

Optimal Policies of Social Infrastructures Maintenance using Shock and Damage Model

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Abstract

Social infrastructures such as roads and bridges are indispensable for our lives. They have to be maintained continuously and such maintenance has become a big issue in Japan. Social infrastructures are maintained under strict restrictions such as decreasing in local finance revenue and scarcity of skilful engineers. Various kinds of factors such as inspection periods, maintenance costs, and degradation levels, are necessary to consider in establishing efficient maintenance plans of social infrastructures. Furthermore, the special circumstances of social infrastructures such as the delay of constructions which is caused by the scarcity of budget, must be discussed for the efficient maintenance plan. For such discussion, the stochastic cost model which contains preventive and corrective maintenances is useful. Although these models have been studied in mechanical and electronic systems, unique characteristics of social infrastructures such as their enormous scale and delays due to maintenance budget restrictions must be considered when such social infrastructure models are discussed. In this paper, we establish maintenance models of infrastructures which some of preventive maintenance must be prolonged. The expected maintenance cost rate is established using the cumulative damage model and optimal policies which minimizes them are considered. Three basic models and their extended models which consider natural disasters are discussed.

Keywords- Social infrastructures, Asset management, Maintenance, Optimal policy, Cumulative damage model.

1. Introduction

Social infrastructures such as roads and bridges must maintain high reliability. On the other hand, the national and local governments have various restrictions such as a decrease in budget and technicians, and it is necessary to carry out efficient conservation such as minimizing the cost of conservation. The maintenance of social infrastructures has become a big national issue in 2023 because 39% of road bridges and 27% of tunnels in Japan are to exceed 50 years since their construction (Ministry of Land, Infrastructure, Transport and Tourism, 2017). Such budget restrictions cause delays in preventive maintenance (PM) of social infrastructures. When there are multiple social infrastructures which are needed to undergo the PM, the selection which to delay and which to prioritize is a quite difficult issue because multiple factors such as the frequency of infrastructure use and the risk of procrastination must be considered. Furthermore, prioritizing

between multiple facilities is the quite difficult issue because the scale of social infrastructure is huge, and the maintenance cost varies case by case depending on the degree of deterioration. For undergoing such consideration, the stochastic cost model which contains PM and corrective maintenances (CM) is needed. Although these models have been studied in mechanical and electronic systems, unique characteristics of social infrastructures such as their enormous scale and delays due to maintenance budget restrictions must be considered when such models are considered.

Maintenances are divided into PM and CM (Nakagawa, 2007). Establishing effective PM is important and there exist lots of studies concerning the social infrastructures maintenance plan in the past. The tunnel structure degradation model using Poisson process was recommended by Sakai et al. (2014). Kaito et al. (2005) showed that the Life Cycle Cost of bridge maintenance can be divided into fixed costs and variable costs (Kaito et al., 2005). Obama et al. (2012) discussed a methodology to design optimal scrapping and maintenance policies of infrastructures which are to be crashed due to the demand reduction or the advance of degradation (Obama et al., 2012). Kobayashi et al. (2012) considered the pavement degradation process of both the load surface integrity and the load carrying capacity using the combined Markov deterioration model, and suggested the optimal repair and improve model for minimizing the expected LCC (Kobayashi et al., 2012). Tsuda et al. (2015) represented a methodology for predicting bridge component degradation by using the Markovian transition probability model (Tsuda et al., 2005). Ito et al. (2018) proposed optimal policies of social infrastructure maintenance using Markov Chains (Ito et al., 2018).

The deterioration hazard model is useful for establishing effective maintenance policies because complex deterioration processes can be set in the form of hazard functions and optimal policies can be mathematically discussed. So, such models for road pavement (Kobayashi et al., 2013; Shin and Madanat, 2003) and bridges (Tsuda et al., 2006) was considered. Because the maintenance of social infrastructures is performed annually by local governments and budgets of these local governments are restricted, all of necessary PM cannot be performed by the end of the year and some of them are postponed to the next year. Ito et al. (2018) studied social infrastructure hazard models considering such delayed maintenance (Ito and Nakagawa, 2018) and Kishida et al. (2019) expanded it considering natural disasters (Kishida et al., 2019). In the model, the PM is performed when the cumulative damage surpasses the prespecified managerial level Z and the CM is undergone when the cumulative damage surpasses the prespecified failure level $K(>Z)$. In practice, there exist some damage levels between Z and K because the scale of social infrastructures is huge, and PM costs change depends on the levels. When we consider the damage level of delayed maintenance, we should consider those multiple damage levels.

In this paper, we discuss the social infrastructure maintenance model which has several damage levels. PM costs vary when these damage levels vary, and the PM time is exceeded when the damage level is lower than a certain damage level. The expected cost rate is established and the optimal damage level which minimizes the expected cost rate is discussed. Model 1 is a basic multi-stage model and Model 3 is an extended model which considers the maintenance delay. Because deriving the formula of Model 3 directly from Model 1 is desperate, a particular case of Model 3 was considered as Model 2 and the mathematical formula form was clarified by the analysis of Model 2. Model 4, 5, and 6 are extensions of Model 1, 2, and 3 in consideration of natural disasters.

2. Model 1

First, we consider the fundamental model (Kishida et al., 2020). Following maintenance policies of infrastructures are assumed:

- (i) The social infrastructure is deteriorated with traffics, and its amount of damage is proportional to time jT ($j = 1, 2, \dots$) for a specified $T > 0$. An amount W_j of damage due to the j th shock has an identical distribution $G(x) \equiv P_r \{W_j \leq x\}$ with mean time $1/\mu$.
- (ii) The infrastructure damage levels are denoted as discrete levels Z_i ($i = 1, 2, \dots, K - 1, K$), where $Z_1 < Z_2 < \dots < Z_K$ for $K \geq 2$. The CM is undergone at time jT when the cumulative damage surpasses a threshold level Z_K at time jT . When the cumulative damage is between $Z_1 (< Z_K)$ and Z_K , the PM is undergone at time jT .
- (iii) When the cumulative damage is between Z_i and Z_{i+1} ($i = 1, 2, \dots, K - 1$), CM cost is c_K and PM cost is $c_i (< c_{i+1})$.

