0\left((\vartheta+\varsigma)+(ð0+\tilde{\beta})(1-\mathcal{V}_w^*(\vartheta+\varsigma))\right)}{(\vartheta+\varsigma)} \tag{36}$$

(vi) Let $\mathcal{T}_f$ be the stationary possibility of the operator being failure, then

$$\mathcal{T}_f = (\alpha^{\sim}+\tilde{\beta})\ \Phi_b\ (1) = (\alpha^{\sim}+\tilde{\beta})\ \frac{Num_b'(1)}{Deno'(1)} \times \overline{\mathcal{V}_b^*}(\mathcal{U}_b\ (q))$$

$$= (\tilde{\alpha}+\tilde{\beta})\ \frac{\mathcal{B}_0\ ((ð0+\tilde{\beta})ð2\ \mathcal{D}\prime(1)\ (\mathcal{V}_w^*(\vartheta+\varsigma)+\overline{\mathcal{V}_w^*}(\vartheta^{\sim}+\varsigma))+\mathcal{A}+\mathcal{L}+\mathcal{I}+\mathcal{F})}{(1+\mathfrak{R}^*(ð0+\tilde{\beta})\ ð1\ \mathcal{D}\prime(1)+\bar{\mathfrak{R}}*(ð0+\tilde{\beta})(ð0\ ð1\ \mathcal{D}\prime(1)-ð0+\tilde{\beta}\ ð1\ \mathcal{D}\prime(1)-\tilde{\beta})))} \tag{37}$$

where, $\mathcal{A} = (ð_0+\tilde{\beta})\ (1-\mathcal{V}_w^*(\vartheta^\sim+\varsigma)+\varsigma)$,

$\mathcal{L} = ð_2\ \mathcal{D}_1'\ (1)\ (ð_0+\tilde{\beta})\ (\mathcal{V}_w^*)'\ (\vartheta^\sim+\tilde{\beta}))\ \Re^*(ð_0+\tilde{\beta}) + ð_0\ \bar{\Re}^*(ð_0+\tilde{\beta})$,
$\mathcal{J} = ð_0\,(ð_0+\tilde{\beta})\ \bar{\Re}^*(ð_0+\tilde{\beta})\ [\ (\mathcal{V}_w^*(\vartheta+\varsigma)-1)-\varsigma)]$, and
$\mathcal{F} = -\tilde{\beta}\ \ \bar{\Re}^*(\ ð_0 + \tilde{\beta})\ \{(ð0+\tilde{\beta})(1-\mathcal{V}_w^*(\vartheta+\varsigma)) - ð_1\mathcal{D}'(1)(ð_0+\tilde{\beta})(1-\mathcal{V}_w^*(\vartheta+\varsigma)+\varsigma) + ð_2\mathcal{D}'(1)(ð_0+\tilde{\beta})(\mathcal{V}_w'^*(\vartheta+\varsigma)+\overline{\mathcal{V}_w^*}(\vartheta+\varsigma))\}$.

6.2 Average System Size

***Theorem* 6.1** The average number of consumers in the orbit and the precise anticipated waiting time which together make up the average queue length $'\mathcal{L}_0$' is provided by,

$\mathcal{L}_0 = \lim_{m=1} \mathcal{K}_0'(m) = \mathcal{K}_0'(1) = \Re'(1) + \Phi_1'(1) + \Phi_b' + \mathcal{B}'(1)$

where, $\Re'(1) = \frac{\Re_r''(1)}{\mathcal{D}_r''(1)}$; $\Re_r''(1) = \frac{(1-\Re^*(ð0+\tilde{\beta}))}{(ð0+\tilde{\beta})}\ (\mathfrak{G}'' + \mathfrak{G}')$;

$\mathfrak{G}' = -ð_1\mathcal{D}'(1)(ð_0+\tilde{\beta})(1-\mathcal{V}_w^*(\vartheta+\varsigma)+\varsigma) + (ð_0+\tilde{\beta})\ ð_2\,\mathcal{D}'(1)\ ((\mathcal{V}_w^*)'(\vartheta^\sim+\varsigma) + \overline{\mathcal{V}_w^*}(\vartheta^\sim+\varsigma))$- $ð_2\,\mathcal{D}'(1)$ $\mathcal{V}_w'^*(\vartheta+\varsigma)$,

$\mathfrak{G}'' = -ð_1\mathcal{D}'(1)\mathcal{V}_b^{*''}(\alpha^\sim+\tilde{\beta}) - ð_1\mathcal{D}''(1)\mathcal{V}_b'^*\left(\tilde{\alpha}+\tilde{\beta}+\mathcal{S}'(1)\right)((ð_0+\tilde{\beta})\mathcal{H}(1)+\varsigma) + 2(-ð_1\mathcal{D}'(1),$
$\mathcal{V}_w^{*'}(\alpha^\sim+\tilde{\beta}) + \mathcal{S}'(1)) + (ð_0+\tilde{\beta})\mathcal{H}'(1) + (ð_0+\tilde{\beta})\mathcal{H}''(1)(\mathcal{V}_b^{*''}(\alpha^\sim+\tilde{\beta}) + \mathcal{S}(1)) + (ð0+\tilde{\beta})\mathcal{V}_w^{*''}(\vartheta+\varsigma)(\ ð_2$
$(\mathcal{D}'(1))^2$- $ð_2\,\mathcal{D}''(1)\mathcal{V}^{*''}(\vartheta+\varsigma)$,

$\mathcal{D}_r''(1) = -(\Re^*(ð_0+\tilde{\beta}) + ð_0\ \bar{\Re}^*(\ ð_0+\tilde{\beta}) + \mathcal{E}''(1)$-2 $ð_0\ \bar{\Re}^*(\ ð_0+\tilde{\beta}\mathcal{E}'(1))$- $\tilde{\beta}\ \bar{\Re}^*(\ ð_0+\tilde{\beta}(\ \mathcal{G}''(1) + \mathcal{G}'(1))$

where $\mathcal{E}'(1) = -ð_1\mathcal{D}'(1)$,

$\mathcal{E}''(1) = \mathcal{V}_b^{*''}(\alpha^\sim+\tilde{\beta})\ (\ ð_1\mathcal{D}'(1))^2 - ð_1\ \mathcal{D}''(1)\ \mathcal{V}_b^{*'}(\alpha^\sim+\tilde{\beta}) + (\alpha^\sim+\tilde{\beta})[\ (\ ð_1\mathcal{D}'(1))^2\ \overline{\mathcal{V}_b^{*''}}((\alpha^\sim+\tilde{\beta})$- $ð_1$,
$\mathcal{D}''(1)\overline{\mathcal{V}_b^{*'}}\ (\alpha^\sim+\tilde{\beta})$,

$\mathcal{G}'(1) = ð_1\mathcal{D}'(1)$,

$\mathcal{G}''(1) = (\ ð_1\mathcal{D}'(1))^2 - ð_1\ \mathcal{D}''(1))$,

$\mathcal{B}'(1) = (ð_0+\tilde{\beta})\ \mathcal{B}_0 \left\{\frac{ð_2\mathcal{D}_1'(1)(\vartheta+\varsigma)\mathcal{U}_w^{*'}(\vartheta+\varsigma) + ð_2\mathcal{D}_1'(1)(1-\mathcal{U}_w^*(\vartheta+\varsigma))}{((\vartheta+\varsigma))^2}\right\}$,

$\Phi_b'(1) = \frac{\mathcal{D}_r''(1)\ Num_b'(1)}{2(Deno'(1))^2}$, $\Phi_1'(1) = \frac{\tilde{\beta}\ \overline{V_b^*}(\tilde{\alpha})\ Num'(1)}{Deno'(1)}$,

$\mathcal{L}_q = \lim_{m=1} \mathcal{K}_q'(m) = \mathcal{K}_q'(1) = \mathcal{K}_0'(1) + \Phi_1(1) + \Phi b\ (1) + \mathcal{B}(1)$.