The probability which CM is undergone at time $(j + 1) T$ is:

$$\sum_{j=0}^{\infty} \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \tag{1}$$

where, $\phi^{(j)}(x)$ ($j = 1, 2, \dots$) denotes the j -fold Stieltjes convolution of distribution $\phi(x)$,

$\phi^{(0)}(x) \equiv 1$. The probabilities which PM is undergone at time $(j + 1) T$ are;

$$\sum_{j=0}^{\infty} \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \quad (i = 1, 2, \dots, K - 1) \tag{2}$$

The mean time to maintenance is:

$$\begin{aligned} & \sum_{j=0}^{\infty} (j + 1)T \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) + \sum_{j=0}^{\infty} (j + 1)T \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\ & = T[1 + M_G(Z_1)] \end{aligned} \tag{3}$$

where, $M_G(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$. The total expected cost until maintenance is:

$$\begin{aligned} & c_K \sum_{j=0}^{\infty} \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) + \sum_{i=1}^{K-1} c_i \sum_{j=0}^{\infty} \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\ & = c_K - \sum_{j=2}^K (c_j - c_{j-1}) \left[G(Z_j) - \int_0^{Z_1} \bar{G}(Z_j - x) dM_G(x) \right] \end{aligned} \tag{4}$$

Thus, the expected cost rate is, from (3) and (4),

$$TC_1(Z_1) = \frac{c_K - \sum_{j=2}^K (c_j - c_{j-1}) \left[G(Z_j) - \int_0^{Z_1} \bar{G}(Z_j - x) dM_G(x) \right]}{1 + M_G(Z_1)} \tag{5}$$

$TC_1(Z_1)$ is expressed as a function of Z_1 because PM is undergone when degradation level is greater than Z_1 . We seek an optimal Z_1^* which minimizes $C_1(Z_1)$. Differentiating $C_1(Z_1)$ with respect to Z_1 and setting it equal to zero;

$$\sum_{j=2}^K (c_j - c_{j-1}) \int_{Z_j - Z_1}^{Z_j} [1 + M_G(Z_j - x)] dG(x) = c_1 \tag{6}$$

Denote the left-hand side of equation (6) by $Q_1(Z_1)$,
 $Q_1(0) = 0$,

$$Q_1(Z_2) = \sum_{j=2}^K (c_j - c_{j-1}) \int_{Z_j - Z_2}^{Z_j} [1 + M_G(Z_j - x)] dG(x).$$

Thus, if $Q_1(Z_2) > c_1$, then there exists a unique and finite Z_1^* ($0 < Z_1^* < Z_2$) which satisfies (6). When $K = 2$, (6) is:

$$\int_{Z_2 - Z_1}^{Z_2} [1 + M_G(Z_2 - x)] dG(x) = \frac{c_1}{c_2 - c_1} \tag{7}$$

whose left-hand side increases strictly with Z_1 from 0 to $M_G(Z_2)$. Thus, if $M_G(Z_2 - x) > c_1 / (c_2 - c_1)$, then there exists a unique and finite Z_1^* which satisfies (7). It can be easily shown that Z_1^* in (6) decreases with K to Z_1 in (7).

When $c_K > c_{K-1} = c_{K-2} = \dots = c_2 > c_1$, the expected cost rate in (5) is:

$$TC_1(Z_1) = \frac{c_1 - (c_K - c_2) \left[\bar{G}(Z_K) - \int_0^{Z_1} \bar{G}(Z_K - x) dM_G(x) \right] + (c_2 - c_1) \left[\bar{G}(Z_2) - \int_0^{Z_1} \bar{G}(Z_2 - x) dM_G(x) \right]}{1 + M_G(Z_1)} \tag{8}$$

and (6) is rewritten as;

$$(c_K - c_2) \int_{Z_K - Z_1}^{Z_K} [1 + M_G(Z_K - x)] dG(x) + (c_2 - c_1) \int_{Z_2 - Z_1}^{Z_2} [1 + M_G(Z_2 - x)] dG(x) = c_1 \tag{9}$$

whose left-hand side $Q_1(Z_1)$ increases strictly with Z_1 from 0 to

$$\begin{aligned} Q_1(Z_2) &= (c_K - c_2) \int_{Z_K - Z_2}^{Z_K} [1 + M_G(Z_K - x)] dG(x) \\ &\quad + (c_2 - c_1) \int_0^{Z_2} [1 + M_G(Z_2 - x)] dG(x) \\ &= (c_K - c_2) \int_{Z_K - Z_2}^{Z_K} [1 + M_G(Z_K - x)] dG(x) + (c_2 - c_1) M_G(Z_2). \end{aligned}$$

Thus, if $Q_1(Z_2) > c_1$, then there exists a unique and finite Z^*_1 ($0 < Z^*_1 < Z_2$) which satisfies (9). It can be easily shown that Z^*_1 in (6) is smaller than Z^*_1 in (9). When $G(x) = 1 - \exp(-\mu x)$, the expected cost rate (5) is rewritten as;

$$TC_1(Z_1) = \frac{c_1 + \sum_{j=2}^K (c_j - c_{j-1})e^{-\mu(Z_j - Z_1)}}{1 + \mu Z_1} \tag{10}$$

and (6) is rewritten as;

$$\sum_{j=2}^K (c_j - c_{j-1})\mu Z_1 e^{-\mu(Z_j - Z_1)} = c_1 \tag{11}$$

Letting denote the left-hand side of (11) by $Q_1(Z_1)$, $Q_1(Z_1)$ increases strictly with Z_1 from 0 to

$$Q_1(Z_2) = \sum_{j=2}^K (c_j - c_{j-1})\mu Z_2 e^{-\mu(Z_j - Z_2)} \tag{12}$$

Thus, if $Q_1(Z_2) > c_1$, then there exists a unique and finite Z^*_1 ($0 < Z^*_1 < Z_2$) which satisfies (11).

In particular, when $K = 2$, (11) is:

$$\mu Z_1 e^{-\mu(Z_2 - Z_1)} = \frac{c_1}{c_2 - c_1} \tag{13}$$

If $\mu Z_2 > c_1 / (c_2 - c_1)$, then there exists a unique and finite Z^*_1 ($0 < Z^*_1 < Z_2$) which satisfies (13).

3. Model 2

We consider the following maintenance policy and assumptions (1) and (3) in Section 2.

2') The infrastructure damage levels are denoted as discrete levels Z_i ($i = 1, 2, \dots, K - 1, K$). The CM is undergone at time jT when the cumulative damage surpasses a threshold level Z_K at time jT . When the cumulative damage is between $Z_1 (< Z_K)$ and Z_K , the PM is undergone at time $(j + 1) T$.

The probability that CM is undergone at time $(j + 2) T$ is:

$$\sum_{j=0}^{\infty} \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_i - x}^{Z_{i+1} - x} \overline{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \tag{14}$$

the probabilities that PM is undergone at time $(j + 2) T$ is:

$$\begin{aligned} & \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left[\int_{Z_i - x}^{Z_{i+1} - x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\ & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i - x}^{Z_{n+1} - x} [G(Z_{n+1} - x - y) - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \end{aligned} \tag{15}$$

where, $\phi^{(j)}(x)$ ($j = 1, 2, \dots$) denotes the j -fold Stieltjes convolution of distribution $\phi(x)$, $\phi^{(0)}(x) \equiv 1$, and $(1) + (14) + (15) = 1$. The mean time to maintenance is:

$$\begin{aligned}
 & \sum_{j=0}^{\infty} (j+1)T \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\
 & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} (j+2)T \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} (j+2)T \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} (j+2)T \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & = T \left[1 + G(Z_K) + \int_0^{Z_1} G(Z_K - x) dM_G(x) \right] \tag{16}
 \end{aligned}$$

where, $M_G(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$. The total expected cost until maintenance is:

$$\begin{aligned}
 & c_K \sum_{j=0}^{\infty} \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) + c_K \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} c_i \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} c_n \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & = c_K - \sum_{i=1}^{K-1} (c_K - c_i) \sum_{j=0}^{\infty} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & - \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} (c_K - c_n) \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & = c_K - \sum_{j=2}^K (c_j - c_{j-1}) \left\{ \int_{Z_1}^{Z_j} G(Z_j - x) dG(x) \right. \\
 & \left. + \int_0^{Z_1} \left[\int_{Z_1-x}^{Z_j-x} G(Z_j - x - y) dG(y) \right] dM_G(x) \right\} \tag{17}
 \end{aligned}$$

Thus, the expected cost rate is from (16) and (17),

$$TC_2(Z_1) = \frac{c_K - \sum_{j=2}^K (c_j - c_{j-1}) \left\{ \int_{Z_1}^{Z_j} G(Z_j - x) dG(x) + \int_0^{Z_1} \left[\int_{Z_1-x}^{Z_j-x} G(Z_j - x - y) dG(y) \right] dM_G(x) \right\}}{1 + G(Z_K) + \int_0^{Z_1} G(Z_K - x) dM_G(x)} \tag{18}$$

Because it is hard to consider optimal policies analytically, it is assumed that $G(x) = 1 - \exp(-\mu x)$. Then, (18) is rewritten as;

$$TC_2(Z_1) = \frac{c_1 + \sum_{j=2}^K (c_j - c_{j-1}) [1 + \mu(Z_j - Z_1)] e^{-\mu(Z_j - Z_1)}}{2 + \mu Z_1 - e^{-\mu(Z_K - Z_1)}} \tag{19}$$

We seek an optimal Z_1^* which minimizes $C_2(Z_1)$. Differentiating $TC_2(Z_1)$ with respect to Z_1 and setting it equal to zero;

$$\sum_{j=2}^K (c_j - c_{j-1}) e^{-\mu(Z_j - Z_1)} \left[(1 + \mu Z_1) \frac{\mu(Z_j - Z_1)}{1 - e^{-\mu(Z_K - Z_1)}} - 1 \right] = c_1 \tag{20}$$

When $\mu(Z_j - Z_1) = 1 - \exp[-\mu(Z_j - Z_1)]$, the left-hand side of (20) agrees with that of (11). In particular, When $K = 2$, (20) is:

$$e^{-\mu(Z_2 - Z_1)} \left[(1 + \mu Z_1) \frac{\mu(Z_2 - Z_1)}{1 - e^{-\mu(Z_2 - Z_1)}} - 1 \right] = \frac{c_1}{c_2 - c_1} \tag{21}$$

(21) is equivalent to eq. (13) of (see Ito and Nakagawa, 2018). Comparing (21) with (13), Z_1^* in (21) is smaller than that in (13), because $\mu(Z_2 - Z_1) \geq 1 - \exp[-\mu(Z_2 - Z_1)]$.

In general, we cannot prove that the monotonous property of the left-hand side of (20) for Z_1 . So that, we consider the following three approximate \tilde{Z}_1 cases as follows:

3.1 Case I:

$$\sum_{j=2}^K (c_j - c_{j-1}) e^{-\mu(Z_j - Z_1)} \left[(1 + \mu Z_1) \frac{Z_j - Z_1}{Z_K - Z_1} - 1 \right] = c_1 \tag{22}$$

Letting denote the left-hand sides of (20) and (22) by $Q_2(Z_1)$ and $Q_{21}(Z_1)$, respectively. $Q_2(Z_1) > Q_{21}(Z_1)$ because $\mu(Z_K - Z_1) > 1 - \exp[-\mu(Z_K - Z_1)]$. $Z_1^* < \tilde{Z}_1$ where Z_1^* satisfies (20) and \tilde{Z}_1 satisfies (22). When $K = 2$, \tilde{Z}_1 is equal to Z_1^* which satisfies (13).

3.2 Case II:

$$\sum_{j=2}^K (c_j - c_{j-1}) e^{-\mu(Z_j - Z_1)} \left[(1 + \mu Z_1) \frac{\mu(Z_j - Z_1)}{1 - e^{-\mu(Z_K - Z_1)}} - 1 \right] = c_1 \tag{23}$$

Letting denote the left-hand sides of (23) by $Q_{22}(Z_1)$, $Q_2(Z_1) < Q_{22}(Z_1)$ because $1 - \exp[-\mu(Z_K - Z_1)] > 1 - \exp[-\mu(Z_j - Z_1)]$. $Z_1^* > \tilde{Z}_1$ where Z_1^* satisfies (20) and \tilde{Z}_1 satisfies (23).

3.3 Case III:

$$\left[(1 + \mu Z_1) \frac{\mu(Z_j - Z_1)}{1 - e^{-\mu(Z_K - Z_1)}} - 1 \right] \sum_{j=2}^K (c_j - c_{j-1}) e^{-\mu(Z_j - Z_1)} = c_1 \tag{24}$$

Letting denote the left-hand sides of (23) by $Q_{23}(Z_1)$, $Q_2(Z_1) < Q_{23}(Z_1)$ because $\mu(Z_K - Z_1) > \mu(Z_j - Z_1)$. $Z_1^* > \tilde{Z}_1$ where Z_1^* satisfies (20) and \tilde{Z}_1 satisfies (24).

Because

$$\frac{\mu(Z_j - Z_1)}{1 - e^{-\mu(Z_j - Z_1)}} \geq \left[\frac{\mu(Z_K - Z_1)}{1 - e^{-\mu(Z_K - Z_1)}} \right].$$

\tilde{Z}_1 which satisfies (23) is greater than \tilde{Z}_1 which satisfies (24).

4. Model 3

We consider the following maintenance policy and assumptions 1) and 3) in Section 2.