***Corollary* 6.2** The average amount of time a consumer stays in the queueing framework and in the orbit, respectively is calculated by utilizing Little's formula

$\mathcal{W}_s = \frac{\mathcal{L}_q}{\partial}$, $\mathcal{W}_0 = \frac{\mathcal{L}_0}{\partial}$.

***Corollary* 6.3** The availability of the service provider and the frequency of failures are the system's reliability measurements for the steady state and consequently, are provided by:

Proof: $\mathfrak{U}_s = = 1 - \mathcal{T}_f$ and $\mathfrak{F}_s = = \alpha^\sim[\ \Phi_1(1) + \Phi_b(1) + \mathcal{B}(1) + \mathcal{B}_{wv}(1)]$.

7. Maximum Entropy Concept

Jaynes (1957) proposed the maximum entropy approach. Entropy usually pertains to how a framework or a probabilistic distribution varies. It occurs frequently that we are incapable of estimating a suitable distribution in probabilistic analysis. The information about the waiting line length, steady state possibilities, etc. that is provided in queueing systems can be viewed of as constraints in obtaining maximum entropy results. Nonetheless, there are numerous distributions that satisfy them. With the facts at hand, we arrive at the best distribution for them using this strategy. The ideal probability distribution is one that optimises the entropy function.

7.1 Formation of Entropy Function

Given that 'n' is the total number of users in the framework, consider $\mathfrak{P}_1(\mathrm{n})$, $\mathfrak{P}_2(\mathrm{n})$, $\mathfrak{P}_3(\mathrm{n})$, $\mathfrak{P}_4(\mathrm{n})$ that reflect the possibilities that the server (operator) is in the reiterate period, disperse period with priority consumer, disperse with ordinary consumer, engaged with lower speed period, respectively.

We construct the entropy function $'\mathfrak{J}'$ as per El-Affendi and Kouvatsos (1983) work as,

$$\mathfrak{J} = -\sum_{n=1}^{\infty}\mathfrak{P}_1(\mathrm{n})\,\log\mathfrak{P}_1(\mathrm{n}) - \sum_{n=1}^{\infty}\mathfrak{P}_2(\mathrm{n})\,\log\mathfrak{P}_2(\mathrm{n}) - \sum_{n=1}^{\infty}\mathfrak{P}_3(\mathrm{n})\,\log\mathfrak{P}_3(\mathrm{n}) - \sum_{n=1}^{\infty}\mathfrak{P}_4(\mathrm{n})\,\log\mathfrak{P}_4(\mathrm{n}) \tag{38}$$

With the list of constraints listed below, we will estimate the steady state probabilities as

$$\sum_{n=1}^{\infty}\mathfrak{P}_1(\mathrm{n}) = \mathfrak{E}_1 \tag{39}$$

$$\sum_{n=1}^{\infty}\mathfrak{P}_2(\mathrm{n}) = \mathfrak{E}_2 \tag{40}$$

$$\sum_{n=1}^{\infty}\mathfrak{P}_3(\mathrm{n}) = \mathfrak{E}_3 \tag{41}$$

$$\sum_{n=1}^{\infty}\mathfrak{P}_4(\mathrm{n}) = \mathfrak{E}_4 \tag{42}$$

$$\sum_{n=1}^{\infty} n[\mathfrak{P}_1(\mathrm{n}) + \mathfrak{P}_2(\mathrm{n}) + \mathfrak{P}_3(\mathrm{n}) + \mathfrak{P}_4(\mathrm{n})] = \mathfrak{E}_5 \tag{43}$$

where $\mathfrak{E}_1=\mathfrak{R}$, $\mathfrak{E}_2=\Phi_1$, $\mathfrak{E}_3=\Phi_b$, $\mathfrak{E}_4=\mathcal{B}$, $\mathfrak{E}_5=\mathcal{L}_q$.

Therefore, in order to maximise the entropy function subject to the constraints (39)-(43), the Lagrange's function is constructed by us i.e., $\mathfrak{J}$ ($\mathfrak{P}_1(\mathrm{n})$, $\mathfrak{P}_2(\mathrm{n})$, $\mathfrak{P}_3(\mathrm{n})$, $\mathfrak{P}_4(\mathrm{n})$) using Lagrange multipliers θ_1, θ_2, θ_3, θ_4, θ_5 as

$$\mathfrak{J}(\mathfrak{P}_1(\mathrm{n}), \mathfrak{P}_2(\mathrm{n}), \mathfrak{P}_3(\mathrm{n})) = -\sum_{n=1}^{\infty}\mathfrak{P}_1(\mathrm{n})\log\mathfrak{P}_1(\mathrm{n}) - \sum_{n=1}^{\infty}\mathfrak{P}_2(\mathrm{n})\log\mathfrak{P}_2(\mathrm{n}) - \sum_{n=1}^{\infty}\mathfrak{P}_3(\mathrm{n})\log\mathfrak{P}_3(\mathrm{n}) - \sum_{n=1}^{\infty}\mathfrak{P}_4(\mathrm{n})\log\mathfrak{P}_4(\mathrm{n}) - \theta_1\left(\sum_{n=1}^{\infty}\mathfrak{P}_1(\mathrm{n}) - \mathfrak{E}_1\right) - \theta_2\left(\sum_{n=1}^{\infty}\mathfrak{P}_2(\mathrm{n}) - \mathfrak{E}_2\right) - \theta_3\left(\sum_{n=1}^{\infty}\mathfrak{P}_3(\mathrm{n}) - \mathfrak{E}_3\right) - \theta_4\left(\sum_{n=1}^{\infty}\mathfrak{P}_4(\mathrm{n}) - \mathfrak{E}_4\right) - \theta_5\left(\sum_{n=1}^{\infty} n[\mathfrak{P}_1(\mathrm{n}) + \mathfrak{P}_2(\mathrm{n}) + \mathfrak{P}_3(\mathrm{n}) + \mathfrak{P}_4(\mathrm{n})] - \mathfrak{E}_5\right) \tag{44}$$

After formulating the maximum entropy model and using the maximum entropy approach we will develop the steady state probabilities. We will also estimate the average expected amount of time that the consumer waits in queue.

***Lemma* 1:** The approximate steady state possibilities and the maximum entropy solution, subject to constraints, are determined by

$$\mathfrak{P}_1(\mathrm{n}) = \tilde{\sigma}\,\mathfrak{R}\,\frac{(\mathfrak{E}5-\tilde{\sigma})^{\mathrm{n}-1}}{(\mathfrak{E}5)^{\mathrm{n}}},\ \mathfrak{P}_2(\mathrm{n}) = \tilde{\sigma}\,\Phi_1\,\frac{(\mathfrak{E}5-\tilde{\sigma})^{\mathrm{n}-1}}{(\mathfrak{E}5)^{\mathrm{n}}},\ \mathfrak{P}_3(\mathrm{n}) = \tilde{\sigma}\,\Phi_b\,\frac{(\mathfrak{E}5-\tilde{\sigma})^{\mathrm{n}-1}}{(\mathfrak{E}5)^{\mathrm{n}}},\ \mathfrak{P}_4(\mathrm{n}) = \tilde{\sigma}\,\mathcal{B}\,\frac{(\mathfrak{E}5-\tilde{\sigma})^{\mathrm{n}-1}}{(\mathfrak{E}5)^{\mathrm{n}}},$$

where, $\tilde{\sigma} = \mathfrak{R} + \Phi_1 + \Phi_b + \mathcal{B}$.