2''). The infrastructure damage levels are denoted as discrete levels Z_i ($i = 1, 2, \dots, L-1, L, \dots, K - 1, K$). The CM is undergone at time jT when the cumulative damage surpasses a threshold level Z_K at time jT . When the cumulative damage is between $Z_L (< Z_K)$ and Z_K the PM is undergone at time $(j + 1) T$ when the cumulative damage is between $Z_1 (< Z_L)$ and Z_L , and is undergone at time jT .

The probability that CM is undergone at time $(j + 2) T$ is:

$$\sum_{i=1}^{L-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left[\int_{Z_i - x}^{Z_{i+1} - x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \tag{25}$$

The probabilities that PM is undergone at time $(j + 2) T$ is:

$$\begin{aligned} & \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left[\int_{Z_i - x}^{Z_{i+1} - x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\ & + \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i - x}^{Z_{i+1} - x} [G(Z_{n+1} - x - y) - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\ & + \sum_{i=L}^{K-1} \sum_{j=0}^{\infty} \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \end{aligned} \tag{26}$$

where, (1) + (25) + (26) = 1. The mean time to maintenance is:

$$\begin{aligned} & \sum_{j=0}^{\infty} (j + 1)T \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\ & + \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} (j + 2)T \int_0^{Z_1} \left[\int_{Z_i - x}^{Z_{i+1} - x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} (j+2)T \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} G(Z_{i+1}-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} (j+2)T \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1}-x-y) - G(Z_n-x-y)] dG(y) \right\} dG^{(j)}(x) \\
 & + \sum_{i=L}^{K-1} \sum_{j=0}^{\infty} (j+1)T \int_0^{Z_1} [G(Z_{i+1}-x) - G(Z_i-x)] dG^{(j)}(x) \\
 & = T \left[1 + G(Z_L) + \int_0^{Z_1} G(Z_L-x) dM_G(x) \right] \tag{27}
 \end{aligned}$$

The total expected cost until maintenance is:

$$\begin{aligned}
 & c_K \sum_{j=0}^{\infty} \int_0^{Z_1} \bar{G}(Z_K-x) dG^{(j)}(x) + c_K \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} c_i \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} G(Z_{i+1}-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{i=L}^{K-1} c_i \sum_{j=0}^{\infty} \int_0^{Z_1} [G(Z_{i+1}-x) - G(Z_i-x)] dG^{(j)}(x) \\
 & + \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} c_n \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1}-x-y) - G(Z_n-x-y)] dG(y) \right\} dG^{(j)}(x) \\
 & = c_K - \sum_{i=L}^{K-1} \sum_{j=0}^{\infty} (c_K - c_i) \int_0^{Z_1} [G(Z_{i+1}-x) - G(Z_i-x)] dG^{(j)}(x) \\
 & - \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} (c_K - c_n) \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1}-x-y) - G(Z_n-x-y)] dG(y) \right\} dG^{(j)}(x) \\
 & - \sum_{i=1}^{L-1} \sum_{j=0}^{\infty} (c_K - c_i) \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} G(Z_{i+1}-x-y) dG(y) \right] dG^{(j)}(x) \\
 & = c_K - \sum_{i=L+1}^K (c_i - c_{i+1}) \left\{ G(Z_i) - G(Z_L) + \int_0^{Z_1} [G(Z_i-x) - G(Z_L-x)] dM_G(x) \right\} \\
 & - \sum_{i=2}^L \sum_{j=i}^K (c_j - c_{j-1}) \left\{ \int_{Z_{i-1}}^{Z_i} G(Z_j-x) dG(x) \right\}
 \end{aligned}$$

$$+ \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_j - x - y) dG(y) \right] dM_G(x) \} \tag{28}$$

Thus, the expected cost rate is from (27) and (28),

$$\begin{aligned}
 & c_K - \sum_{i=L+1}^K (c_i - c_{i-1}) \{G(Z_i) - G(Z_L) \\
 & + \int_0^{Z_1} [G(Z_i - x) - G(Z_L - x)] dM_G(x) \} \\
 & - \sum_{i=2}^L \sum_{j=i}^K (c_j - c_{j-1}) \left\{ \int_{Z_{i-1}}^{Z_i} G(Z_j - x) dG(x) \right. \\
 TC_3(Z_1) = & \left. + \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_j - x - y) dG(y) \right] dM_G(x) \right\} \\
 & \frac{1 + G(Z_L) + \int_0^{Z_1} G(Z_L - x) dM_G(x)}{\tag{29}
 \end{aligned}$$

Because it is difficult to discuss optimal policies analytically, it is assumed that $G(x) = 1 - \exp(-\mu x)$. Then, (29) is rewritten as;

$$TC_3(Z_1) = \frac{c_1 + \sum_{i=2}^K (c_i - c_{i-1}) e^{-\mu(Z_i - Z_1)} + \sum_{i=2}^L \sum_{j=i}^K (c_j - c_{j-1}) \mu(Z_i - Z_{i-1}) e^{-\mu(Z_j - Z_1)}}{2 + \mu Z_1 - e^{-\mu(Z_L - Z_1)}} \tag{30}$$

We seek an optimal Z_1^* which minimizes $C_3(Z_1)$. Differentiating (30) with respect to Z_1 and setting it equal to zero,

$$\begin{aligned}
 & \sum_{i=2}^L (c_i - c_{i-1}) e^{-\mu(Z_i - Z_1)} \left[\frac{(1 + \mu Z_1) \mu(Z_i - Z_1)}{1 - e^{-\mu(Z_L - Z_1)}} - 1 \right] \\
 & - \left[\frac{(1 + \mu Z_1) \mu(Z_L - Z_1)}{1 - e^{-\mu(Z_L - Z_1)}} - 1 \right] \sum_{i=L+1}^K (c_i - c_{i-1}) e^{-\mu(Z_i - Z_1)} = c_1 \tag{31}
 \end{aligned}$$

When $L = K$, (31) is equivalent to (20) where $\sum_{i=K+1}^K \equiv 0$.

5. Extended Models

We expand Model 1, 2, and 3 to models which consider the natural disaster.

5.1 Model 4

We make assumptions 1), 2), and 3) of Section 2 and add the following assumption:

4). The social infrastructure is damaged by the natural disaster and the severe weather which occur independently of normal degradations, and the disaster recovery cost is much higher than the CM cost. The occurrence time of the natural disaster has a general distribution $F(t)$, and the average disaster recovery cost is $c_D (> c_K)$.

The probability that CM is undergone at time $(j + 1) T$ is:

$$\sum_{j=0}^{\infty} \bar{F}((j + 1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \tag{32}$$

the probability that PM is undergone at time $(j + 1) T$ is:

$$\sum_{i=1}^{K-1} \sum_{j=0}^{\infty} \bar{F}((j + 1)T) \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \tag{33}$$

and the probability that the disaster recovery occurs at time $(j + 1) T$ is:

$$\sum_{j=0}^{\infty} F((j + 1)T) \int_0^{Z_1} \bar{G}(Z_1 - x) dG^{(j)}(x) \tag{34}$$

The mean time to maintenance is:

$$\begin{aligned} & \sum_{j=0}^{\infty} (j + 1)T \bar{F}((j + 1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\ & + \sum_{i=1}^{K-1} \sum_{j=0}^{\infty} (j + 1)T \bar{F}((j + 1)T) \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\ & + \sum_{j=0}^{\infty} \int_0^{(j+1)T} t dF(t) \int_0^{Z_1} \bar{G}(Z_1 - x) dG^{(j)}(x) = \sum_{j=0}^{\infty} G^{(j)}(Z_1) \int_{jT}^{(j+1)T} \bar{F}(t) dt \end{aligned} \tag{35}$$

and the total expected cost until maintenance is:

$$\begin{aligned} & c_K \sum_{j=0}^{\infty} \bar{F}((j + 1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\ & + \sum_{i=1}^{K-1} c_i \sum_{j=0}^{\infty} \bar{F}((j + 1)T) \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\ & + c_D \sum_{j=0}^{\infty} F((j + 1)T) \int_0^{Z_1} \bar{G}(Z_1 - x) dG^{(j)}(x) \\ & = c_K - \sum_{i=1}^{K-1} (c_K - c_i) \sum_{j=0}^{\infty} \bar{F}((j + 1)T) \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\ & + (c_D - c_K) \sum_{j=0}^{\infty} F((j + 1)T) [G^{(j)}(Z_1) - G^{(j+1)}(Z_1)] \\ & = c_K - \sum_{i=2}^K (c_i - c_{i-1}) \sum_{j=0}^{\infty} \bar{F}((j + 1)T) \int_0^{Z_1} G(Z_i - x) dG^{(j)}(x) + (c_K - c_1) \sum_{j=1}^{\infty} \bar{F}(jT) G^{(j)}(Z_1) \\ & + (c_D - c_K) \sum_{j=0}^{\infty} F((j + 1)T) [G^{(j)}(Z_1) - G^{(j+1)}(Z_1)] \end{aligned} \tag{36}$$

Thus, the expected cost rate is, from (35) and (36),

$$C_4(Z_1) = \frac{c_K - \sum_{i=2}^K (c_i - c_{i-1}) \sum_{j=0}^{\infty} \bar{F}((j+1)T) \int_0^{Z_1} G(Z_i - x) dG^{(j)}(x) + (c_K - c_1) \sum_{j=1}^{\infty} \bar{F}(jT) G^{(j)}(Z_1) + (c_D - c_K) \sum_{j=0}^{\infty} F((j+1)T) [G^{(j)}(Z_1) - G^{(j+1)}(Z_1)]}{\sum_{j=0}^{\infty} G^{(j)}(Z_1) \int_{jT}^{(j+1)T} \bar{F}(t) dt} \tag{37}$$

When $F(t) = 1 - \exp(-\lambda t)$ and $G(x) = 1 - \exp(-\mu x)$, (37) is:

$$\frac{C_4(Z_1)}{\lambda} = \frac{c_K - \alpha e^{-(1-\alpha)\mu Z_1} \sum_{i=2}^K (c_i - c_{i-1}) [1 - e^{-\mu(Z_i - Z_1)}]}{1 - \alpha e^{-(1-\alpha)\mu Z_1}} + c_D - c_K \tag{38}$$

where, $\alpha \equiv \exp(-\lambda T)$. We find optimal Z_1^* to minimize $C_4(Z_1)$. Differentiating $C_4(Z_1)$ with Z_1 and putting it equal to zero,

$$\frac{\alpha}{1 - \alpha} [1 - e^{-(1-\alpha)\mu Z_1}] \sum_{i=2}^K (c_i - c_{i-1}) e^{-\mu(Z_i - Z_1)} = c_1 \tag{39}$$

Letting denote the left-hand side of equation (39) by $Q_4(Z_1; \alpha)$, $Q_4(0; \alpha) = 0$,

$$Q_4(Z_2; \alpha) = \frac{\alpha}{1 - \alpha} [1 - e^{-(1-\alpha)\mu Z_2}] \left[c_2 - c_1 - \sum_{i=3}^K (c_i - c_{i-1}) e^{-\mu(Z_i - Z_2)} \right].$$

Thus, if $Q_4(Z_2) > c_1$, then there exists a unique and finite Z_1^* ($0 < Z_1^* < Z_2$) which satisfies (39).

When $K = 2$, (12) is equivalent to eq. (22) of (see Ito et al., 2018).

5.2 Model 5

We make assumptions 1) and 3) of Section 2, assumption 2') of Section 3, and assumption 4) of Section 5.1. The mean time to maintenance is:

$$\begin{aligned} & \sum_{j=0}^{\infty} (j+1)T \bar{F}((j+1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\ & + \sum_{j=0}^{\infty} (j+2)T \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\ & + \sum_{j=0}^{\infty} (j+2)T \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\ & + \sum_{j=0}^{\infty} (j+2)T \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) \right. \end{aligned}$$

$$\begin{aligned}
 & -G(Z_n - x - y)]dG(y)\}dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+1)T} t dF(t) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+2)T} t dF(t) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+2)T} t dF(t) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+2)T} t dF(t) \sum_{i=1}^{K-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) \right. \\
 & \left. - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & = \int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_{(j+1)T}^{(j+2)T} \bar{F}(t) dt \int_0^{Z_1} G(Z_K - x) dG^{(j)}(x) \tag{40}
 \end{aligned}$$

and the total expected cost until maintenance is:

$$\begin{aligned}
 & c_K \sum_{j=0}^{\infty} \bar{F}((j+1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\
 & + c_K \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} c_i \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} c_n \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) \right. \\
 & \left. - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & + c_D \left(\sum_{j=0}^{\infty} \bar{F}((j+1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \right. \\
 & \left. + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1}-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1}-x-y) - G(Z_n-x-y)] dG(y) \right\} dG^{(j)}(x) \\
 & = c_K + (c_D - c_K) \left\{ F(T) + \sum_{j=0}^{\infty} [\bar{F}((j+1)T) - \bar{F}((j+2)T)] \int_0^{Z_1} G(Z_K-x) dG^{(j)}(x) \right\} \\
 & - \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{n=2}^K (c_n - c_{n-1}) \int_0^{Z_1} \left[\int_{Z_1-x}^{Z_n-x} G(Z_n-x-y) dG(y) \right] dG^{(j)}(x) \tag{41}
 \end{aligned}$$