Proof: We get the requisite results by partially differentiating Lagrange's function $'\mathfrak{J}'$ with regard to $\mathfrak{P}_1(\mathrm{n})$, $\mathfrak{P}_2(\mathrm{n})$, $\mathfrak{P}_3(\mathrm{n})$, $\mathfrak{P}_4(\mathrm{n})$ correspondingly and further equate the outcome to '0'. Also subsequently manipulating the equations we obtain the required results therein.

***Lemma* 2:** According to maximum entropy technique, the average expected amount of time that the clients wait in the line is

$$\mathcal{T}_a = \sum_{n=1}^{\infty}\{\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}\,\mathfrak{P}_1(\mathrm{n}) + \sum_{n=1}^{\infty}\{\frac{\eta}{\tilde{\beta_0}}+\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}\,\mathfrak{P}_2(\mathrm{n}) + \sum_{n=1}^{\infty}\{\frac{\eta}{\tilde{\beta_0}}+\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}\mathfrak{P}_3(\mathrm{n}) + \sum_{n=1}^{\infty}\{\frac{\eta(\mathrm{bw})^2}{\tilde{\beta_0}(\mathrm{bw})}+\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}\,\mathfrak{P}_4(\mathrm{n}) \quad (45)$$

Proof: Assume a specific consumer, let's say $\mathfrak{W}$, who is in queue behind 'n' other people. Currently, the server could be inactive, busy with priority consumer, engaged with ordinary consumer and busy at lower speed service.

(a) Stagnant (Idle) state: The consumer will find this case to be especially accessible. The operator may deliver service whatever they would like. Hence, the group of arriving consumers receives immediate service. This suggests that the estimated waiting time will roughly correspond to the service time of the consumer prior to consumer $\mathfrak{W}$.

$$\tilde{\mathcal{W}_1} = \sum_{n=1}^{\infty}\{\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}.$$

(b) Busy state with priority and ordinary consumers: In this scenario, the server is already dispersed; thus, the group of incoming consumers joins the orbit. First, the 'n' customers in front of them in line receive service, and then the consumers in front of the randomly chosen customer $\mathfrak{W}$ do as well. Hence, the average anticipated waiting time is

$$\tilde{\mathcal{W}_2} = \sum_{n=1}^{\infty}\{\frac{\eta}{\tilde{\beta_0}}+\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\},$$

and,

$$\tilde{\mathcal{W}_3} = \sum_{n=1}^{\infty}\{\frac{\eta}{\tilde{\beta_0}}+\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}.$$

(c) Working leisure and Working malfunction state: The client $\mathfrak{W}$ is awaiting the completion of the working leisure time. Consumers who already in the orbit are attended first when the server resumes operating in its normally busy state. The clients before M then receive their services after that,

$$\tilde{\mathcal{W}_4} = \sum_{n=1}^{\infty}\{\frac{\eta(\mathrm{b_w})^2}{\tilde{\beta_0}(\mathrm{b_w})}+\frac{1}{2\tilde{\beta_0}}(\frac{E(X^2)}{E(X)}-1)\}.$$

We get the needed outcome by adding the waiting times for each of the four situations mentioned above as given in Equation (45).

8. Sensitivity Analysis with Simulation of Practical Application

Sensitivity analysis helps to examine how changes in the input values for a specific variable influence various finding of the mathematical model. It enables people to make appropriate judgements about businesses, economy or their investments and enables decision-makers to know exactly what changes may be made in their respective system to improve the grade of service. Extremely complex models may make it difficult to assess the inputs. By doing sensitivity analysis, users can find out more about which factors are not so much important and can be removed from the model because they are not relevant.

8.1 Simulating Practical Application

To estimate the impact of the parameters that are included in our methodology on the performance measures, numerical examples are absolutely vital. Any response of a model in a specific circumstance can be anticipated by performing a numerical analysis on it.

The Problem

In this particular segment, we have highlighted how the recommended framework can really be used in actual-life circumstances involving congestion in the BusStop framework as discussed in section 3. The BusStop service is accountable for collecting, maintaining, and delivering users with details regarding public transportation. The system described in this part facilitates and regulates the management mechanism. BusStop framework offers its service as well as a number of other facilities over the cell phone, enabling a consumer to call and investigate for themselves the routes, schedules and location of the stoppage. The request for information is sent by mobile users (or consumers), and BusStop framework handles it using a web-based user interface. This BusStop framework functions as only single server. Our ultimate ambition is to assess certain sensitive system parameters in order to calculate the cost of the BusStop framework.

The Solution

In order to illustrate the problem discussed above, we explicitly use the case from Section 3, i.e., by executing a numerical simulation how various eventualities are handled by the BusStop framework. Consumer asks to access the bus route at a rate of $ð_2$= 0.99/min, which we receive affirmatively. The controller interacts to the consumers at a rate of u_1= 3.5/min. The controller gives clients first priority through voice messages, therefore voice messages come at a rate of $\tilde{\beta}$=0.13/min. Although the BusStop framework takes working leisure, the controller works less quickly at a rate of u_2= 3.2 per minute. The BusStop framework (server) is typically thought of as always being reachable. Unfortunately, in many circumstances that occur in real practice, the controller has failure that causes service interruptions. We will employ a working distortion with rate ς= 0.07. The application server could crumble at any moment when the regularly busy controller is running due to disasters. We utilise this technology in our laptops and smartphones that makes it possible for viruses to enter the system. Here, the consumer rate is negative and equals 0.05. These are some potential outcomes: $\Re$ = 0.2185, Φ_1= 0.1578, Φ_b= 0.2320, $\mathcal{B}$ =0.1779, $\hat{P}_0$ = 0.0044, $\mathcal{W}_s$= 21.2925, TC=526.3166 $.

8.2 Impact of System Parameters on Average Queue Length and Long Run Chance

We have also looked into several more numerical outcomes. These findings demonstrate the potential influence of various variables on system performance indicators. The outcome below is demonstrated using MATLAB R2022a software and we have done sensitivity analysis by plotting the graphs through MS EXCEL.

Table 1. Impact of reduced service (lower service) rate u_2 on system performance measures, when $ð_0$ =3.2, $ð_1$= 0.25, $ð_2$= 1.9, $\tilde{\alpha}$=0.10, u_1=3.5, $\tilde{\vartheta}$=0.012, ς=0.05, $\tilde{\beta}$=0.15.

L.W.S ($\mathbf{u_2}$)	$\hat{P}_0$	$\Re$	Φ_1	Φ_b	$\mathcal{B}_{wv}$	$\mathcal{T}_f$	$\mathcal{B}$
3.3	0.0022	0.2591	0.1554	0.2871	0.2962	0.0287	0.1478
3.4	0.0023	0.2579	0.1547	0.2858	0.2993	0.0286	0.1472
3.5	0.0023	0.2567	0.1540	0.2846	0.3023	0.0285	0.1465
3.6	0.0024	0.2555	0.1533	0.2834	0.3054	0.0283	0.1459
3.7	0.0024	0.2532	0.1526	0.2821	0.3084	0.0282	0.1453

Table 2. Impact of vacation rate $\tilde{\vartheta}$ on system performance measures, when ð$_0$ =3.2, ð$_1$=0.25, ð$_2$=1.9 $\tilde{\alpha}$=0.10, u$_1$=3.5, u$_2$ =3.2, ς=0.05.

V.R ($\tilde{\vartheta}$)	$\hat{P}_0$	$\Re$	Φ_1	Φ_b	$\mathcal{B}_{wv}$	$\mathcal{T}_f$	$\mathcal{B}$
0.013	0.0022	0.2602	0.1561	0.2884	0.2931	0.0288	0.1485
0.018	0.0022	0.2601	0.1560	0.2884	0.2933	0.0288	0.1485
0.023	0.0022	0.2599	0.1559	0.2884	0.2936	0.0288	0.1485
0.028	0.0022	0.2598	0.1559	0.2884	0.2938	0.0288	0.1485
0.033	0.0022	0.2596	0.1558	0.2884	0.2940	0.0288	0.1485

Table 3. Impact of negative (adverse) arrival rate $\tilde{\alpha}$ on system performance measures, when ð$_0$ =3.2, ð$_1$=0.25 ð2=1.9 u$_1$=3.5, u$_2$=3.2, $\tilde{\vartheta}$=0.012, ς=0.05, $\tilde{\beta}$=0.15.

N.A.R ($\tilde{\alpha}$)	$\hat{P}_0$	$\Re$	Φ_1	Φ_b	$\mathcal{B}_{wv}$	$\mathcal{T}_f$	$\mathcal{B}$
0.12	0.0022	0.2637	0.1465	0.2906	0.2969	0.0349	0.1504
0.14	0.0022	0.2668	0.1380	0.2925	0.3004	0.0409	0.1522
0.16	0.0022	0.2697	0.1305	0.2940	0.3036	0.0470	0.1538
0.18	0.0022	0.2722	0.1237	0.2953	0.3065	0.0532	0.1553
0.20	0.0022	0.2746	0.1177	0.2063	0.3092	0.0593	0.1566

Table 4. Impact of priority rate $\tilde{\beta}$ on system performance measures, when ð$_0$ =3.2, ð$_1$=0.25 ð$_2$=1.9 u$_1$=3.5, u$_2$=3.2, $\tilde{\vartheta}$=0.012, ς=0.05.

P.R ($\tilde{\beta}$)	$\hat{P}_0$	$\Re$	Φ_1	Φ_b	$\mathcal{B}_{wv}$	$\mathcal{T}_f$	$\mathcal{B}$
0.16	0.0021	0.2594	0.1596	0.2866	0.2923	0.0287	0.1483
0.21	0.0021	0.2559	0.1733	0.2791	0.2896	0.0279	0.1480
0.26	0.0020	0.2536	0.1831	0.2731	0.2882	0.0273	0.1483
0.31	0.0020	0.2520	0.1905	0.2679	0.2876	0.0268	0.1491
0.36	0.0019	0.2508	0.1963	0.2634	0.2875	0.0263	0.1501

From all the aforementioned tables we will conclude the few remarks which are listed below:

For Tables 1, 2, and 3, the value of probabilities enhances as we upsurge the value of any of these rates i.e., lower service rates, vacation rates, negative arrival rates and priority rates. Such probabilities include operator in indolent (or idle) periods, priority user absorbed with normal service periods, engaged with ordinary users in normal service period. Consequently, as we enhance the value of reduced service rates, vacation rates, negative arrival *(advent*) rates, and priority rates, the steady state possibility for operator in reiterate phase for Tables 1, 2, and 4 is amplified. The operator is in a failure state, which is amplified for Tables 1, 4 as we raise the value of lower service rates, vacation rates, negative arrival rates, and priority rates. Additionally, for Table 4, the possibilities for operator are dormant, indulge with priority users with normal service period is declined as we increase the values of any u$_2$, $\tilde{\vartheta}$, $\tilde{\alpha}$ or $\tilde{\beta}$. Finally, the chances for operator being failure is augmented due to upgrading the value of parameters same as before.

In addition, we highlight how the queue length of the customers fluctuates based on various criteria using Figures 2, 3, 4, 5. The suggested attribute values are ð$_0$ = 0.3.2; ð$_1$ = 0.19; ð$_2$ = 0.99; $\tilde{\vartheta}$ = 0.01; ς = 0.05; $\tilde{\beta}$ =0.12; u$_1$ = 3.5; u$_2$ = 3.2; $\tilde{\gamma}$ = 0.14; $\tilde{\alpha}$ = 0.12. We conclude that Figures 2, 3 and 4, 5 display the anticipated queue length curve by altering for various values of (a) ð$_1$ (b) ς (c) $\tilde{\vartheta}$ (d) $\tilde{\beta}$.

We also predict the entire set of figures which reflect a significant drop in queue length, proving that our technique is applicable and consistent in the actual world as well.

Note: μ_2 represents the lower service rate during working leisure and working malfunction and μ_1 represents regular service rate.

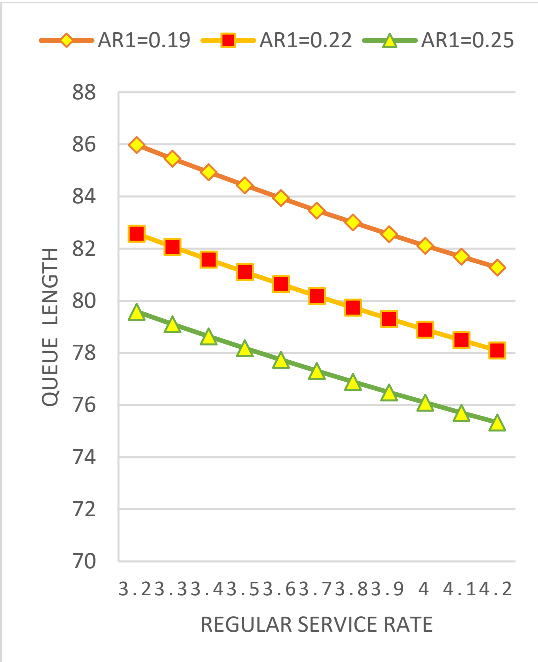

Figure 2. μ_1 V s ð$_1$ by fluctuating ð$_1$.

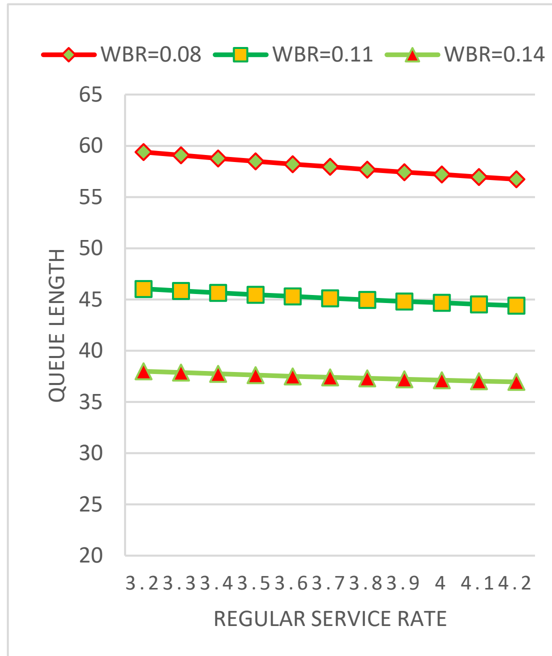

Figure 3. μ_1 V s ς by fluctuating ς.

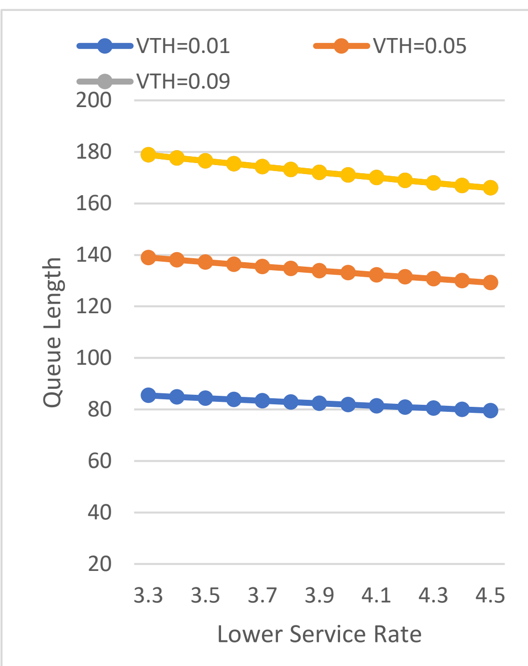

Figure 4. μ_2 V s $\tilde{\vartheta}$ by fluctuating $\tilde{\vartheta}$.