Thus, the expected cost rate is, from (40) and (41),

$$\begin{aligned}
 & c_K + (c_D - c_K) \left\{ F(T) + \sum_{j=0}^{\infty} [\bar{F}((j+1)T) - \bar{F}((j+2)T)] \right. \\
 & \quad \left. \times \int_0^{Z_1} G(Z_K-x) dG^{(j)}(x) \right\} \\
 & \quad - \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{n=2}^K (c_n - c_{n-1}) \\
 & \quad \times \int_0^{Z_1} \left[\int_{Z_1-x}^{Z_n-x} G(Z_n-x-y) dG(y) \right] dG^{(j)}(x) \\
 C_5(Z_1) & = \frac{\int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_{(j+1)T}^{(j+2)T} \bar{F}(t) dt \int_0^{Z_1} G(Z_K-x) dG^{(j)}(x)}{\int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_{(j+1)T}^{(j+2)T} \bar{F}(t) dt \int_0^{Z_1} G(Z_K-x) dG^{(j)}(x)}.
 \end{aligned}$$

When $F(t) = 1 - \exp(-\lambda t)$ and $G(x) = 1 - \exp(-\mu x)$, (42) is:

$$\frac{C_5(Z_1)}{\lambda} = \frac{c_K - \alpha^2 e^{\alpha \mu Z_1} \sum_{i=2}^K (c_i - c_{i-1}) [e^{-\mu Z_1} - e^{-\mu Z_i} - \mu(Z_i - Z_1)e^{-\mu Z_i}]}{1 - \alpha e^{\alpha \mu Z_1} [\alpha e^{-\mu Z_1} + (1 - \alpha)e^{-\mu Z_K}]} + c_D - c_K \tag{42}$$

We find optimal Z_1^* to minimize $C_5(Z_1)$. Differentiating $C_5(Z_1)$ with respect to Z_1 and putting it equal to zero,

$$\begin{aligned}
 & \frac{1}{1 - e^{-\mu(Z_K - Z_1)}} \sum_{n=2}^K (c_n - c_{n-1}) \{ (1 - \alpha e^{\alpha \mu Z_1} e^{-\mu Z_K}) [1 - e^{-\mu(Z_i - Z_1)}] \\
 & + \frac{\alpha}{1 - \alpha} [1 - \alpha e^{-(1-\alpha)\mu Z_1}] \mu(Z_i - Z_1) e^{-\mu(Z_i - Z_1)} \} = c_K \tag{43}
 \end{aligned}$$

When $K = 2$, (43) is equivalent to eq. (31) of (Ito et al., 2018).

5.3 Model 6

We make assumptions 1) and 3) of Section 2, assumption 2”) of Section 4, and assumption 4) of Section 5.1. The mean time to maintenance is:

$$\sum_{j=0}^{\infty} (j+1)T \bar{F}((j+1)T) \int_0^{Z_1} \bar{G}(Z_K-x) dG^{(j)}(x)$$

$$\begin{aligned}
 & + \sum_{j=0}^{\infty} (j+2)T\bar{F}((j+2)T) \sum_{i=1}^{L-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} (j+2)T\bar{F}((j+2)T) \sum_{i=1}^{L-1} \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} (j+2)T\bar{F}((j+2)T) \sum_{i=1}^{L-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) \right. \\
 & \left. - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} (j+1)T\bar{F}((j+1)T) \sum_{i=L}^{K-1} \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+1)T} tdF(t) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+2)T} tdF(t) \sum_{i=1}^{L-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+2)T} tdF(t) \sum_{i=1}^{L-1} \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1} - x - y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+2)T} tdF(t) \sum_{i=1}^{L-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1} - x - y) \right. \\
 & \left. - G(Z_n - x - y)] dG(y) \right\} dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \int_0^{(j+1)T} tdF(t) \sum_{i=L}^{K-1} \int_0^{Z_1} [G(Z_{i+1} - x) - G(Z_i - x)] dG^{(j)}(x) \\
 & = \int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_{(j+1)T}^{(j+2)T} \bar{F}(t) dt \int_0^{Z_1} G(Z_L - x) dG^{(j)}(x) \tag{44}
 \end{aligned}$$

and the total expected cost until maintenance is:

$$\begin{aligned}
 & c_K \sum_{j=0}^{\infty} \bar{F}((j+1)T) \int_0^{Z_1} \bar{G}(Z_K - x) dG^{(j)}(x) \\
 & + c_K \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{L-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K - x - y) dG(y) \right] dG^{(j)}(x)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{L-1} c_i \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1}-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{L-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} c_n \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1}-x-y) \right. \\
 & \left. - G(Z_n-x-y)] dG(y) \right\} dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+1)T) \sum_{i=L}^{K-1} c_i \int_0^{Z_1} [G(Z_{i+1}-x) - G(Z_i-x)] dG^{(j)}(x) \\
 & + c_D \left(\sum_{j=0}^{\infty} \bar{F}((j+1)T) \int_0^{Z_1} \bar{G}(Z_K-x) dG^{(j)}(x) \right. \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_i-x}^{Z_{i+1}-x} \bar{G}(Z_K-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_{i+1}-x-y) dG(y) \right] dG^{(j)}(x) \\
 & + \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=1}^{K-1} \int_0^{Z_1} \left\{ \sum_{n=i+1}^{K-1} \int_{Z_i-x}^{Z_{i+1}-x} [G(Z_{n+1}-x-y) \right. \\
 & \left. - G(Z_n-x-y)] dG(y) \right\} dG^{(j)}(x) \\
 & \left. + \sum_{j=0}^{\infty} \bar{F}((j+1)T) \sum_{i=L}^{K-1} \int_0^{Z_1} [G(Z_{i+1}-x) - G(Z_i-x)] dG^{(j)}(x) \right) = c_K + (c_D - c_K)F(T) \\
 & + \sum_{j=0}^{\infty} [(c_D - c_L)\bar{F}((j+1)T) - (c_D - c_K)\bar{F}((j+2)T)] \int_0^{Z_1} G(Z_L-x) dG^{(j)}(x) \\
 & - \sum_{j=0}^{\infty} \bar{F}((j+1)T) \sum_{i=L+1}^K (c_i - c_{i-1}) \int_0^{Z_1} G(Z_i-x) dG^{(j)}(x) \\
 & - \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=2}^L \sum_{n=i}^K (c_n - c_{n-1}) \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_n-x-y) dG(y) \right] dG^{(j)}(x) \tag{45}
 \end{aligned}$$