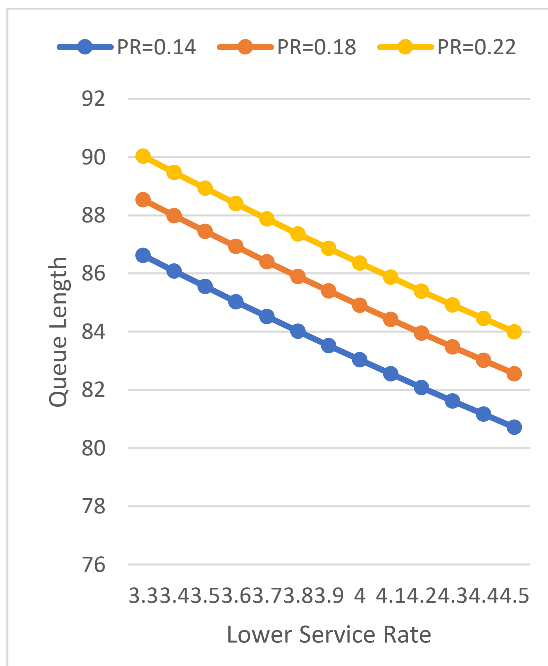

Figure 5. u$_2$ V s $\tilde{\beta}$ by fluctuating $\tilde{\beta}$.

8.3 Comparable Analysis Amidst Precise (Exact) Waiting Time and Imminent (Approximate) Waiting Time

We demonstrate a correlative analysis in the subsequent tables that includes the precise and imminent (approximative) waiting times ($\mathcal{T}_s$ and $\mathcal{T}_a$, respectively) of our developed model as well as the absolute percentage error (APE) specified by APE= [($\mathcal{T}_s - \mathcal{T}_a$) / $\mathcal{T}_s$] × 100. We assign the defaults settings as $ð_0$ = 0.5, $ð_1$ =0.6, $ð_2$ = 0.7 for Case-1. Also, for Case-2 we select the values $ð_0$ = 0.42, $ð_1$ = 0.63, $ð_2$ = 0.81.

Table 5. Impact of ς, β˘, and α˘ on $\mathcal{T}_s$ and $\mathcal{T}_a$ for various values of $ð_0$, $ð_1$, $ð_2$.

Case-1				Case-2			
(ς)	$\mathcal{T}_s$	$\mathcal{T}_a$	APE	(ς)	$\mathcal{T}_s$	$\mathcal{T}_a$	APE
0.14	110.11	106.84	2.9714	0.16	121.79	118.95	2.3324
0.15	107.93	97.99	9.2117	0.17	121.7938	118.9530	2.3324
0.16	106.51	90.77	14.7792	0.18	119.21	104.94	11.9696
$\tilde{\beta}$	$\mathcal{T}_s$	$\mathcal{T}_a$	APE	$\tilde{\beta}$	$\mathcal{T}_s$	$\mathcal{T}_a$	APE
0.15	104.63	99.07	5.3181	0.175	119.40	111.34	6.7472
0.16	102.89	92.89	9.7249	0.185	117.94	111.56	5.4110
0.17	102.00	93.33	8.4921	0.195	116.53	111.78	4.0809
$\tilde{\alpha}$	$\mathcal{T}_s$	$\mathcal{T}_a$	APE	$\tilde{\alpha}$	$\mathcal{T}_s$	$\mathcal{T}_a$	APE
0.07	112.27	107.47	4.2720	0.10	118.78	114.28	3.7909
0.08	110.46	105.51	4.4829	0.11	117.59	112.96	3.9407
0.09	108.95	103.85	4.6772	0.12	116.53	111.78	4.0809

Table 6. Impact of ς, $\tilde{\beta}$, $\tilde{\alpha}$ and on $\mathcal{T}_s$ and $\mathcal{T}_a$ for various values of $ð_0$, $ð_1$, $ð_2$.

Case-1				Case-2			
μ_2	$\mathcal{T}_s$	$\mathcal{T}_a$	APE	μ_2	$\mathcal{T}_s$	$\mathcal{T}_a$	APE
2.50	108.95	103.85	4.6772	2.65	134.96	125.33	7.1311
2.55	110.90	101.90	7.4329	2.70	136.45	123.38	9.5737
2.60	111.23	100.03	10.0734	2.75	137.97	121.53	11.9180
$\tilde{\gamma}$	$\mathcal{T}_s$	$\mathcal{T}_a$	APE	$\tilde{\gamma}$	$\mathcal{T}_s$	$\mathcal{T}_a$	APE
0.11	110.05	99.07	99778	0.14	127.25	122.45	3.7771
0.12	108.90	98.14	9.8845	0.15	125.79	121.28	3.5870
0.13	107.78	97.23	9.7932	0.16	124.38	120.15	3.4012
μ_1	$\mathcal{T}_s$	$\mathcal{T}_a$	APE	μ_1	$\mathcal{T}_s$	$\mathcal{T}_a$	APE
3.6	108.04	100.31	7.1555	3.3	132.81	119.31	10.1589
3.7	108.29	103.40	4.5165	3.4	133.16	123.34	7.3723
3.8	108.53	106.49	1.8763	3.5	133.49	127.37	4.5844

We acknowledge from Tables 5 and 6 that the waiting time for the consumers in waiting framework is inversely proportional to ς, $\tilde{\beta}$, $\tilde{\alpha}$, $\tilde{\gamma}$ for Case-1 and for the remaining parameters i.e., u_2, u_1 it is completely the reverse. Additionally, Case 2 follows the same trend with respect to precise waiting time and imminent waiting time. Nevertheless, APE is significantly lower in Case 2 than that in Case 1 in relative terms. We can infer from the numerical analysis outlined above that the outcomes are consistent with actual situations that are seen in cyber cafes and other service systems.

9. Bi-Objective Optimization: Joint Optimal Values of Expected Cost of the Framework and Expected Waiting Time of Consumers in System

Multi-Objective optimization aims to discover an effective solution that balances two functions so that the system generates the lowest cost and minimize the waiting times of the consumers simultaneously for a particular queueing system. Administrators are able to accomplish the best design over a wide range by the help of this technique. In this portion, we determine an optimisation problem with two objectives in which anticipated cost function TC^* and anticipated waiting time $\mathcal{W}_s^*$ in the waiting line must both be optimised simultaneously over the domain D. Assume the subsequent bi-objective optimization.

$(\widetilde{B}, \widetilde{P})$ $(TC^*, \mathcal{L}_s^*)$ = min $(TC, \mathcal{W}_s)$,

where D: {(ð, $\tilde{\vartheta}$): ð ∈ (0.2, 1.7), $\tilde{\vartheta}$ ∈ (0.65, 0.75)} ⊂ R^2.

The Multi-Objective Genetic Algorithm (MOGA) has been employed to identify a worthwhile solution to the problem. The satisfactory result found using **MOGA** is shown in Figure 6 where it can be observed that the minimal cost and minimal user waiting time are TC=32.62$ and $\mathcal{W}_s$=0.7514 mins for the pareto rates (ð, $\tilde{\vartheta}$) = (0.2, 0.65). By altering the working malfunction rate i.e., 'ς', we have found Pareto optimal solutions in Table 7 for various values of joining probabilities for distinct cost sets. Also, in Table 8 we have also find the Pareto optimal solution for various values of joining probabilities values values for different cost sets by modifying the priority rate i.e., '$\tilde{\beta}$'.

Table 7. Pareto optimal solutions for various values of joining probabilities for different cost sets by varying 'ς' i.e., working malfunction rate.

Parameters	Join optimal parameters and joint optimal objective functions	Cost Set A	Cost Set B	Cost Set B
ς=0.07	(ð, $\tilde{\vartheta}$); (TC, $\mathcal{W}_s$)	(0.8955, 0.6008); (0.7521, 36.2911)	(0.6007, 0.6004); (0.7532, 28.6003)	(0.5917, 0.6001); (0.7529, 28.3700)
ς=0.09	(ð, $\tilde{\vartheta}$); (TC, $\mathcal{W}_s$)	(0.7567, 0.6001); (0.7514, 32.62550)	(0.2791, 0.6000); (0.7749, 20.8989)	(0.2012, 0.6003); (0.7933, 19.6079)
ς=0.11	(ð, $\tilde{\vartheta}$); (TC, $\mathcal{W}_s$)	(0.2863, 0.6001); (0.7738, 21.0463)	(0.2320, 0.6017); (0.7877, 20.0308)	(0.2641, 0.6017); (0.7804, 20.6108)

Table 8. Pareto optimal solutions for various values of joining probabilities for different cost sets by varying $\tilde{\beta}$ i.e., priority rate.

Parameters	Join optimal parameters and joint optimal objective functions	Cost Set A	Cost Set B	Cost Set C
$\tilde{\beta}$=0.11	(ð, $\tilde{\vartheta}$); (TC, $\mathcal{W}_s$)	(0.2641, 0.6017); (0.7804, 20.6108)	(0.2320, 0.6017); (0.7877, 20.0380)	(0.2791, 0.6000); (0.7749, 20.8989)
$\tilde{\beta}$=0.14	(ð, $\tilde{\vartheta}$); (TC, $\mathcal{W}_s$)	(0.2791, 0.6000); (0.7749, 20.8989)	(0.2320, 0.6017); (0.7877, 20.0380)	(0.8955, 0.6008); (0.7521, 36.2911)
$\tilde{\beta}$=0.15	(ð, $\tilde{\vartheta}$); (TC, $\mathcal{W}_s$)	(0.7567, 0.6001); (0.7514, 32.6255)	(0.8955, 0.6008); (0.7521, 36.2911)	(0.6007, 0.6004); (0.7532, 28.6003)

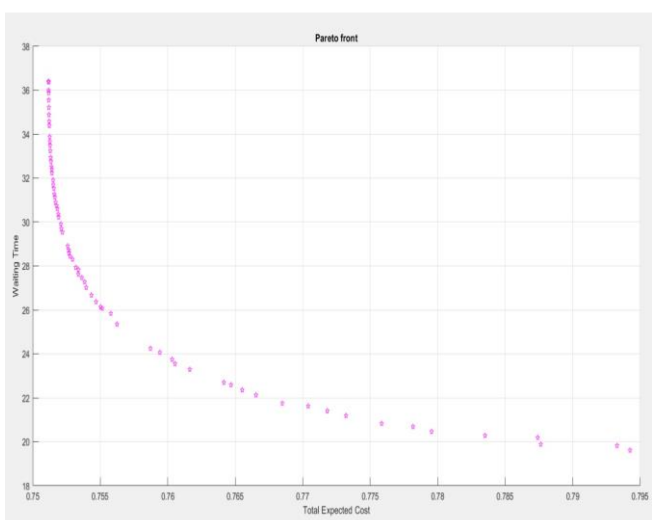

Figure 6. Waiting time vs total expected cost.

10. Conclusion

The whole inquiry includes a bi-objective optimization and maximum entropy findings of bulk general service reiterate priority G-queue with state dependent arrivals, working malfunction and working leisure. Performance metrics that can regulate and diminish queue size, such as steady state possibilities and the system's average queue size are addressed. Our major objective is to offer a correlative analysis that corresponds to the exact and approximative anticipated waiting times for the proposed model with the intention that executive of the framework and managers can shorten the amount of time that users spend and also successively raises the quality of service provided by their corresponding organizations. The waiting faced in manufacturing sector, communication networks, supermarkets, industrial sector and other fields can also be reduced using this proposed framework. Fundamentally, it is nearly impossible to construct a framework in which the server never malfunctions or disengages in all these enormous sectors. In order to better utilise resources, this inquiry is pertinent to and supportive of those regions where a server (operator) can be sluggish. This model is created in a way that aids in preventing the daily congestion issues that networking and communication platforms confront. The good illustration and the numerical simulation covered in this work show how the outcomes are supported in actual rush hours. The inclusion of ideas such as optional services, multiple vacations, etc. could further enhance this study by including optional service or bulk service. However, doing so would make the model more arduous and sophisticated and would need a significant amount of computing effort.

Conflict of Interest

The authors affirm to the absence of any conflicts of interest.

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References

Agarwal, R., Agarwal, D., & Upadhyaya, S. (2021). Study on optimization of unreliable server queueing systems: A PSO based survey. In: Singh, D., Awasthi, A.K., Zelinka, I., Deep, K. (eds) *Proceedings of International Conference on Scientific and Natural Computing: Proceedings of SNC 2021*. Springer, Singapore, pp. 189-197. https://doi.org/10.1007/978-981-16-1528-3_16.

Berman, O., Krass, D., & Wang, J. (2006). Locating service facilities to reduce lost demand. *IIE Transactions*, *38*(11), 933-946. https://doi.org/10.1080/07408170600856722.

Bounkhel, M., Tadj, L., & Hedjar, R. (2020). Entropy analysis of a flexible Markovian queue with server breakdowns. *Entropy*, *22*(9), 979. https://doi.org/10.3390/e22090979.

Chaleshtori, A.E., Jahani, H., & Aghaie, A. (2020). Bi-objective optimization approach to a multi-layer location-allocation problem with jockeying. *Computers & Industrial Engineering*, *149*, 106740. https://doi.org/10.1016/j.cie.2020.106740.

Chauhan, D. (2018). Maximum entropy analysis of unreliable $M^X/(G1, G2)/1$ queue with Bernoulli vacation schedule. *International Journal of Statistics and Applied Mathematics*, *3*(6), 110-118.

Choudhury, G., & Ke, J.C. (2012). A batch arrival retrial queue with general retrial times under Bernoulli vacation schedule for unreliable server and delaying repair. *Applied Mathematical Modelling, 36*(1), 255-269. https://doi.org/10.1016/j.apm.2011.05.047.

Choudhury, G., & Tadj, L. (2011). The optimal control of an $M^x/G/1$ unreliable server queue with two phases of service and Bernoulli vacation schedule. *Mathematical and Computer Modelling*, *54*(1-2), 673-688. https://doi.org/10.1016/j.mcm.2011.03.010.

Demircioglu, M., Bruneel, H., & Wittevrongel, S. (2021). Analysis of a discrete-time queueing model with disasters. *Mathematics*, *9*(24), 3283. https://doi.org/10.3390/math9243283.

Dimitriou, I. (2013). A preemptive resume priority retrial queue with state dependent arrivals, unreliable server and negative customers. *Top*, *21*(3), 542-571. https://doi.org/10.1007/s11750-011-0198-4.

Dimitriou, I., & Langaris, C. (2010). A repairable queueing model with two-phase service, start-up times and retrial customers. *Computers & Operations Research*, *37*(7), 1181-1190. https://doi.org/10.1016/j.cor.2009.03.003.