Thus, the expected cost rate is, from (44) and (45),

$$\begin{aligned}
 & c_K + (c_D - c_K)F(T) \\
 & + \sum_{j=0}^{\infty} [(c_D - c_L)\bar{F}((j+1)T) - (c_D - c_K)\bar{F}((j+2)T)] \\
 & \quad \times \int_0^{Z_1} G(Z_L - x) dG^{(j)}(x) \\
 & - \sum_{j=0}^{\infty} \bar{F}((j+1)T) \sum_{i=L+1}^K (c_i - c_{i-1}) \int_0^{Z_1} G(Z_i - x) dG^{(j)}(x) \\
 & \quad - \sum_{j=0}^{\infty} \bar{F}((j+2)T) \sum_{i=2}^L \sum_{n=i}^K (c_n - c_{n-1}) \\
 & \quad \times \int_0^{Z_1} \left[\int_{Z_{i-1}-x}^{Z_i-x} G(Z_n - x - y) dG(y) \right] dG^{(j)}(x) \\
 C_6(Z_1) = & \frac{\hspace{10em}}{\int_0^T \bar{F}(t) dt + \sum_{j=0}^{\infty} \int_{(j+1)T}^{(j+2)T} \bar{F}(t) dt \int_0^{Z_1} G(Z_L - x) dG^{(j)}(x)} \hspace{10em} (46)
 \end{aligned}$$

When $F(t) = 1 - \exp(-\lambda t)$ and $G(x) = 1 - \exp(-\mu x)$, (46) is:

$$\begin{aligned}
 & c_K - \alpha e^{\alpha\mu Z_1} \left\{ \alpha \sum_{i=2}^L \sum_{j=i}^K (c_j - c_{j-1}) \right. \\
 & \quad \times [e^{-\mu Z_{i-1}} - e^{-\mu Z_i} - \mu(Z_i - Z_{i-1})e^{-\mu Z_j}] \\
 C_6(Z_1) = & \frac{-\sum_{i=L+1}^K (c_i - c_{i-1})(e^{-\mu Z_i} - e^{-\mu Z_L})}{\lambda} + c_D - c_K \hspace{10em} (47) \\
 & \left. \frac{1 - \alpha e^{\alpha\mu Z_1}}{1 - e^{-\mu(Z_L - Z_1)}} \right\}
 \end{aligned}$$

We find optimal Z_1^* to minimize $C_6(Z_1)$. Differentiating $C_6(Z_1)$ with respect to Z_1 and putting it equal to zero,

$$\begin{aligned}
 & \frac{1 - \alpha e^{-(1-\alpha)\mu Z_1}}{1 - e^{-\mu(Z_L - Z_1)}} \left\{ \sum_{i=2}^L (c_i - c_{i-1}) \left[1 - e^{-\mu(Z_i - Z_1)} + \frac{\alpha}{1 - \alpha} \mu(Z_i - Z_1) e^{-\mu(Z_i - Z_1)} \right] \right. \\
 & + \sum_{i=L+1}^K (c_i - c_{i-1}) \left[1 - e^{-\mu(Z_L - Z_1)} + \frac{\alpha}{1 - \alpha} \mu(Z_L - Z_1) e^{-\mu(Z_i - Z_1)} \right] \left. \right\} \\
 & + \alpha e^{-(1-\alpha)\mu Z_1} \sum_{i=2}^K (c_i - c_{i-1}) [1 - e^{-\mu(Z_i - Z_1)}] = c_K \hspace{10em} (48)
 \end{aligned}$$

When $L = K$, (48) is equivalent to (43).

6. Numerical Calculation

We assume that $G(x) = 1 - \exp(-\mu x)$, $c_i = i$ ($i=1, 2, 3, \dots, K$), and $Z_i = i$ ($i=2, 3, 4, \dots, K$). Table 1 gives the optimal managerial cumulative damage Z_1^* ($0 < Z_1^* < 2$) and the minimum cost rate $TC_1(Z_1^*)$ of Model 1 when $T = 1$, $K=25, 30, 35$, and $\mu = 1, 1.5, 2$. In this illustration, Z_1^* decreases and $TC_1(Z_1^*)$ increases when c_K and $1/\mu$ increases.

Table 1. Optimal Z_1^* and $TC_1(Z_1^*)$ of Model 1 for $K=25, 30, 35$ and $\mu = 1, 1.5, 2$ when $T = 1$.

c_K	μ	Z_1^*	$TC_1(Z_1^*)$
25	1	1.444	1.385
30	1	1.387	1.442
35	1	1.335	1.496
25	1.5	1.602	0.832
25	2	1.657	0.604

Table 2 presents the optimal managerial cumulative damage Z^*_1 and the minimum cost rate $TC_2(Z^*_1)$ of Model 2 when $T = 1, K=25, 30, 35, \mu = 1, 1.5, 2$. In this illustration, Z^*_1 decreases and $TC_2(Z^*_1)$ increases when c_K and $1/\mu$ increases which are similar to Table 1. Z^*_1 in Table 2 are smaller than those in Table 1.

Table 2. Optimal Z^*_1 and $TC_2(Z^*_1)$ of Model 2 for $K=25, 30, 35$ and $\mu = 1, 1.5, 2$ when $T = 1$.

c_K	μ	Z^*_1	$TC_2(Z^*_1)$
25	1	0.981	1.080
30	1	0.925	2.494
35	1	0.924	2.822
25	1.5	0.981	0.884
25	2	0.996	0.469

Table 3 gives the optimal managerial cumulative damage Z^*_1 and the minimum cost rate $TC_3(Z^*_1)$ of Model 3 when $T = 1, L=10, K=25, 30, 35, \mu = 1, 1.5, 2$. In this illustration, Z^*_1 decreases and $TC_3(Z^*_1)$ increases when c_K and $1/\mu$ increases which are similar to Table 1 and 2. Z^*_1 in Table 3 are smaller than those in Table 1 and 2.

Table 3. Optimal Z^*_1 and $TC_3(Z^*_1)$ of Model 3 for $K=25, 30, 35$ and $\mu = 1, 1.5, 2$ when $T = 1$ and $L=10$.

c_K	μ	Z^*_1	$TC_3(Z^*_1)$
25	1	0.878	2.754
30	1	0.875	3.223
35	1	0.873	3.691
25	1.5	0.973	0.992
25	2	0.995	0.483

We assume that $F(t) = 1 - \exp(-\lambda t)$. Table 4 presents the optimal managerial cumulative damage Z^*_1 and the minimum cost rate $TC_4(Z^*_1)$ of Model 4 when $T = 1, K=25, 30, 35, \mu = 1, 1.5, 2$, and $\lambda = 0.1, 0.02, 0.01$. In this illustration, Z^*_1 decreases and $TC_4(Z^*_1)$ increases when c_K and $1/\mu$ increases which are similar to Table 1, 2, and 3. Z^*_1 and $TC_4(Z^*_1)$ decrease when λ decreases. Z^*_1 and $TC_4(Z^*_1)$ in Table 4 are greater than those in Table 1.