Falin, G.I., & Templeton, J.G.C. (1997). Retrial queues. Chapman & Hall, London.

Gao, S. (2015). A preemptive priority retrial queue with two classes of customers and general retrial times. *Operational Research*, *15*, 233-251. https://doi.org/10.1007/s12351-015-0175-z.

Gao, S., Wang, J., & Li, W.W. (2014). An M/G/1 retrial queue with general retrial times, working vacations and vacation interruption. *Asia-Pacific Journal of Operational Research*, *31*(02), 1440006. https://doi.org/10.1142/S0217595914400065.

Gelenbe, E. (1989). Random neural networks with negative and positive signals and product form solution. *Neural Computation*, *1*(4), 502-510. https://doi.org/10.1162/neco.1989.1.4.502.

Gelenbe, E. (1991a). Product-form queueing networks with negative and positive customers. *Journal of Applied Probability*, *28*(3), 656-663. https://doi.org/10.2307/3214499.

Gelenbe, E., & Tucci, S. (1991b). Performances d'un système informatique dupliqué. *Comptes rendus de l'Académie des sciences. Série 2, Mécanique, Physique, Chimie, Sciences de l'univers, Sciences de la Terre*, *312*(1), 27-30.

Henderson, W., Northcote, B.S., & Taylor, P.G. (1994). Networks of customer queues and resource queues. *Teletraffic Science and Engineering, 1*, 853-864. https://doi.org/10.1016/b978-0-444-82031-0.50089-5.

Jain, M., & Bhargava, C. (2008). Bulk arrival retrial queue with unreliable server and priority subscribers. *International Journal of Operations Research*, *5*(4), 242-259.

Jain, M., & Upadhyaya, S. (2012). Optimal repairable $M^X/G/1$ queue with Bernoulli feedback and setup. *International Journal of Mathematics in Operational Research*, *4*(6), 679-702. https://doi.org/10.1504/ijmor.2012.049939.

Jaynes, E.T. (1957). Information theory and statistical mechanics. *Physical Review Journals Archive.*, *106*(4), 620. https://doi.org/10.1103/physrev.106.620.

Kalidass, K., & Kasturi, R. (2012). A queue with working breakdowns. *Computers & Industrial Engineering*, *63*(4), 779-783. https://doi.org/10.1016/j.cie.2012.04.018.

Kapur, J.N. (1989). *Maximum-entropy models in science and engineering*. John Wiley & Sons.

Katayama, T., & Kobayashi, K. (2007). Analysis of a nonpreemptive priority queue with exponential timer and server vacations. *Performance Evaluation*, *64*(6), 495-506. https://doi.org/10.1016/j.peva.2006.06.003.

Ke, J.C., Lin, C.H., Yang, J.Y., & Zhang, Z.G. (2009). Optimal (d, c) vacation policy for a finite buffer M/M/c queue with unreliable servers and repairs. *Applied Mathematical Modelling*, *33*(10), 3949-3962. https://doi.org/10.1016/j.apm.2009.01.008.

Khodemani-Yazdi, M., Tavakkoli-Moghaddam, R., Bashiri, M., & Rahimi, Y. (2019). Solving a new bi-objective hierarchical hub location problem with an M⁄ M⁄ c queuing framework. *Engineering Applications of Artificial Intelligence*, *78*, 53-70. https://doi.org/10.1016/j.engappai.2018.10.004.

Kim, B.K., & Lee, D.H. (2014). The M/G/1 queue with disasters and working breakdowns. *Applied Mathematical Modelling*, *38*(5-6), 1788-1798. https://doi.org/10.1016/j.apm.2013.09.016.

Li, T., & Zhang, L. (2017). An M/G/1 retrial G-queue with general retrial times and working breakdowns. *Mathematical and Computational Applications, 22*(1), 15. https://doi.org/10.3390/mca22010015.

Liou, C.D. (2015). Markovian queue optimisation analysis with an unreliable server subject to working breakdowns and impatient customers. *International Journal of Systems Science*, *46*(12), 2165-2182. https://doi.org/10.1080/00207721.2013.859326.

Liu, T.H., Hsu, H.Y., Ke, J.C., & Chang, F.M. (2023). Preemptive priority Markovian queue subject to server breakdown with imperfect coverage and working vacation interruption. *Computation*, *11*(5), 89. https://doi.org/10.3390/computation11050089.

Liu, W.Y., Xu, X.L., & Tian, N.S. (2007). Stochastic decompositions in the M/M/1 queue with working vacations. *Operations Research Letters*, *35*(5), 595-600. https://doi.org/10.1016/j.orl.2006.12.007.

Liu, Z., & Gao, S. (2011). Discrete-time Geo1, GeoX2/G1, G2/1 retrial queue with two classes of customers and feedback. *Mathematical and Computer Modelling*, *53*(5-6), 1208-1220. https://doi.org/10.1016/j.mcm.2010.11.090.

Liu, Z., Wu, J., & Yang, G. (2009). An M/G/1 retrial G-queue with preemptive resume and feedback under N-policy subject to the server breakdowns and repairs. *Computers & Mathematics with Applications*, *58*(9), 1792-1807. https://doi.org/10.1016/j.camwa.2009.07.077.

Malik, G., Upadhyaya, S., & Sharma, R. (2021a). Cost inspection of a Geo/G/1 retrial model using particle swarm optimization and Genetic algorithm. *Ain Shams Engineering Journal*, *12*(2), 2241-2254. https://doi.org/10.1016/j.asej.2020.11.012.

Malik, G., Upadhyaya, S., & Sharma, R. (2021b). Particle swarm optimization and maximum entropy results for $M^X/G/1$ retrial G-queue with delayed repair. *International Journal of Mathematical, Engineering and Management Sciences*, *6*(2), 541-563. https://doi.org/10.33889/ijmems.2021.6.2.033.

Manoharan, P., & Subathra, S. (2023). Non-Markovian retrial queue, balking, disaster under working breakdown and working vacation. *Journal of Computational Analysis & Applications*, *31*(2), 244-255.

Mitrany, I.L., & Avi-Itzhak, B. (1968). A many-server queue with service interruptions. *Operations Research*, *16*(3), 628-638. https://doi.org/10.1287/opre.16.3.628.

Muthusamy, S., Devadoss, N., & Ammar, S.I. (2022). Reliability and optimization measures of retrial queue with different classes of customers under a working vacation schedule. *Discrete Dynamics in Nature and Society*, *2022*. https://doi.org/10.1155/2022/6806104.

Nithya, R.P., & Haridass, M. (2020a). Cost optimisation and maximum entropy analysis of a bulk queueing system with breakdown, controlled arrival and multiple vacations. *International Journal of Operational Research*, *39*(3), 279-305. https://doi.org/10.1504/ijor.2020.110476.

Nithya, R.P., & Haridass, M. (2020b). Stochastic modelling and analysis of maximum entropy of $M^X/G/1$ queuing system with balking, startup and vacation interruption. *International Journal of Services and Operations Management*, *37*(3), 343-371. https://doi.org/10.1504/ijsom.2020.111034.

Parkash, O., & Mukesh. (2016). Contribution of maximum entropy principle in the field of queueing theory. *Communications in Statistics-Theory and Methods*, *45*(12), 3464-3472. https://doi.org/10.1080/03610926.2013.875574.

Rajadurai, P. (2018). Sensitivity analysis of an M/G/1 retrial queueing system with disaster under working vacations and working breakdowns. *RAIRO-Operations Research*, *52*(1), 35-54. https://doi.org/10.1051/ro/2017091.

Rajadurai, P., Chandrasekaran, V.M., & Saravanarajan, M.C. (2018a). Analysis of an unreliable retrial G-queue with working vacations and vacation interruption under Bernoulli schedule. *Ain Shams Engineering Journal*, *9*(4), 567-580. https://doi.org/10.1016/j.asej.2016.03.008.

Rajadurai, P., Saravanarajan, M.C., & Chandrasekaran, V.M. (2015). Analysis of M/G/1 retrial queue with balking, negative customers, working vacation and server breakdown. *International Journal of Applied Engineering Research, 10*(55), 4130-4136.

Rajadurai, P., Saravanarajan, M.C., & Chandrasekaran, V.M. (2018b). A study on M/G/1 feedback retrial queue with subject to server breakdown and repair under multiple working vacation policy. *Alexandria Engineering Journal*, *57*(2), 947-962. https://doi.org/10.1016/j.aej.2017.01.002.

Rajadurai, P., Yuvarani, S., & Saravanarajan, M.C. (2016). Performance analysis of preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption. *Songklanakarin Journal of Science & Technology, 38*(5), 507-520.

Seenivasan, M., & Abinaya, R. (2022). Markovian queueing model with single working vacation and server breakdown. *Journal of Computational Analysis & Applications*, *30*(2), 210.

Servi, L.D., & Finn, S.G. (2002). M/M/1 queues with working vacations (M/M/1/WV). *Performance Evaluation*, *50*(1), 41-52. https://doi.org/10.1016/S0166-5316(02)00057-3.

Shannon, C.E. (1948). A mathematical theory of communication. *The Bell System Technical Journal, 27*(3), 379-423. https://doi.org/10.1002/j.1538-7305.1948.tb01338.x.

Song, B., & Roy-Chowdhury, A.K. (2008). Robust tracking in a camera network: A multi-objective optimization framework. *IEEE Journal of Selected Topics in Signal Processing, 2*(4), 582-596. https://doi.org/10.1109/jstsp.2008.925992.

Tadj, L., & Choudhury, G. (2005). Optimal design and control of queues. *Top*, *13*, 359-412. https://doi.org/10.1007/bf02579061.

Tavakkoli-Moghaddam, R., Vazifeh-Noshafagh, S., Taleizadeh, A.A., Hajipour, V., & Mahmoudi, A. (2017). Pricing and location decisions in multi-objective facility location problem with M/M/m/k queuing systems. *Engineering Optimization, 49*(1), 136-160. https://doi.org/10.1080/0305215x.2016.1163630.

Thiruvengadam, K. (1963). Queuing with breakdowns. *Operations Research*, *11*(1), 62-71. https://doi.org/10.1287/opre.11.1.62.

Upadhyaya, S. (2014). Performance analysis of a batch arrival retrial queue with Bernoulli feedback. *International Journal of Mathematics in Operational Research, 6*(6), 680-703. https://doi.org/10.1504/ijmor.2014.065423.

Upadhyaya, S. (2018). Performance analysis of a discrete-time Geo/G/1 retrial queue under J-vacation policy. *International Journal of Industrial and Systems Engineering, 29*(3), 369-388. https://doi.org/10.1504/ijise.2018.093052.

Upadhyaya, S. (2020). Investigating a general service retrial queue with damaging and licensed units: an application in local area networks. *Opsearch, 57*(3), 716-745. https://doi.org/10.1007/s12597-020-00440-1.

Upadhyaya, S., & Kushwaha, C. (2020). Performance prediction and ANFIS computing for unreliable retrial queue with delayed repair under modified vacation policy. *International Journal of Mathematics in Operational Research, 17*(4), 437-466. https://doi.org/10.1504/ijmor.2020.110843.

Upadhyaya, S., Ghosh, S., & Malik, G. (2022). Cost investigation of a batch arrival retrial G-queue with working malfunction and working vacation using particle swarm optimization. *Nonlinear Studies, 29*(3), 809.

Vahdani, B., & Mohammadi, M. (2015). A bi-objective interval-stochastic robust optimization model for designing closed loop supply chain network with multi-priority queuing system. *International Journal of Production Economics, 170*(A), 67-87. https://doi.org/10.1016/j.ijpe.2015.08.020.

Vijayalakshmi, V., Kalidass, K., & Deepa, B. (2021). Cost analysis of M/M/1/N queue with working breakdowns and a two-phase service. *Journal of Physics: Conference Series, 1850*(1), 012026. 10.1088/1742-6596/1850/1/012026.

Wang, J., & Zhang, P. (2009). A discrete-time retrial queue with negative customers and unreliable server. *Computers & Industrial Engineering, 56*(4), 1216-1222. https://doi.org/10.1016/j.cie.2008.07.010.

Wang, K.H., & Huang, K.B. (2009). A maximum entropy approach for the< p, N>-policy M/G/1 queue with a removable and unreliable server. *Applied Mathematical Modelling*, *33*(4), 2024-2034. https://doi.org/10.1016/j.apm.2008.05.007.

Wang, K.H., Chan, M.C., & Ke, J.C. (2007). Maximum entropy analysis of the $M^{[x]}/M/1$ queueing system with multiple vacations and server breakdowns. *Computers & Industrial Engineering*, *52*(2), 192-202. https://doi.org/10.1016/j.cie.2006.11.005.

Wang, K.H., Chuang, S.L., & Pearn, W.L. (2002). Maximum entropy analysis to the N policy M/G/1 queueing system with a removable server. *Applied Mathematical Modelling*, *26*(12), 1151-1162. https://doi.org/10.1016/s0307-904x(02)00056-2.

Wang, K.H., Yang, D.Y., & Pearn, W.L. (2011). Comparative analysis of a randomized N-policy queue: an improved maximum entropy method. *Expert Systems with Applications*, *38*(8), 9461-9471. https://doi.org/10.1016/j.eswa.2011.01.153.

Wu, C.H., Yang, D.Y., & Yong, C.R. (2023). Performance evaluation and bi-objective optimization for F -policy queue with alternating service rates. *Journal of Industrial and Management Optimization*, *19*(5), 3819-3839. https://doi.org/10.3934/jimo.2022111.

Wu, D.-A., & Takagi, H. (2006). M/G/1 queue with multiple working vacations. *Performance Evaluation, 63*(7), 654-681. https://doi.org/10.1016/j.peva.2005.05.005.

Wu, J., & Lian, Z. (2013). A single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. *Computers & Industrial Engineering*, *64*(1), 84-93. https://doi.org/10.1016/j.cie.2012.08.015.

Wu, J., & Yin, X. (2011). An M/G/1 retrial G-queue with non-exhaustive random vacations and an unreliable server. *Computers & Mathematics with Applications*, *62*(5), 2314-2329. https://doi.org/10.1016/j.camwa.2011.07.018.

Yang, S.-p., Wu, J.-b., & Liu, Z.-m. (2013). An $M^{[x]}/G/1$ retrial g-queue with single vacation subject to the server breakdown and repair. *Acta Mathematicae Applicatae Sinica, English Series, 29*(3), 579-596. https://doi.org/10.1007/s10255-013-0237-z.

Yang, T., & Templeton, J.G.C. (1987). A survey on retrial queues. *Queueing Systems, 2*(3), 201-233.

Yang, W.S., Kim, J.D., & Chae, K.C. (2002). Analysis of M/G/1 stochastic clearing systems. *Stochastic Analysis and Applications, 20*(5), 1083-1100. https://doi.org/10.1081/sap-120014554.

Zhang, M., & Liu, Q. (2015). An M/G/1 G-queue with server breakdown, working vacations and vacation interruption. *Opsearch*, *52*(2), 256-270.

Zhang, Z.J., & Xu, X.L. (2008). Analysis for the M/M/1 queue with multiple working vacations and N-policy. *International Journal of Information and Management Sciences*, *19*(3), 495-506.

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