Table 4. Optimal Z^*_1 and $TC_4(Z^*_1)$ of Model 4 for $K=25, 30, 35, \mu = 1, 1.5, 2$, and $\lambda = 0.1, 0.02, 0.01$ when $T = 1$.

c_K	μ	λ	Z^*_1	$TC_4(Z^*_1)$
25	1	0.1	1.548	11.260
30	1	0.1	1.487	11.316
35	1	0.1	1.433	11.369
25	1.5	0.1	1.725	10.716
25	2	0.1	1.762	10.502
25	1	0.02	1.465	3.359
25	1	0.01	1.455	2.372

Table 5 gives the optimal managerial cumulative damage Z^*_1 and the minimum cost rate $TC_5(Z^*_1)$ of Model 5 when $T = 1, K=25, 30, 35, \mu = 1, 1.5, 2$, and $\lambda = 0.1, 0.02, 0.01$. In this illustration, Z^*_1 decreases and $TC_5(Z^*_1)$ increases when c_K and $1/\mu$ increases which are similar to Table 1, 2, 3, and

4. Z^*_1 and $TC_5(Z^*_1)$ decrease when λ decreases which are similar to Table 4. Z^*_1 and $TC_5(Z^*_1)$ in Table 5 are greater than those in Table 2.

Table 5. Optimal Z^*_1 and $TC_5(Z^*_1)$ of Model 5 for $K=25, 30, 35, \mu = 1, 1.5, 2$, and $\lambda = 0.1, 0.02, 0.01$ when $T = 1$.

c_K	μ	λ	Z^*_1	$TC_5(Z^*_1)$
25	1	0.1	1.458	11.709
30	1	0.1	1.272	11.870
35	1	0.1	1.124	12.019
25	1.5	0.1	1.998	10.852
25	2	0.1	1.999	10.555
25	1	0.02	1.329	3.913
25	1	0.01	1.313	2.940

Table 6 presents the optimal managerial cumulative damage Z^*_1 and the minimum cost rate $TC_6(Z^*_1)$ of Model 6 when $T = 1, L=10, K=25, 30, 35, \mu = 1, 1.5, 2$, and $\lambda = 0.1, 0.02, 0.01$. In this illustration, Z^*_1 decreases and $TC_6(Z^*_1)$ increases when c_K and $1/\mu$ increases which are similar to Table 1, 2, 3, 4, and 5. Z^*_1 and $TC_6(Z^*_1)$ decrease when λ decreases which are similar to Table 4, 5, and 6. Z^*_1 and $TC_6(Z^*_1)$ in Table 6 are greater than those in Table 3.

Table 6. Optimal Z^*_1 and $TC_6(Z^*_1)$ of Model 6 for $K=25, 30, 35, \mu = 1, 1.5, 2$, and $\lambda = 0.1, 0.02, 0.01$ when $T = 1$ and $L=10$.

c_K	μ	λ	Z^*_1	$TC_6(Z^*_1)$
25	1	0.1	1.252	11.834
30	1	0.1	1.078	12.010
35	1	0.1	0.940	12.174
25	1.5	0.1	1.997	10.887
25	2	0.1	1.999	10.561
25	1	0.02	1.138	4.045
25	1	0.01	1.124	3.072

Table 7 gives the optimal managerial cumulative damage Z^*_1 and the minimum cost rate $TC_6(Z^*_1)$ of Model 6 when $T = 1, K=5, \mu = 1, \lambda = 0.1$, and $L=1, 2, 3, 4, 5$. In this illustration, Z^*_1 increases and $TC_6(Z^*_1)$ decreases when L increases.

Table 7. Optimal Z^*_1 and $TC_6(Z^*_1)$ of Model 6 for $L=1, 2, 3, 4, 5$ when $T = 1, K=5, \mu = 1, \lambda = 0.1$.

L	Z^*_1	$TC_6(Z^*_1)$
1	0.911	12.120
2	0.976	12.057
3	1.084	11.960
4	1.165	11.897
5	1.458	11.709

The fluctuation tendencies of optimal managerial levels and their cost rates due to the fluctuations in parameters c_K and μ are the same in Model 1, 2, 3, 4, 5, and 6, and are consistent with intuitive results. It is evident that optimal managerial levels of Model 2, 3, 5, and 6 which consider the maintenance delay, are smaller than those of Model 1 and 4 which do not consider the maintenance

delay, because the maintenance delay raises the occurrence probability of CM. It is evident that cost rates of Models 4, 5 and 6 which consider the natural disaster are higher than those of Models 1, 2 and 3 which do not consider it. Managerial levels of Models 4, 5 and 6 are higher than those of Models 1, 2 and 3 because the risk of occurrence probability of CM due to natural disasters reduces the risk due to non-natural disasters. When L increases, optimal managerial levels increase because the risk of occurrence probability of CM increases.

7. Conclusions

When the maintenance of infrastructure is planned, some unique characteristics which are different from mechanical ones must be considered. Because the scale of infrastructure is larger than that of mechanical systems, the relationship between damage and maintenance costs of infrastructure is complicated. Maintenances of some facilities must be postponed to the next fiscal year because the budget of local government is restricted. Because damages caused by natural disasters are inevitable, their effects must be considered when the maintenance plan is established. Based on a multiple cost model (Model 1) of Kishida et al. (2019) two expanded models (Model 2 and 3) which consider the maintenance delay have been considered. Furthermore, expanded models of these 3 models (Model 4, 5, and 6) which consider natural disasters have been established. We showed that there existed optimal solutions of Models 1 and 4 analytically and numerical calculation showed that there exist optimum values which minimizes expected cost rates of 6 Models. By performing sensitivity analysis of each Model, reasonable results could be obtained. By using these Models, the multiple managerial maintenance damage of infrastructure can be considered and the optimal maintenance can be clarified. These Models clarify which infrastructure should be prioritized under a limited budget. In particular, by considering the impact of natural disasters, these Models can be used effectively when requesting budgets to the central government and explaining to local residents. In the future, we plan to apply this maintenance model to an actual infrastructure system and confirm its usefulness.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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